Section 5.1

5. a. The solution is given in the book.
   b. Yes. Every element in \( C \) is contained in \( A \).
   c. Yes. Every element in \( C \) is contained in \( A \).
   d. The solution is given in the book.

8. The solution is given in the book.

12. a. The solution is given in the book.
    b. The solution is given in the book.
   c. No. 8 \( \in \) \( D \) because \( 8 = 3 \cdot 3 - 1 \). But 8 \( \not\in \) \( A \).
       For if 8 were in \( A \), then \( 8 = 2i - 1 \) for some integer \( i \).
       Solving for \( i \) would give \( 2i = 9 \), or \( i = 9/2 \), which is not an integer.
       Hence 8 \( \not\in \) \( A \).
   d. Yes.
      \([\text{Show } B \subseteq D \text{ and } D \subseteq B:]\)

Suppose \( n \in B \).
By definition of \( B \), \( n = 3j + 2 \) for some integer \( j \).
But then \( n = 3j + 2 = 3j + 3 - 1 = 3(j + 1) - 1 \).
Let \( s = j + 1 \).
Then \( s \) is an integer and \( n = 3s - 1 \).
So by definition of \( D \), \( n \in D \).
Hence any element of \( B \) is in \( D \), or, symbolically, \( B \subseteq D \).

Conversely, suppose \( q \in D \).
By definition of \( D \), \( q = 3s - 1 \) for some integer \( s \).
But then \( q = 3s - 1 = 3s - 3 + 2 = 3(s - 1) + 2 \) for some integer \( s \).
Let \( j = s - 1 \).
Then \( j \) is an integer and \( q = 3j + 2 \).
So by definition of \( B \), \( q \in B \).
Hence any element of \( D \) is in \( B \), or, symbolically, \( D \subseteq B \).

Since \( B \subseteq D \) and \( D \subseteq B \), by definition of set equality \( B = D \).
15. b. 

\[ (A \cup B)^c \]

Shaded region is 

18. c. 

\[ A \times B \times C = \{(1, u, m), (1, u, n), (1, v, m), (1, v, n), \\
(2, u, m), (2, u, n), (2, v, m), (2, v, n), \\
(3, u, m), (3, u, n), (3, v, m), (3, v, n)\} \]

20 

\[ L = \{11s, 111, 12s, 21s, 211, 22s, 221\} \]

22. 

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</table>

Section 5.2 

4. a. A \cup B \subseteq B 

b. A \cup B 

c. x \in B 

d. A 

e. or 

f. B 

g. A 

h. B 

i. B
9. **Proof:**

Suppose \( A \) and \( B \) are arbitrarily chosen sets.

*Show \((A - B) \cup (A \cap B) \subseteq A*: 

Suppose \( x \in (A - B) \cup (A \cap B) \).

By definition of union, \( x \in (A - B) \) or \( x \in A \cap B \).

*Case 1 \((x \in A - B)\):*

Then by definition of set difference \( x \in A \) and \( x \notin B \).

In particular, \( x \in A \).

*Case 2 \((x \in A \cap B)\):*

Then by definition of intersection \( x \in (A \cap B) \), and \( x \in A \).

In particular, \( x \in A \).

Hence in either case, \( x \in A \), and so by definition of subset, \((A - B) \cup (A \cap B) \subseteq A\).

*Show \( A \subseteq (A - B) \cup (A \cap B)\):*

Suppose \( x \in A \).

Either \( x \in B \) or \( x \notin B \).

*Case 1 \((x \in B)\):*

Then since \( x \in A \) also, by definition of intersection \( x \in (A \cap B) \), and so by the inclusion in union property, \( x \in (A - B) \cup (A \cap B) \).

*Case 2 \((x \notin B)\):*

Then since \( x \in A \), by definition of set difference \( x \in A - B \), and so by the inclusion in union property, \( x \in (A - B) \cup (A \cap B) \).

Hence in either case, \( x \in (A - B) \cup (A \cap B) \), and so by definition of subset, \( A \subseteq (A - B) \cup (A \cap B) \).

Since both set containments have been proved, \((A - B) \cup (A \cap B) = A\) by definition of set equality.

11. **True.**

**Proof:**

Let \( A, B \) and \( C \) be any sets.

*Show \((A - B) \cap (C - B) \subseteq (A \cap C) - B and (A \cap C) - B \subseteq (A - B) \cap (C - B)\):*

*Show \((A - B) \cap (C - B) \subseteq (A \cap C) - B\):

Suppose \( x \in (A - B) \cap (C - B) \).

By definition of intersection, \( x \in A - B \) and \( x \in C - B \), and so by definition of set difference, \( x \in A \) and \( x \notin B \) and \( x \in C \) and \( x \notin B \).

Thus \( x \in A \) and \( x \in C \) and \( x \notin B \).

So by definition of intersection, \( x \in (A \cap C) - B \), and by definition of set difference \( x \in (A \cap C) - B \).
[Thus \((A - B) \cap (C - B) \subseteq (A \cap C) - B\) by definition of subset.]

Show \((A \cap C) - B \subseteq (A - B) \cap (C - B)\):
Suppose \(x \in (A \cap C) - B\).
By definition of set difference, \(x \in A \cap C\) and \(x \notin B\),
and by definition of intersection, \(x \in A\) and \(x \in C\) and \(x \notin B\).
Thus it is true that \(x \in A\) and \(x \notin B\) and \(x \in C\) and \(x \notin B\),
and so by definition of set difference, \(x \in A - B\) and \(x \in C - B\),
and by definition of intersection, \(x \in (A - B) \cap (C - B)\).
[Thus \((A \cap C) - B \subseteq (A - B) \cap (C - B)\) by definition of subset.]

[Since both subset containments have been proved, \((A - B) \cap (C - B) = (A \cap C) - B\) by definition of set equality.]

15. The solution is given in the book.

20. False. Counterexample:
Let \(A = \{1\}\), \(B = \{2\}\) and \(C = \{1, 3\}\).
Then \(A \notin B\) and \(B \notin C\) but \(A \subseteq C\).

24. True.
Proof:
Suppose \(A\), \(B\) and \(C\) are sets.

Show \(A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)\):
Suppose \((x, y) \in A \times (B \cap C)\).
By definition of Cartesian product, \(x \in A\) and \(y \in B \cap C\).
By definition of intersection, \(y \in B\) and \(y \in C\).
It follows that both statements “\(x \in A\) and \(y \in B\)” and “\(x \in A\) and \(y \in C\)” are true.
Hence by definition of Cartesian product, \((x, y) \in A \times B\) and \((x, y) \in A \times C\),
and so by definition of intersection, \((x, y) \in (A \times B) \cap (A \times C)\).
[Thus \(A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)\) by definition of subset.]

Show \((A \times B) \cap (A \times C) \subseteq A \times (B \cap C)\):
Suppose \((x, y) \in (A \times B) \cap (A \times C)\).
By definition of intersection, \((x, y) \in A \times B\) and \((x, y) \in A \times C\),
and so by definition of Cartesian product \(x \in A\) and \(y \in B\) and also \(x \in A\)
and \(y \in C\).
Consequently, the statement “\(x \in A\) and both \(y \in B\) and \(y \in C\)” is true.
It follows by definition of intersection that \(x \in A\) and \(y \in B \cap C\),
and so by definition of Cartesian product, \((x, y) \in A \times (B \times C)\).
[Thus \((A \times B) \cap (A \times C) \subseteq A \times (B \cap C)\) by definition of subset.]
[Since both subset containments have been proved, \( (A \times B) \cap (A \times C) = A \times (B \cap C) \) by definition of set equality.]

26. The solution is given in the book.

29. The solution is given in the book.

36. Proof:

Let sets \( A \) and \( B \) be given. Then

\[
B^c \cup (B^c \setminus A)^c = (B^c \cup (B^c \cap A^c))^c \quad \text{by the alternate representation for set difference laws}
\]
\[
= (B^c)^c \cap (B^c \cap A^c)^c \quad \text{by De Morgan's law}
\]
\[
= B \cap (B^c \cap A^c)^c \quad \text{by the double complement law}
\]
\[
= B \cap ((B^c)^c \cup (A^c)^c) \quad \text{by De Morgan's law}
\]
\[
= B \cap (B \cup A) \quad \text{by the double complement law}
\]
\[
= B \quad \text{by the absorption law.}
\]

Section 5.3

4. a. For any subset \( A \) of a universal set \( U \), \( A \cap A^c = \emptyset \).

Proof:
Let \( A \) be a subset of the universal set \( U \). Suppose \( A \cap A^c \neq \emptyset \), that is, suppose there were an element \( x \) such that \( x \in A \cap A^c \).

Then by definition of intersection, \( x \in A \) and \( x \in A^c \). But by definition of the complement, \( x \in A^c \) implies \( x \notin A \). This is a contradiction. \([Hence the supposition is false, and we conclude that A \cap A^c = \emptyset.]

b. For any subset \( A \) of a universal set \( U \), \( A \cup A^c = U \).

Proof:
Suppose \( A \) is a subset of the universal set \( U \).

\( A \cup A^c \subseteq U \): Let \( x \in A \cup A^c \). By definition of union, \( x \in A \) or \( x \in A^c \). But \( A \) is a subset of \( U \) by hypothesis and \( A^c \) is also a subset of \( U \) by definition of the complement. Thus by definition of subset, \( x \in U \) regardless of whether \( x \in A \) or \( x \in A^c \) \([and hence A \cup A^c \subseteq U.]

\( U \subseteq A \cup A^c \): Let \( x \in U \). It is certainly true that \( x \in A \) or \( x \notin A \) (this is a tautology). But by definition of the complement, if \( x \notin A \) then \( x \in A^c \).

Thus \( x \in A \) or \( x \in A^c \), and so by definition of union \( x \in A \cup A^c \) \([and hence U \subseteq A \cup A^c.]

5
5. a. If $U$ is a universal set, then $U^c = \emptyset$.

Proof:
Let $U$ be a universal set. Suppose $U^c \neq \emptyset$, that is, suppose there were an element $x$ such that $x \in U^c$.
Then by definition of the complement, $x \not\in U$. But, by definition, a universal set contains all elements under discussion, and thus it is impossible that $x \not\in U$. [Hence the supposition is false, and we conclude that $U^c = \emptyset$.]

b. If $U$ is a universal set, then $\emptyset^c = U$.

Proof:
Let $U$ be a universal set.

$\emptyset \subseteq U$: Let $x \in \emptyset$. By definition of complement, $x \in U$ and $x \not\in \emptyset$. In particular, $x \in U$. [Hence $\emptyset^c \subseteq U$ by definition of subset.]

$U \subseteq \emptyset$: Let $x \in U$. By definition of $\emptyset$, $x \not\in \emptyset$ because no element is in $\emptyset$. Thus $x \in U$ and $x \not\in \emptyset$, and so by definition of complement $x \in \emptyset$. [Hence $U \subseteq \emptyset^c$ by definition of subset.]

[Since both set containments $\emptyset^c \subseteq U$ and $U \subseteq \emptyset^c$ have been proved, $\emptyset^c = U$ by definition of equality.]

8. This is true. Illustration:

```
A   B   C
A - B
B - C
```

Proof: Let $A$, $B$ and $C$ be any sets, and suppose that

$$(A - B) \cap (B - C) \cap (A - C) \neq \emptyset.$$ 

Then there is an element $x$ such that

$$x \in (A - B) \cap (B - C) \cap (A - C).$$

By definition of intersection, $x \in A$ and $x \not\in B$ and $x \in B$ and $x \not\in C$. In particular, $x \in B$ and $x \not\in B$ which is a contradiction. Hence the supposition is false. That is, $(A - B) \cap (B - C) \cap (A - C) = \emptyset$.

10. The solution is given in the book.
13. This is true. Illustration:

![Illustration](image)

**Proof:** Let $A$, $B$ and $C$ be any sets such that $A \subseteq B$ and $B \cap C = \emptyset$. Suppose $A \cap C \neq \emptyset$, that is, suppose there were an element $x$ in $A \cap C$. By definition of intersection, $x \in A$ and $x \in C$.

By hypothesis, $A \subseteq B$, and so since $x \in A$, $x \in B$ also. Hence $x \in B \cap C$, which implies $B \cap C \neq \emptyset$. But $B \cap C = \emptyset$ by hypothesis. This is a contradiction. [Thus the supposition that $A \cap C \neq \emptyset$ is false, or equivalently, $A \cap C = \emptyset$.]

21. The solution is given in the book.

24. Let $A$ and $B$ be sets. Then

$$(A \cup B) - B$$

$$= (A \cup B) \cap B^c$$

by the alternate representation

for set difference law

$$= B^c \cap (A \cup B)$$

by the commutative law for $\cap$

$$= (B^c \cap A) \cup (B^c \cap B)$$

by the distributive law

$$= (A \cap B^c) \cup (B^c \cap B)$$

by the commutative law for $\cap$

$$= (A \cap B^c) \cup \emptyset$$

by the intersection with the complement law

$$= (A \cap B^c)$$

by the union with $\emptyset$ law

$$= A \cap B^c$$

by the alternate representation

for set difference law
27. 

\[(A - (A \cap B)) \cap (B - (A \cap B))\]

\[= (A \cap (A \cap B)^c) \cap (B \cap (A \cap B)^c)\]  
by the alternate representation for set difference law

\[= A \cap ((A \cap B)^c \cap [B \cap (A \cap B)^c])\]  
by the associative law for \(\cap\)

\[= A \cap ((A \cap B)^c \cap B \cap (A \cap B)^c)\]  
by the associative law for \(\cap\)

\[= A \cap (B \cap [(A \cap B)^c \cap (A \cap B)^c])\]  
by the commutative law for \(\cap\)

\[= A \cap (B \cap (A \cap B)^c)\]  
by the idempotent law for \(\cap\)

\[= (A \cap B) \cap (A \cap B)^c\]  
by the associative law for \(\cap\)

\[= U\]  
by the intersection with the complement law

\[33. \quad \text{d. Proof: Let } A \text{ be any set. Then}\]

\[A \oplus A^c = (A - A^c) \cup (A^c - A)\]  
by definition of \(\oplus\)

\[= (A \cap (A^c)^c) \cup (A^c \cap A)^c\]  
by the alternate representation for set difference law

\[= (A \cap A) \cup A^c\]  
by double negative law and the idempotent law for \(\cap\)

\[= A \cup A^c\]  
by the idempotent law for \(\cap\)

\[= U\]  
by the union with the complement law

34. c. Yes, because by definition of set difference, no element in \(A - (B \cup C)\) can be in \(B \cup C\), which implies (by definition of union) that no element of \(A - (B \cup C)\) can be in \(B\), whereas all elements in \(B - (A \cup C)\) are in \(B\).

Hence there can be no elements simultaneously in the sets \(A - (B \cup C)\) and \(B - (A \cup C)\), and so these sets are disjoint.

35. b. Yes. Every element in \(\{p, q, u, v, w, x, y, z\}\) is in one of the sets of the partition and no element is in more than one set of the partition.

37. Yes. Every real number \(x\) satisfies exactly one of the conditions: \(x > 0\) or \(x = 0\) or \(x < 0\). (See property T16 of Appendix A.)

40. The solution is given in the book.

42. c. \(\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \{ \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\} \}\)

43. d. False. The elements of \(\mathcal{P}(A \times B)\) are subsets of \(A \times B\), whereas the elements of \(\mathcal{P}(A) \times \mathcal{P}(B)\) are ordered pairs whose first element is a subset of \(A\) and whose second element is a subset of \(B\).
To be concrete, let \( A = B = \{1\} \). Then

\[
P(A) = \{ \emptyset, \{1\} \},
\]

\[
P(B) = \{ \emptyset, \{1\} \},
\]

\[
P(A) \times P(B) = \{ (\emptyset, \emptyset), (\emptyset, \{1\}), (\{1\}, \emptyset), (\{1\}, \{1\}) \}
\]

On the other hand, \( A \times B = \{(1,1)\} \), and so

\[
P(A \times B) = \{ \emptyset, \{(1,1)\} \}
\]

45. No, the sets \( S_a, S_b, S_c \), and \( S_d \) are not mutually disjoint. For example, \( \{a,b\} \in S_a \) and \( \{a,b\} \in S_b \).

Section 5.4

4. Since there are no real numbers with negative squares, this sentence is vacuously true, and hence it is a statement.

5. The solution is given in the textbook.

7. Suppose Nixon says (ii) and the only utterance Jones makes about Watergate is (i). Suppose also that apart from (ii) all of Nixon’s other assertions about Watergate are evenly split between true and false.

Case 1 (Statement (i) is true):
In this case, more than half of Nixon’s assertions about Watergate are false, and so (since all of Nixon’s other assertions about Watergate are evenly split between true and false) statement (ii) must be false (because it is an assertion about Watergate). So at least one of Jones’ statements about Watergate is false. But the only statement Jones makes about Watergate is (i). So statement (i) is false.

Case 2 (statement (i) is false):
In this case, half or more of Nixon’s assertions about Watergate are true, and so (since all of Nixon’s other assertions about Watergate are evenly split between true and false) statement (ii) must be true. But statement (ii) asserts that everything Jones says about Watergate is true. And so in particular, statement (i) is true.

The above arguments show that under the given circumstances, statements (i) and (ii) are contradictory.

9. No. Suppose there were such a book. If such a book did not refer to itself, then it would belong to the set of all books that do not refer to themselves. But it is supposed to refer to all books in this set, and so it would refer to itself. On the other hand, if such a book referred to itself, then it would belong to the set of books to which it refers and this set only contains books that do not refer to themselves. Thus it would not refer to itself. It follows that the assumption that such a book exists leads to a contradiction, and so there is no such book.