CSE 20: Discrete Mathematics
Spring Quarter 1998

Solutions for Quiz #3 - April 22, 1998

1. Consider the following (wrong) “proof” that the product of any two consecutive integers is even. Find the type of mistake (as given on page 120 and 121 in the book) in it.

Proof: Consider two consecutive integers. Call the smaller one \( n \). By the parity theorem, either \( n \) is even or \( n \) is odd.

Case 1 (\( n \) is even):
Let \( n = 4 \). Then \( n \cdot (n + 1) = 4 \cdot 5 = 20 \) which is even.

Case 2 (\( n \) is odd):
Let \( n = 3 \). Then \( n \cdot (n + 1) = 3 \cdot 4 = 12 \) which is even.

Hence, in either case the product \( n \cdot (n + 1) \) is even as was to be shown.

This incorrect proof is arguing from examples. It just shows the theorem to be true in two cases. A real proof must show it’s true for all consecutive integers.

2. A two-dimensional array \( m \) has 5 rows and 3 columns

\[
\begin{bmatrix}
m[0,0] & m[0,1] & m[0,2] \\
m[1,0] & m[1,1] & m[1,2] \\
m[2,0] & m[2,1] & m[2,2] \\
m[3,0] & m[3,1] & m[3,2] \\
m[4,0] & m[4,1] & m[4,2] \\
\end{bmatrix}
\]

The 15 entries in the array are to be stored in column major form in locations 2,816 to 2,830 in a computer’s memory. This means that the entries in the first column (reading top to bottom) are stored first, then the entries in the second column, and finally the entries in the third column.

a. Which location will \( m[3,1] \) be stored in?

\( m[3,1] \) will be stored in location 2,824.

b. Write a formula (in \( r \) and \( s \)) that gives the offset \( n \) such that \( m[r, s] \) is stored in location 2,816 + \( n \).

\( m[i,j] \) is stored in location 2,816 + 5\( j \) + \( i \). Thus,

\[ n = 5j + i \]

c. Find formulas (in \( n \)) for \( i \) and \( j \) so that \( m[i, j] \) is stored in location 2,816 + \( n \).

\[ i = n \mod 5 \]

and

\[ j = n \div 5 \]
3. Prove that for any odd integer $n$,
\[
\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}.
\]

Proof:

Let $n$ be an odd integer.

[We must show that $\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}$.]

By definition of odd, $n = 2k + 1$ for some integer $k$.

Substituting into the left-hand side of the equation to be proved gives
\[
\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(2k + 1)^2}{4} \right\rfloor = \left\lfloor \frac{4k^2 + 4k + 1}{4} \right\rfloor = \left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k + 1
\]
where $\left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k + 1$ by definition of ceiling because $k^2 + k + 1$ is an integer and $k^2 + k < k^2 + k + \frac{1}{4} < k^2 + k + 1$.

On the other hand, substituting into the right-hand side of the equation to be proved gives
\[
\frac{n^2 + 3}{4} = \frac{(2k + 1)^2 + 3}{4} = \frac{4k^2 + 4k + 1 + 3}{4} = \frac{4k^2 + 4k + 4}{4} = k^2 + k + 1
\]
also.

Thus the left- and right-hand side of the equation to be proved both equal $k^2 + k + 1$, and so the two sides are equal to each other.

In other words, $\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}$ [as was to be shown].

4. Prove by contradiction that the sum of any rational number and any irrational number is irrational.

Proof:

Suppose not.

Suppose $a$ is a rational number, $b$ is an irrational number and $a + b$ is rational.

[We must derive a contradiction.]

By definition of rational, $a = p/q$ and $a + b = r/s$ where $p$, $q$, $r$ and $s$ are integers and $q \neq 0$ and $s \neq 0$.

By substitution, we have
\[
a + b = \frac{p}{q} + b = \frac{r}{s}
\]
or, equivalently,
\[
b = \frac{r}{s} - \frac{p}{q} = \frac{qs - pq}{qs}
\]
Now $qs - pq$ and $sq$ are both integers [because products and differences of integers are integers] and $sq \neq 0$ because $s \neq 0$ and $q \neq 0$.

Hence by definition of rational, $b$ is a rational number.

But this contradicts the supposition that $b$ is irrational.

[Hence the supposition is false and the statement is true.]