1. Suppose $S$ is a set containing $k$ elements, and $x \in S$. Let $A$ denote the set of all subsets of $S$ that contain $x$, and $B$ denote the set of all subsets of $S$ that do not contain $x$.

Answer the following questions and give a brief justification or a proof for your answers.

a. Are $A$ and $B$ disjoint (i.e. is $A \cap B = \emptyset$)?
Yes, they are disjoint. Proof by contradiction:
Suppose not, i.e. let the subset $X \in A \cap B$. Then by definition of $A$ and $B$, $X$ contains $x$ and $X$ does not contain $x$. This is a contradiction, and hence $A$ and $B$ are disjoint.

b. Compare the sizes of $A$ and $B$.
The sizes are equal. For each set $X \in A$, it is $X \leftrightarrow \{x\} \in B$.
Conversely, for each set $Y \in B$, $Y \cup \{x\} \in A$.
Furthermore, the mapping $A \ni X \mapsto Y = X \leftrightarrow \{x\} \in B$ is a bijection, because $X$ uniquely defines $X \leftrightarrow \{x\}$ and $Y$ uniquely defines $Y \cup \{x\}$, and $A$ and $B$ are finite.

c. How many elements are in $A \cup B$?
As $A \cup B$ contains all subsets of $S$ that contain $x$, and all subsets of $S$ that do not contain $x$, it contains all subsets of $S$, i.e. $A \cup B = \mathcal{P}(S)$.
Being equal to the power set of $S$, by theorem 5.3.6 it must contain $2^k$ elements (recall $S$ has $k$ elements).

2. Consider the function $f : D \to \mathbb{R}$, where $D = \{x \in \mathbb{R} | x > 0\}$

$$f(x) = \frac{2x \Leftrightarrow 3}{5x}$$

a. Is $f$ one-to-one? Give a proof or a counterexample.
$f$ is one-to-one. Proof:
Let $x, y \in D$ and suppose $f(x) = f(y)$. Show $x = y$. Indeed, it is

$$\frac{2x \Leftrightarrow 3}{5x} = \frac{2y \Leftrightarrow 3}{5y}$$

$$\Leftrightarrow (2x \Leftrightarrow 3)5y = (2y \Leftrightarrow 3)5x$$

$$\Leftrightarrow 10xy \Leftrightarrow 15y = 10yx \Leftrightarrow 15x$$

$$\Leftrightarrow 15y = 15x$$

$$\Leftrightarrow y = x$$
b. Find $f^{-1}: f(D) \to \mathbb{R}$.

Let $y = f(x)$. Then

$$y = \frac{2}{5} \Leftrightarrow \frac{3}{5x}$$
$$\Leftrightarrow \frac{2}{5} = \frac{3}{5x}$$
$$\Leftrightarrow 5x = \frac{3}{\frac{2}{5}} \Leftrightarrow y$$
$$\Leftrightarrow x = \frac{3}{2} \Leftrightarrow 5y$$

So $f^{-1}: f(D) \ni y \mapsto f^{-1}(y) = \frac{3}{2} \Leftrightarrow 5y \in \mathbb{R}$.

3. Determine whether the given binary relation $\mathcal{R}$ is reflexive, symmetric, transitive, or none of these. Justify your answers.

$$x, y \in \mathbb{R}, \quad x \mathcal{R} y \iff y = x^2 + 5$$

$\mathcal{R}$ is not reflexive:

Counterexample: Let $x = 1$, then $x = 1 \neq 6 = x^2 + 5$, so not $x \mathcal{R} x$.

$\mathcal{R}$ is not symmetric:

Counterexample: Let $x = 1$, $y = 6$.

Then $x \mathcal{R} y$ because $x^2 + 5 = 1 + 5 = 6 = y$, but $6^2 + 5 = 41 \neq 1$, so not $y \mathcal{R} x$.

$\mathcal{R}$ is not transitive:

Counterexample: Let $x = 0$, $y = 5$, $z = 30$.

Then $x \mathcal{R} y$ and $y \mathcal{R} z$ because $x^2 + 5 = 5 = y$ and $y^2 + 5 = 25 + 5 = 30 = z$, but $x^2 + 5 = 5 \neq 30 = z$, and so not $x \mathcal{R} z$.

4. Design a finite state automaton with input alphabet $\{0, 1, 2\}$ to accept exactly the set of all words which end with $0^*12^*1$ and do not contain $12$.

(Note that the symbol $2^*$ stands for an arbitrary number of $2$'s, including zero.)

It is sufficient if you draw the transition diagram, you do not need to give the next-state-table.
5. Draw a Venn diagram and shade the area corresponding to the set:

\[
[(C \cap A) \leftrightarrow B] \cup [(C \cap B) \leftrightarrow A]
\]