1. Design a finite state automaton with input alphabet \( \{a, b, c\} \) to accept exactly the set of all words which end with \( c^*ab^*a \) and do not contain \( ab \).
(Note that the symbol \( b^* \) stands for an arbitrary number of \( b \)'s, including zero.)
It is sufficient if you draw the transition diagram, you do not need to give the next-state-table.

2. Draw a Venn diagram and shade the area corresponding to the set:
\[
[(A \cap B) \Leftrightarrow C] \cup [(A \cap C) \Leftrightarrow B]
\]
3. Suppose $A$ is a set containing $n$ elements, and $w \in A$. Let $S_1$ denote the set of all subsets of $A$ that contain $w$, and $S_2$ denote the set of all subsets of $A$ that do not contain $w$.

Answer the following questions and give a brief justification or a proof for your answers.

a. Are $S_1$ and $S_2$ disjoint (i.e. is $S_1 \cap S_2 = \emptyset$)?

Yes, they are disjoint. Proof by contradiction:
Suppose not, i.e. let the subset $X \in S_1 \cap S_2$. Then by definition of $S_1$ and $S_2$, $X$ contains $w$ and $X$ does not contain $w$. This is a contradiction, and hence $S_1$ and $S_2$ are disjoint.

b. Compare the sizes of $S_1$ and $S_2$.

The sizes are equal. For each set $X \in S_1$, it is $X \leftrightarrow \{w\} \in S_2$.
Conversely, for each set $Y \in S_2$, $Y \cup \{w\} \in S_1$.
Furthermore, the mapping $S_1 \ni X \mapsto Y = X \leftrightarrow \{w\} \in S_2$ is a bijection, because $X$ uniquely defines $X \leftrightarrow \{w\}$ and $Y$ uniquely defines $Y \cup \{w\}$, and $S_1$ and $S_2$ are finite.

c. How many elements are in $S_1 \cup S_2$?

As $S_1 \cup S_2$ contains all subsets of $A$ that contain $w$, and all subsets of $A$ that do not contain $w$, it contains all subsets of $A$, i.e. $S_1 \cup S_2 = \mathcal{P}(A)$.
Being equal to the power set of $A$, by theorem 5.3.6 it must contain $2^n$ elements (recall $A$ has $n$ elements).

4. Determine whether the given binary relation $R$ is reflexive, symmetric, transitive, or none of these. Justify your answers.

$x, y \in \mathbb{R}$, $x \, R \, y \iff y^2 = x + 7$

$R$ is not reflexive:
Counterexample: Let $x = 1$, then $x^2 = 1 \neq 8 = x + 7$, so not $x \, R \, x$.

$R$ is not symmetric:
Counterexample: Let $y = 1$, $x = \sqrt[3]{6}$.
Then $x \, R \, y$ because $y^2 = 1 = \sqrt[3]{6^2} = x + 7$, but $\sqrt[3]{6^2} = 36 \neq 8 = 1 + 7$, so not $y \, R \, x$.

$R$ is not transitive:
Counterexample: Let $x = 2$, $y = \sqrt[3]{3}$, $z = 2$.
Then $x \, R \, y$ and $y \, R \, z$ because $x + 7 = 2 + 7 = 9 = (\sqrt[3]{3})^2 = y^2$ and $y + 7 = \sqrt[3]{3} + 7 = 4 = 2^2 = z^2$, but $x + 7 = 9 \neq 4 = z^2$, and so not $x \, R \, z$.

5. Consider the function $f : D \to \mathbb{R}$, where $D = \{x \in \mathbb{R} \mid x > 0\}$

$$f(x) = \frac{3 \sqrt[5]{5x}}{2x}$$

a. Is $f$ one-to-one? Give a proof or a counterexample.

$f$ is one-to-one. Proof:
Let $x, y \in D$ and suppose $f(x) = f(y)$. Show $x = y$. Indeed, it is

\[
\frac{3 \Leftrightarrow 5x}{2x} = \frac{3 \Leftrightarrow 5y}{2y} \\
\Leftrightarrow (3 \Leftrightarrow 5x)2y = (3 \Leftrightarrow 5y)2x \\
\Leftrightarrow 6y \Leftrightarrow 10xy = 6x \Leftrightarrow 10yx \\
\Leftrightarrow 6y = 6x \\
\Leftrightarrow y = x
\]

b. Find $f^{-1} : f(D) \to \mathbb{R}$.

Let $y = f(x)$. Then

\[
y = \frac{3 \Leftrightarrow 5x}{2x} \\
\Leftrightarrow y + \frac{5}{2} = \frac{3}{2x} \\
\Leftrightarrow 2x = \frac{3}{y + \frac{5}{2}} \\
\Leftrightarrow x = \frac{3}{2y + 5}
\]

So $f^{-1} : f(D) \ni y \mapsto f^{-1}(y) = \frac{3}{2y + 5} \in \mathbb{R}$.