Section 3.6

1. The solution is in the book.

4. statement: There is no least positive rational number.

   negation: There is a least positive rational number.

Proof (by contradiction): Suppose not.
That is, suppose there is a least positive rational number.
Call this number \( r \).
Then \( r \) is a real number such that \( r > 0 \), \( r \) is rational, and for all positive rational numbers \( x \), \( x > r \).

Let \( s = r/2 \). [We will show that \( s \) is a positive rational number with \( s < r \).]
[Show that \( 0 < s < r \).] Note that if we divide both sides of the inequality
\( 0 < r \) by 2, we obtain \( 0 < r/2 = s \),
and if we add \( r \) to the inequality \( 0 < r \) and then divide by 2, we obtain \( r + r/2 < r/2 + r \), or, equivalently, \( s = r/2 < r \).
Hence \( 0 < s < r \).
[Show that \( s \) is rational:] Note also that since \( r \) is rational, \( r = a/b \) for
some integers \( a \) and \( b \) with \( b \neq 0 \) and so \( s = r/2 = a/(2b) = a/2b \).
Since \( a \) and \( 2b \) are integers and \( 2b \neq 0 \), \( s \) is rational.
Thus we have found a positive rational number \( s \) such that \( s < r \).
This contradicts the supposition that \( r \) is the least positive rational number.
Therefore, there is no least positive rational number.

8. Proof (by contraposition): Suppose \( a \) and \( b \) are [particular but arbitrarily chosen] real numbers such that \( a \geq 25 \) and \( b \geq 25 \).
Then \( a + b \geq 25 + 25 = 50 \).
Hence if \( a + b < 50 \), then \( a < 25 \) or \( b < 25 \).

15. Counterexample: Let \( a = 4 \) and \( n = 2 \).
Then \( 4 \mid n^2 \) because \( 4 \mid 4 \) (since \( 4/4 \) is an integer),
but \( a \nmid n \) because \( 4 \nmid 2 \) (since \( 2/4 \) is not an integer).

20. Counterexample: \( \sqrt{2} \) is irrational.
Also \( \sqrt{2} - \sqrt{2} = 0 \) and 0 is rational (by Theorem 3.2.1).
Thus 3 irrational numbers whose difference is rational.
Section 3.7

6. This statement is false.
\( \sqrt{2}/4 = (1/4) \cdot \sqrt{2} \), which is a product of a nonzero rational number and an irrational number.
By exercise 18 of section 3.6, such a product is irrational.

That is, suppose \( \exists \) an irrational number \( x \) such that \( \sqrt{x} \) is rational.
By definition of rational, \( \sqrt{x} = a/b \) for some integers \( a \) and \( b \) with \( b \neq 0 \).
Then \( x = (\sqrt{x})^2 = (a/b)^2 = a^2/b^2 \).
But \( a^2 \) and \( b^2 \) are both integers (being products of integers), and \( b \neq 0 \)
by the zero product property.
Hence, \( x \) is rational (by definition of rational).
This contradicts the supposition that \( x \) is irrational, and so the supposition
is false.
Therefore, the square root of an irrational number is irrational.

24. There are (at least) two possible ways of proving this result:

Proof (by contradiction): Suppose not.
Suppose there exist two distinct real numbers \( b_1 \) and \( b_2 \) such that for all
real numbers \( r \), (1) \( b_1 r = r \) and (2) \( b_2 r = r \).
Then \( b_1 b_2 = b_2 \) (by (1) with \( r = b_1 \)) and \( b_2 b_1 = b_1 \) (by (2) with \( r = b_2 \)).
Consequently, \( b_2 = b_1 b_2 = b_2 b_1 = b_1 \) by substitution and the commutative
law of multiplication.
But this implies that \( b_1 = b_2 \), which contradicts the supposition that \( b_1 \)
and \( b_2 \) are distinct.
[Thus the supposition is false and there exists at most one real number \( b \)
such that \( br = r \) for all real numbers \( r \).]

Proof (direct): Suppose \( b_1 \) and \( b_2 \) are real numbers such that \( b_1 \)
and \( b_2 \) are distinct.
(1) \( b_1 r = r \) and
(2) \( b_2 r = r \) for all real numbers \( r \).
By (1) \( b_1 b_2 = b_2 \),
and by the commutative law of multiplication and (2), \( b_1 b_2 = b_2 b_1 = b_1 \).
Since both \( b_1 \) and \( b_2 \) are equal to \( b_1 b_2 \), we conclude that \( b_1 = b_2 \).

Section 3.8

3. \( b. \ z = 4 \)

5. \( c = 41/24 \)

17. Proof: Let \( a \) and \( b \) be any positive integers.

Part 1 (proof that if \( \gcd(a, b) = a \) then \( a \mid b \)):
Suppose that \( \gcd(a, b) = a \).
By definition of greatest common divisor, \( \gcd(a, b) \mid b \),
and so by substitution, \( a \mid b \).
Part 2 (proof that if \( a \mid b \) then \( \gcd(a, b) = a \)):

Suppose that \( a \mid b \).

Then since it is also the case that \( a \mid a \), \( a \) is common divisor of \( a \) and \( b \). Thus by definition of greatest common divisor, \( a \leq \gcd(a, b) \).

On the other hand, since no integer greater than \( a \) divides \( a \), the greatest common divisor of \( a \) and \( b \) is less than or equal to \( a \).

In symbols, \( \gcd(a, b) \leq a \).

Therefore, since \( a \leq \gcd(a, b) \) and \( \gcd(a, b) \leq a \), then \( \gcd(a, b) = a \).

22. a. Proof: Suppose \( a, d, q \) and \( r \) are integers such that \( a = d \cdot q + r \) and \( 0 \leq r < d \).

[We must show that \( q = \lfloor a/d \rfloor \) and \( r = a - \lfloor a/d \rfloor \cdot d \).]

Solving \( a = d \cdot q + r \) for \( r \) gives \( r = a - d \cdot q \),

and substitution into \( 0 \leq r < d \) gives \( 0 \leq a - d \cdot q < d \).

Add \( d \cdot q \): Then \( d \cdot q \leq a < d + d \cdot q = d \cdot (q + 1) \),

and so \( q \leq a/d < q + 1 \).

Thus by definition of floor, \( q = \lfloor a/d \rfloor \),

and by substitution into \( r = a - d \cdot q \), we have \( r = a - \lfloor a/d \rfloor \cdot d \) [as was to be shown].

b. \( r := B, a := A, b := -B \)

\[ \textbf{while} \ (b \neq 0) \]
\[ r := a - \lfloor a/b \rfloor \cdot b \]
\[ a := b \]
\[ b := r \]
\[ \textbf{end while} \]
\[ \gcd := a \]

c. \( \text{lcm}(2 \cdot 3^2 \cdot 5, \ 2^3 \cdot 3) = 2^3 \cdot 3^2 \cdot 5 = 360 \)
\( \text{lcm}(3500, 1960) = \text{lcm}(2^2 \cdot 5^2 \cdot 7, \ 2^3 \cdot 5 \cdot 7^2) = 2^3 \cdot 5^2 \cdot 7^2 = 49,000 \)