Top-down Calculus Workbook

Integrals Intuition Practice

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Preface


Reviewing calculus is much more fun than learning it for the first time. You can go over the material in any order, including backwards. Make one pass through the material fairly quickly, focusing on concepts and choosing carefully which problems to work. Additional problems can be worked later to improve technique. Excellent online software is available for calculus (differentiation and integration) and graphing functions.

To facilitate navigation, important elements of each chapter (e.g., definition, lemma, theorem, exercise, figure, etc.) are labeled in order of appearance using a shared numbering system. For example, 4.10 is a figure and 4.11 a theorem. A list of all such labeled elements is in the index.

I wish to thank Professor Emeritus Michael Sharpe, Department of Mathematics, UCSD, for his LaTeX Scanpages Package which made this project feasible.

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Chapter 4

INTEGRALS

Imagine that some students have taken a test on differentiation. The professor has taken the exam papers home. After dinner, she sits down to grade the exam and, looking at the first answer on the first student’s exam, she sees

\[ 2x \cos(x^2). \]

It is then that she realizes, with great disgust, that she left the exam at the office and has forgotten what the first problem on the exam was! She sees that all of the other students got the same answer to this problem, so she assumes it is right. The exam problem was of the type “What is the derivative of \( F(x) \)?” But what was \( F(x) \)? This latter question is what the study of “integration” is all about in calculus. The function \( F(x) \) is called the “integral” or “antiderivative” of \( 2x \cos(x^2) \). If you have had a lot of practice at differentiating functions, especially using the chain rule, you might guess \( F(x) = \sin(x^2) \). This is correct as \( (\sin(x^2))' = \cos(x^2)(x^2)' = 2x \cos(x^2) \).

There is a good deal of such guesswork in finding integrals. It is easy to write down functions \( f(x) \) for which no nice expression for the integral is known. An example is \( e^x \).

The symbol “\( \frac{d}{dx} \)” is used to mean “the derivative of.” Thus, \( \frac{d}{dx} \sin(x) \) means “the derivative of \( \sin(x) \).” Correspondingly, we use the strange symbol “\( \int \)” to mean “the integral of.” Thus, we write \( \int 2x \cos(x^2) = \sin(x^2) \) to mean “the integral of \( 2x \cos(x^2) \) is \( \sin(x^2) \).” This idea is summarized in FIGURE 4.1.
FIGURE 4.1 The Derivative vs. the Integral

\[
\begin{align*}
\frac{d}{dx} & \quad 2x \cos(x^2) \quad \rightarrow \quad \int \quad \sin(x^2) \\
\frac{d}{dx} & \quad f(x) \quad \rightarrow \quad \int \quad F(x)
\end{align*}
\]

WHERE \( F'(x) = f(x) \)

The variable in our examples so far has been \( x \). But, if \( f(t) \) is a function of \( t \) then \( \int f(t) \) is a function \( F(t) \) such that \( \frac{d}{dt} F(t) = f(t) \). But what if we walk into an abandoned classroom and see \( \int 2t = ? \) on the blackboard. What was the variable? If it was \( x \), then the answer is \( 2x \). If the variable was \( t \), then the answer is \( 2t \). To avoid this and related confusions, the notation for integrals or antiderivatives is usually \( \int f(x) \, dx \) or \( \int f(t) \, dt \). Thus, if we had seen \( \int 2dt = ? \) then the answer would have been \( 2t \). If we had seen \( \int 2dx = ? \) then the answer would have been \( 2x \).

**F(x) And F(x) + C Have The Same Derivative**

There is another simple but important observation about integrals that needs to be stated. As we have noted, \( \int 2x \cos(x^2) = \sin(x^2) \), which means that \( \frac{d}{dx} \sin(x^2) = 2x \cos(x^2) \). But, of course, \( \frac{d}{dx}(\sin(x^2) + 10) = 2x \cos(x^2) \) also.

In fact, \( \frac{d}{dx}(\sin(x^2) + C) = 2x \cos(x^2) \) for any constant function \( C \). This fact is sometimes incorporated into the notation for integrals by writing
\[ \int 2\cos(x^2)\,dx = \sin(x^2) + C. \]

This notation is intended to remind us that there are infinitely many functions with derivative \(2\cos(x^2)\) and they all differ by a constant function. Once this observation has been made and we understand what we are talking about, it is quite all right to write simply

\[ \int 2\cos(x^2)\,dx = \sin(x^2). \]

We understand in this latter notation that, in fact, \(\sin(x^2)\) is a representative from an infinite class of antiderivatives for \(2\cos(x^2)\) and all of the rest are obtained by adding a constant function to \(\sin(x^2)\).

If The Zero Function Is Its Derivative, The Function Is Constant

The fact that all antiderivatives of a given function \(f(x)\) differ by a constant is a more subtle idea than it seems at first glance. Suppose we have two functions, \(F(x)\) and \(G(x)\), such that \(F'(x) = G'(x) = f(x)\). Let \(H(x) = F(x) - G(x)\). Then \(H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0\). To claim that \(F(x)\) and \(G(x)\) differ by a constant function is the same as claiming that \(H(x)\) is a constant function. This means that the statement that “any two antiderivatives \(F(x)\) and \(G(x)\) differ by a constant” is the same as the statement that “any function \(H(x)\) with derivative function the zero function must be a constant function” (something like \(H(x) = 2\) so \(F(x) = G(x) + 2\)). This latter statement has strong intuitive appeal. Suppose \(H'(x) = 0\) for all \(x\). Let’s try to draw the graph of such an \(H(x)\). Suppose \(H(0) = 2\), for example. Put your pencil at the point (0, 2) and try to imagine what the graph is like near this point. If, in going right or left, you draw the graph with the slightest bit of slope up or down you will construct points on the graph where \(H'(x)\) is not 0. You’re stuck at \(H(x) = 2\) and must draw the graph of the constant function 2. For more advanced courses in mathematical analysis it is essential that this intuitive idea be given a precise analytical formulation. The intuitive idea, however, will suffice for our studies of calculus.

Constant Functions Can Wear Many Disguises

There is another complication that will occur in connection with the ideas associated with the previous paragraph. Suppose John decides that

\[ \int 2\sin(2x)\,dx = -\cos(2x) \quad \text{(plus any added constant)} \]
and suppose that Mary decides that

\[ \int 2\sin(2x)\,dx = 2\sin^2(x). \quad \text{(plus any added constant)} \]

If they are both right (and they are in this case) then \(2\sin^2(x)\) and \(-\cos(2x)\) must differ by a constant (i.e., \(2\sin^2(x) - (-\cos(2x)) = 2\sin^2(x) + \cos(2x)\) is a constant function). If you know your basic trigonometric identities; then you will recognize that this is true and, in fact, \(2\sin^2(x) + \cos(2x) = 1\). Thus, just because two integrals \(F(x)\) and \(G(x)\) for \(f(x)\) must differ by a constant doesn’t mean that they are easily recognizable as differing by a constant (and are both correct answers to the same integration problem). This has obvious bad implications for anyone who has to grade homework and exams of students doing problems on integration.

\[^* \sin^2(x) = (1 - \cos(2x))/2 \quad \text{or} \quad \cos(2x) = 1 - 2\sin^2(x).\]

**Linearity Of The Integral**

The most basic property of integrals is “linearity,” which is stated in Theorem 4.2.

**4.2 Theorem** Let \(f(x)\) and \(g(x)\) be functions and \(\alpha\) and \(\beta\) numbers. Then

\[ \int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx \]

**Proof:** This follows directly from the definition of the integral together with linearity of \(\frac{d}{dx}\). Let \(F(x)\) and \(G(x)\) be the antiderivatives of \(f(x)\) and \(g(x)\).

Then \(\frac{d}{dx}(\alpha F(x) + \beta G(x)) = \alpha f(x) + \beta g(x)\), which means, by definition of the integral, that

\[ \alpha F(x) + \beta G(x) = \int (\alpha f(x) + \beta g(x))\,dx. \]

Substituting \(F(x) = \int f(x)\,dx\) and \(G(x) = \int g(x)\,dx\) gives the result.

Our first task in learning to find antiderivatives or integrals will be to develop some systematic ways to improve our ability to reduce new problems to ones we have already solved. Theorem 4.2 is a start in this direction. We have already noticed that \(\int 2x\cos(x^2)\,dx = \sin(x^2)\) and \(\int 2x\,dx = \sin^2(x)\).
Thus we can, using THEOREM 4.2, evaluate an integral such as $\int (2\pi x \cos(x^2) + 25 \sin(2x)) \, dx$ getting

$$\pi \int 2x \cos(x^2) \, dx + 25 \int \sin(2x) \, dx = \pi \sin(x^2) + 25 \sin^2(x).$$

**You Already Know Many Integrals**

As we begin to compute more integrals, you will discover that you will use the rule of THEOREM 4.2 automatically. Now is the time to take note of the fact that you already know many integrals! Every differentiation formula you have memorized gives rise to a corresponding integration formula:

$$\frac{d}{dx} \sin(x) = \cos(x) \text{ becomes } \sin(x) = \int \cos(x) \, dx$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x} \text{ becomes } \ln(x) = \int \frac{1}{x} \, dx$$

$$\frac{d}{dx} x^n = nx^{n-1} \text{ becomes } x^n = \int nx^{n-1} \, dx$$

This latter integral has been memorized by millions of calculus students as

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \text{ for } n \neq -1.$$  

The reason for not allowing $n$ to be $-1$ is that the right-hand side of this equation is not defined for $n = -1$. We know, however, that for $n = -1$, $\int x^{-1} \, dx = \ln(x)$

**The Chain Rule In Reverse**

By applying THEOREM 4.2, you can now compute an impressive class of integrals such as

$$\int (34 \sin(x) + 23x^3 + 45 \sec^2(x)) \, dx = -34 \cos(x) + 23 \frac{x^4}{4} + 45 \tan(x).$$

But this is not nearly good enough! To really get started on the problem of computing integrals, we must learn how to do the CHAIN RULE in reverse. In particular,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \text{ becomes } f(g(x)) = \int f'(g(x))g'(x) \, dx.$$
This innocent-looking statement seems to be the source of much difficulty for beginning calculus students. For this reason, we shall look at a number of examples of the type that cause headaches for beginning students.

Let's start with an easy example. Consider \( \int \cos(x^2)(2x)\,dx \). It appears that \( f'(x) = \cos(x) \) and \( g(x) = x^2 \) here. This is determined by guessing or by inspection with the aid of past experience. Thus, \( \int \cos(x^2)(2x)\,dx = \int f'(g(x))g'(x)\,dx = f(g(x)) = \sin(x^2) \). We had to guess that \( g(x) = x^2 \) was the proper choice since the function \( g \) is not explicitly mentioned. The way we make such a guess is by knowing our derivative formulas well. We know that \( \frac{d}{dx} x^2 = 2x \). As our eyes scan the expression \( \cos(x^2)(2x) \), we spot the pair \( x^2 \) and its derivative \( 2x \). This is the clue that prompts us to try \( g(x) = x^2 \).

In addition to guessing correctly that \( g(x) = x^2 \) in the previous example, we had to know how to integrate \( f'(x) = \cos(x) \) to get \( f(x) = \sin(x) \). As another example, let's try to calculate \( \int \ln(x^2)/(2x)\,dx \).

Again, we see the pair \( g(x) = x^2 \) and \( g'(x) = 2x \). Thus \( f'(x) = \ln(x) \) so that \( f'(g(x))g'(x) = \ln(x^2)(2x) \). To compute \( f(g(x)) \), we must compute the integral \( f(x) \) of \( f'(x) \). You may not know how to do this, in which case you are stuck! At this point, some students think "If \( f'(x) = \ln(x) \) then \( f(x) = 1/x \ldots \)," but this is going the wrong way as \( f''(x) = 1/x \). We shall learn later how to find the integral \( f(x) \) of \( f'(x) = \ln(x) \). The answer is \( f(x) = x\ln(x) - x \) (check that \( f'(x) = \ln(x) \) for this function). From this fact, we conclude that \( \int \ln(x^2)/(2x)\,dx = (x^2)\ln(x^2) - x^2 \). Check this statement by computing the derivative of the expression on the right.

Another complication that occurs is seen in the following two integrals:

\[
\int x\cos(x^2)\,dx = ? \quad \text{and} \quad \int 5x\ln(x^2)\,dx = ?
\]

In these integrals, we see the \( g(x) = x^2 \) all right, but the \( g'(x) = 2x \) is not there. Instead, we see \( x \) in the first integral and \( 5x \) in the second integral. This may be confusing, but it's not fatal. We know that for any number \( \alpha \) and any function \( h(x) \), \( \int \alpha h(x)\,dx = \alpha \int h(x)\,dx \). Thus,

\[
\int x\cos(x^2)\,dx = \int (1/2)(2x)\cos(x^2)\,dx = (1/2)\int \cos(x^2)(2x)\,dx
\]

We have already discovered that \( \int \cos(x^2)(2x)\,dx = \sin(x^2) \), so we have \( \int x\cos(x^2)\,dx = (1/2)\sin(x^2) \).

Some beginning calculus students like this trick so much that they try the following type of calculation:
\[
\int 2x^2 \cos(x^2) dx = x \int 2 \cos(x^2) dx = x \sin(x^2).
\]

What they do is bring the variable outside the integral sign. This doesn't work as you can check that \( \frac{d}{dx} x \sin(x^2) \neq 2x^2 \cos(x^2) \). Only constants can be brought outside the integral sign in the rule \( \int \alpha h(x) dx = \alpha \int h(x) dx \).

You should now be in a position to understand the following calculation:

\[
\int 5x \ln(x^2) dx = \int (5/2) 2x \ln(x^2) dx = (5/2) \int \ln(x^2)(2x) dx.
\]

This gives, by our previous discussion,

\[
\int 5x \ln(x^2) dx = (5/2)(x^2 \ln(x^2) - x^2).
\]

**Differential Notation**

Certain notations of calculus are designed to help in the task of applying the CHAIN RULE in reverse. We know that \( \int \cos(x) dx = \sin(x) \). There is, of course, nothing special about the \( x \) here:

\[
\int \cos(t) dt = \sin(t) \quad \int \cos(\tau) d\tau = \sin(\tau) \quad \int \cos(A) dA = \sin(A)
\]

and

\[
\int \cos(\text{CALCULUS}) d(\text{CALCULUS}) = \sin(\text{CALCULUS})
\]

\[
\int \cos(\text{JUNK}) d(\text{JUNK}) = \sin(\text{JUNK}).
\]

The general rule is that if \( F(x) \) is an integral or an antiderivative of \( f(x) \) (i.e., \( F'(x) = f(x) \)) then

\[
\int f(\text{JUNK}) d(\text{JUNK}) = F(\text{JUNK})
\]

where JUNK stands for anything for which these formulas make sense. JUNK can be quite complicated. For example

\[
\int \cos \left( \frac{\ln(x)e^x \tan(x)}{x^2 + 2x^3 + 5x + 1} \right) d \left( \frac{\ln(x)e^x \tan(x)}{x^2 + 2x^3 + 5x + 1} \right) = \\
\sin \left( \frac{\ln(x)e^x \tan(x)}{x^2 + 2x^3 + 5x + 1} \right)
\]
In this formula
\[
\text{JUNK} = \left( \frac{\ln(x)e^x \tan(\sin(x))}{x^3 + 2x^3 + 5x + 1} \right). 
\]

Do you get the idea? As long as we know \( F(x) \) and that \( F'(x) = f(x) \), then \( F(JUNK) = \int f(JUNK) \, d(JUNK) \). What does \( d(JUNK) \) mean? Consider, for example, the function \( g(x) = x^3 + x^2 \). We know that \( g'(x) = 3x^2 + 2x \). In our other notation
\[
\frac{dg}{dx} = 3x^2 + 2x. 
\]

By multiplying by \( dx \) on both sides (what this means can be made precise, but for now we think of this as a notational device only) gives
\[
d(x^3 + x^2) = (3x^2 + 2x) \, dx. 
\]

In other words,
\[
d(\text{some function of } x) = (\text{the derivative of that function}) \, dx. 
\]

4.3 DEFINITION (differential notation) Let \( g(x) \) be a function of \( x \) with \( \frac{dg}{dx} = g'(x) \). Then define \( dg = g'(x) \, dx \). This is the differential notation for the derivative of \( g \). The term \( dg \) is called the differential of \( g \) and the term \( dx \) is called the differential of \( x \).

We can state our above discussion as a (trivial) theorem:

4.4 THEOREM Let \( F(x) \) be such that \( F'(x) = f(x) \) and let \( g(x) \) be a function with differential \( dg = g'(x) \, dx \). Then,
\[
\int f(g) \, dg = F(g). 
\]

This "theorem" is, of course just a restatement of the CHAIN RULE as \( (F(g(x)))' = F'(g(x))g'(x) \). For any function \( h(x) \) we have, by the definition of the integral, \( h(x) = \int h'(x) \, dx = \int dh \). Applying this to the CHAIN RULE (where \( F'(x) = f(x) \)) gives \( F(g(x)) = \int f(F(g(x))) = \int F'(g(x))g'(x) \, dx = \int f(g(x)) \, dg \). Thus \( \int f(g) \, dg = F(g) \). We are just playing with notation here, but such "fooling around" is important. If you find this confusing, skip it for now and reread this section after you have worked a number of examples in the exercises that follow.
How We Make Up Hard Problems

Before beginning to work exercises based on THEOREM 4.4, let’s take a look at the sinister process by which calculus instructors the world over torment their students with this type of problem. Using the notation of THEOREM 4.4, we first pick some function that we know how to integrate. Let’s take our old friend, \( \cos(x) \) to be \( f(x) \). Thus, \( F(x) = \sin(x) \). Next we pick any function \( g(x) \) that we can differentiate. Let’s take \( g(x) = \ln(x^2 + 1) \).

Then, \( g'(x) = \frac{2x}{x^2 + 1} \) or \( dg = \frac{2x}{x^2 + 1} dx \). Substituting into the identity \( \int f(g) dg = F(g) \) we obtain

\[ \int \cos(\ln(x^2 + 1)) d(\ln(x^2 + 1)) = \sin(\ln(x^2 + 1)). \]

But this is the same as

\[ \int \cos(\ln(x^2 + 1)) \frac{2x}{x^2 + 1} dx = \sin(\ln(x^2 + 1)). \]

Now, the mean old calculus instructor gives the poor innocent student the following problem:

\[ \int \frac{x \cos(\ln(x^2 + 1))}{x^2 + 1} dx = ? \]

What does the poor student have to know to solve this awful-looking mess? What a good student will do is notice that \( \ln(x^2 + 1) \) has been substituted into \( \cos(x) \) for \( x \). This will suggest that \( g(x) = \ln(x^2 + 1) \). Next, the student will quickly compute \( g'(x) = \frac{2x}{x^2 + 1} \). This will be good news to the student because this is almost, except for a constant multiple of 2, what appears in the expression to be integrated. Using \( g(x) = \ln(x^2 + 1) \) the student will write the mean old calculus instructor’s problem as \( (1/2) \int \cos(g) dg = ? \) But this is easy; the answer is \( (1/2)\sin(g) \). Substituting for \( g \), the student then obtains the correct answer, \( (1/2)\sin(\ln(x^2 + 1)) \). Differentiate this expression to make sure it’s right! This method of solving integrals is called the ‘‘method of substitution’’ and will be the subject of our first set of exercises.

The following exercises are designed to give you practice with the method ‘‘integration by substitution’’ discussed in the previous paragraphs. In EXERCISE 4.5, you are to find functions \( f(x) \) and \( g(x) \) such that the given problem is of the form \( \int f(g) dg \). The phrase ‘‘of the form’’ means that the given problem may differ from \( \int f(g) dg \) by a constant. In problem 1 of
EXERCISE 4.5, for example, take $g(x) = x^2 + 2$ and $f(x) = x^{1/2}$. Then $dg = 2xdx$ and the given integral $\int x(x^2 + 2)^{1/2}dx = (1/2)\int f(g)dg$. In the problems of EXERCISE 4.5, we give hints. After working these problems, try the VARIATIONS on these exercises. There, you are on your own! Now is the time to start getting acquainted with the TABLE OF INTEGRALS in Appendix 1, in particular FUNDAMENTAL FORMS.

4.5 EXERCISES  Evaluate the following integrals:

1. $\int x\sqrt{x^2 + 2} \, dx = \quad (f(x) = x^{1/2} \text{ and } g(x) = x^2 + 2)$
2. $\int \frac{x^{1/4}}{5 + x^{3/4}} \, dx = \quad (f(x) = 1/x \text{ and } g(x) = 5 + x^{3/4})$
3. $\int \frac{\sqrt{x} - 1}{\sqrt{x}} \, dx =$
   (Beware of this type of problem! This is $\int dx - \int x^{-1/2}dx$ and no tricky substitutions are required.)
4. $\int \frac{8x}{1 + e^{2x}} \, dx =$
   (Don’t be fooled by $e^2$.)
5. $\int \frac{x \cos \sqrt{5x^2 + 1}}{\sqrt{5x^2 + 1}} \, dx = \quad (f(x) = \cos(x) \text{ and } g(x) = (5x^2 + 1)^{1/2})$
6. $\int \sin^{3/4} 2x \cos 2x \, dx = \quad (f(x) = x^{3/4} \text{ and } g(x) = \sin(2x))$
7. $\int \tan^2 3t \sec^2 3t \, dt = \quad (f(t) = t^2 \text{ and } g(t) = \tan(3t))$
8. $\int e^x \sec^2(e^x) \, dx = \quad (f(x) = \sec^2(x) \text{ and } g(x) = e^x)$
9. $\int \frac{2 \ln^3(x)}{x} \, dx = \quad (g(x) = \ln(x))$
10. $\int \frac{6x}{(x^2 + 9)^3} \, dx = \quad (g(x) = x^2 + 9)$
11. $\int \frac{e^{\log(x)}}{x} \, dx = \quad (g(x) = \log_2(x) \text{ or try } \log_2(x) = \log_2(e)\log_e(x))$
12. $\int x \sec(x^2) \tan(x^2) dx = \quad (g(x) = x^2)$

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13. \[ \int \frac{u}{(u^2 + 29)^4} \, du = (g(u) = u^2 + 29) \]

14. \[ \int \frac{\csc^2 x}{\sqrt{1 + 5 \cot x}} \, dx = (g(x) = 1 + 5 \cot(x)) \]

15. \[ \int \frac{(\arctan(x))^{3/2}}{1 + x^2} \, dx = (g(x) = \arctan(x)) \]

Now check the solutions to EXERCISE 4.5 in Appendix 3. Note, in particular, the "notational style" used in these solutions. This is a matter of personal taste. Strive to develop your own style.

4.6 VARIATIONS ON EXERCISE 4.5 Either g(x) or f(x) is missing—you find it. Evaluate the integrals.

1. \[ \int \frac{x^7}{(x^8 + 1)^5} \, dx \quad (f(x) = x^{-6}, \, g(x) = ) \]

2. \[ \int \sin^m x \cos x \, dx \quad (f(x) = , \, g(x) = \sin x) \]

3. \[ \int \frac{(x + 1)^2}{x} \, dx \quad \left( \text{reduce } \frac{(x + 1)^2}{x} = x + 2 + \frac{1}{x} \right) \]

4. \[ \int \sin^3 x \, dx \quad (g(x) = \cos(x), \, f(x) = 1 - \cos^2 x) \]

5. \[ \int (x + 1/x)^2 \, dx \quad (\text{square the integrand}) \]

6. \[ \int \csc^2 x \cot x + 1 \, dx \quad (f(x) = 1/x, \, g(x) = ) \]

7. \[ \int \sec^3 x \tan x \, dx \quad (f(x) = x^2, \, g(x) = ) \]

8. \[ \int \frac{d\theta}{\cos \theta} \quad \left( \text{let } g(\theta) = \sin \theta, \text{ then } \int \frac{d\theta}{\cos \theta} = \int \frac{d\sin \theta}{1 - \sin^2 \theta} \right) \]

9. \[ \int x \sin^2 x \cos x^2 \, dx \quad (f(x) = x^2, \, g(x) = ) \]

10. \[ \int e^{\cos x^2} x \sin x^2 \, dx \quad (f(x) = e^x, \, g(x) = ) \]

11. \[ \int \frac{(\arctan x)^a}{1 + x^2} \, dx \quad (f(x) = x^a, \, g(x) = ) \]

12. \[ \int (\alpha \sin rt)^\theta \cos rt \, dt \quad (f(t) = , \, g(t) = \alpha \sin rt) \]
13. \[ \int (x + 1)e^{x^2 + 2x} \, dx \quad (f(x) = \cdots, g(x) = (x + 1)^2) \]
14. \[ \int \frac{\ln^m x}{x} \, dx \quad (m > 0) \quad (f(x) = \cdots, g(x) = \ln x) \]
15. \[ \int (x \ln x \cdot \ln \ln x)^{-1} \, dx \quad (\text{use } g(x) = \ln \ln x) \]

4.7 VARIATIONS ON EXERCISE 4.5 Here are some additional tricks to study. Evaluate the integrals.

1. \[ \int \frac{x^n}{(x^{n+1} + \beta)} \, dx \quad (g(x) = x^{n+1} + \beta, f(x) = 1/x) \]
2. \[ \int \sin 2x \cos^2 x \, dx \quad (\text{use } 2 \sin x = 2 \sin x \cos x, g(x) = \cos x) \]
3. \[ \int \frac{x^2}{x + 1} \, dx \quad (\text{use } x^2 = (x + 1 - 1)^2, \text{ and } g(x) = x + 1) \]
4. \[ \int \cos^2 x \, dx \quad \left( \text{use } \cos^2 x = \frac{1 + \cos 2x}{2} \right) \]
5. \[ \int (x + 1) (x^2 + 2x + 1)^9 \, dx \quad (g(x) = (x + 1)^2) \]
6. \[ \int \frac{\cos x}{\sin x} \, dx \quad (g(x) = \sin x) \]
7. \[ \int \csc^m x \cot x \, dx \quad (g(x) = \csc x) \]
8. \[ \int \frac{1}{\sin \theta} \, d\theta \quad \left( \text{use } \frac{1}{\sin \theta} = \frac{\sin \theta}{1 - \cos^2 \theta}, \text{ and } g(\theta) = \cos \theta \right) \]
9. \[ \int x \tan^2 x^2 \sec^2 x^2 \, dx \quad (g(x) = \tan x^2) \]
10. \[ \int \ln(\sin x^2) \cdot x \cos x^2 \, dx \quad (g(x) = \sin x^2) \]
11. \[ \int (\arccot x^2)^\alpha \frac{x}{1 + x^4} \, dx \quad (g(x) = \arccot x^2) \]
12. \[ \int t^{a-1} \cos t^a \cdot \sin t^a \, dt \quad (g(t) = \cos t^a) \]
13. \[ \int e^{\log_{10} x} \, dx \quad (\text{use } e^{\log_{10} x} = x^{\log_{10} e}) \]
14. \[ \int \frac{1}{x \ln x} \, dx \quad (\text{use } g(x) = \ln x) \]
15. \[ \int \frac{(1 + 1/x^2)^m}{x^3} \, dx \quad (\text{use } g(x) = x^{-2}) \]
4.8 VARIATIONS ON EXERCISE 4.5

1. \[ \int \frac{x}{x^2 + 1} \, dx = \]
2. \[ \int \frac{\cos \sqrt{x} \sin \sqrt{x}}{\sqrt{x}} \, dx = \]
3. \[ \int \frac{x^2 + 1}{x^3} \, dx = \]
4. \[ \int \frac{\sin x}{\cos x} \, dx = \]
5. \[ \int \frac{6x^3 + 5x}{3x^4 + 5x^2} \, dx = \]
6. \[ \int \frac{\sec^2 x}{\tan x + 1} \, dx = \]
7. \[ \int \pi \csc^2 \pi^2 x \cot \pi^2 x \, dx = \]
8. \[ \int \frac{(3x + 4)^{1/5}}{(3x + 4)^{4/5}} \, dx = \]
9. \[ \int (4x + 8)(x^2 + 4) \, dx = \]
10. \[ \int x(x^2 + 1)^{3/2}(x^2 + 1)^{1/2} \, dx = \]
11. \[ \int \frac{1 + \sec x \tan x}{x + \sec x} \, dx = \]
12. \[ \int \sin 4x \cos 4x \, dx = \]
13. \[ \int x e^{x^2} \cdot e^{x^2} \, dx = \]
14. \[ \int \frac{\log_5 4x^3}{x} \, dx = \]
15. \[ \int \frac{2 \cos 3x^2(x \csc^2 3x^2)^2 \cdot 2 \cos 3x^2 \, dx}{x} = \]

4.9 VARIATIONS ON EXERCISE 4.5

1. \[ \int x \cos x^2 (e^{\sin x})^2 \, dx = \]
2. \[ \int \ln(3x^2 + 2) \, dx = \]
3. \[ \int \log_3(\sin x) \cot(x) \, dx = \]
4. \[ \int ((\sin x) \cdot e^{\sin x} + x(\cos x) \cdot e^{\sin x}) \, dx = \]
5. \[ \int x e^{x^2} \cdot \sec^2(e^{x^2}) \tan(e^{x^2}) \, dx = \]
6. \[ \int (x^4 + 1)^{4/3} x^3 \, dx = \]
7. \[ \int x \sin^2(x^2) \cos(x^2) \, dx = \]
8. \[ \int 2 \sqrt{x} \, dx = \]
9. \[ \int 5x^2(4x^3 + 1)^{4/5} \, dx = \]
10. \[ \int \ln^6(\sqrt{2x + 1}) \, dx = \]
11. \[ \int \sqrt{\sin(e^\theta)} \cos(e^\theta) \, d\theta = \]
12. \[ \int (10 \sin t)^7 \cos t \, dt = \]
13. \[ \int (\sin x + x \cos x) e^{\sin x} \, dx = \]
14. \[ \int \frac{\sin(\sin \theta) \cos \theta}{(5 - \cos \theta)^{10}} \, d\theta = \]

**Will Computers Save Us From Integral Calculus?**

The rest of this chapter will be devoted to two tasks. One task is to learn some applications of integral calculus. The other is to learn some additional
techniques for evaluating integrals. We have just learned a few techniques, most notably the method of ‘‘substitution’’ or recognizing the CHAIN RULE in reverse. In order to make things more interesting, we shall first give some applications of integral calculus. Then we’ll learn some more techniques of integration, followed by a few more applications. It is now the case, and it will be even more the case in the future, that sophisticated computer programs will perform the routine computation of integrals. Strange as it may seem, this does not diminish the importance of learning the basic techniques of integration. The reason for this is that the user of algebraic symbol manipulation programs must still intercede to put the input and output into ‘‘canonical forms.’’ In other words, one must still be able to transform problems and answers into equivalent forms that may appear quite different at first glance. The techniques for making these various transformations are exactly the techniques we shall study in this chapter.

**The Fundamental Theorem Of Calculus**

**FIGURE 4.10** shows the graph of a function $f(x)$. Starting with $f$, we compute graphically a function $S_x^a$ (dashed curve), called the **signed area function of $f$ with base point $a$** ($a = -3$ here). For $x \geq a$, let $R_x$ be the planar region between $f$ and the interval $[a, x]$. Let $R_x = A_x \cup B_x$ where $A_x$ is above the $x$-axis and $B_x$ below. Define $S_x^a = A_x - B_x$ where $A_x$ and $B_x$ are the areas of $A_x$ and $B_x$ respectively. Likewise, for $x < a$, let $R_x'$ be the planar region between $f$ and the interval $[x, a]$. Let $R_x' = A'_x \cup B'_x$ where $A'_x$ is above the $x$-axis and $B'_x$ below. Define $S_x'^a = B'_x - A'_x$ where $A'_x$ and $B'_x$ are the areas of $A'_x$ and $B'_x$ respectively. Note the *antisymmetric* property when interchanging the base points’ role: $S_x^b(f) = -S_x^a(f)$. We discuss the signed area function in detail using **FIGURE 4.10**, starting with $x \geq a = -3$.

In computing this area, areas of regions below the horizontal axis are given negative sign and areas of regions above are given positive sign. For example, the region between the graph of $f$ and the interval $[-3, -2]$ on the horizontal axis has area $-1.93$. The region between the graph of $f$ and the interval $[+2, +3]$ has area $+1.52$. The dashed curve in **FIGURE 4.10** is the graph of $S_x^a(f)$ as a function of $x$. Note that at $x = +6$ the value of the signed area function with base point $-3$ is $+5$. In other words, $S_6^a(f) = +5$. This means that the total accumulated signed area between the graph of $f$ and the interval $[-3, +6]$ on the horizontal axis is $+5$.

The values for each unit interval are computed in **FIGURE 4.10** (by tedious inspection!) to get $-1.93$ for the area between $f$ and $[-3, -2], -1.29$ for
$S^6_a(f) = 5$
$S^{6.1}_a(f) = 5.18$
$S^{6.1}_a - S^6_a(f) = 1.8 = f(6)$

Thus, taking limits

$$\frac{d}{dx} S^x_a(f) = f(x)$$

the area between $f$ and $[-2, -1]$, etc. Adding these values, we get approximately +5 for the accumulated signed area $S^6_a(f)$. If $x < a$, we compute the signed area in the same way except that regions below the horizontal axis get positive weight and regions above get negative weight.
We summarize with the following THEOREM 4.11:

Again, look carefully at FIGURE 4.10. In particular, look at +6 on the horizontal axis. At 6, \(f(6) = 1.8\) and \(S_6^6(f) = 5\). A careful check reveals that at 6.1, \(S_{6.1}^6(f) = 5.18\) and the difference, \(S_{6.1}^6(f) - S_6^6(f) = .18\). The number .18 represents the difference between the accumulated area between the graph of \(f\) and the interval \([-3, 6.1]\) and the accumulated area between the graph of \(f\) and the interval \([-3, 6]\). This area, .18, is represented by the shaded region shown in FIGURE 4.10 between the graph of \(f\) and the interval \([6, 6.1]\). This shaded region looks like a tiny rectangle (almost) and has area about \(f(6)\) times .1, or 1.8 times .1, which is .18. If we replace 6 by a generic point \(x\) and .1 by \(\Delta x\) this fact becomes \(S_{x + \Delta x}^x(f) - S_x^x(f) = f(x)\Delta x\) or

\[
\frac{S_{x + \Delta x}^x(f) - S_x^x(f)}{\Delta x} \approx f(x).
\]

The equality in the above expression is approximate but better and better as \(\Delta x\) gets smaller (for graphs of continuous functions such as defined in DEFINITION 3.12). We can write this as

\[
\lim_{\Delta x \to 0} \frac{S_{x + \Delta x}^x(f) - S_x^x(f)}{\Delta x} = \frac{df}{dx} S_x^x(f) = f(x).
\]

We summarize with the following THEOREM 4.11:

4.11 THEOREM (FUNDAMENTAL THEOREM OF CALCULUS)
Let \(f\) be defined and continuous for all \(x\) in the interval \([c, d]\) and let \(a\) be a number in \([c, d]\). Let \(S_a^x(f)\) be the signed area function with base point \(a\). For every point \(x\) in the interval \((c, d) = (tc < t < d)\), the function \(F(x) = S_a^x(f)\) has a derivative and \(\frac{d}{dx} F(x) = f(x)\). Thus the signed area function is an antiderivative or integral of \(f(x)\).

The Envelope Game

It’s time to play the ENVELOPE GAME: Imagine that you have an envelope and inside is the signed area function of \(f(x) = x^2\) based at 0. What will you see when you open the envelope? One possibility is that you will see a graph such as that of \(S_0^x(f)\) shown in FIGURE 4.10. Someone could have drawn a careful graph of the function \(x^2\) and tediously measured the area, making a graph of the resulting signed area function \(F(x) = S_0^x(f)\) with base point \(a = 0\). If this is what you find in the envelope you
will know that whoever put it there wasn’t thinking very hard! We learned
at the very beginning of this chapter that if \( F(x) \) and \( G(x) \) are two antiderivative
s for \( f(x) \) then they differ by the constant function, or \( F(x) = G(x) + C \). We know that \( G(x) = x^{3/3} \) is an antiderivative function for \( x^2 \). This means
that all this measuring of signed area was a total waste of time in this case.
The signed area function must be \( x^{3/3} + C \). In this case we easily see that
\( C \) must be zero since \( F(0) \) is zero by definition of signed area with base point
0 and, obviously, \( G(0) = 0 \) so \( F(0) = G(0) + C \) implies that \( C = 0 \).

You should think very carefully about what we have just done. This basic
idea has many amazing variations and accounts in large part for the usefulness
of calculus. A tedious task of computing areas under graphs of functions \( f(x) \)
has been replaced by a seemingly very different task of guessing antiderivative
s \( F(x) \) for such functions. We can become very clever at guessing an-
tiderivatives and thus very good at computing areas. The area between the
graph of \( f(x) = x^2 \) and the interval \([0, 1]\) is \( S^1_0(f) = (1)^{3/3} = 1/3 \). Try
computing the same area without calculus! Probably, it sounds sort of interesting
but not really all that useful to you to spend your time computing areas
under curves. What happens is that many problems in physics, chemistry,
engineering, astronomy, etc., are just this sort of problem in disguise. Before
going on, we need to develop some notation.

**Some Very Important Ideas**

### 4.12 GENERAL REMARKS AND NOTATION

We have learned that, given a function \( f(x) \), any function \( G(x) \) such that \( \frac{d}{dx} G(x) = f(x) \) is called
an antiderivative or integral of \( f(x) \). We know that all antiderivatives differ
by constant functions. Thus we can construct all other antiderivatives by
forming \( G(x) + C \) where \( C \) is some constant. We learned in **THEOREM
4.11** that antiderivative functions can be constructed graphically. These an-
tiderivative functions are called the ‘‘signed area’’ functions and are denoted
by \( S^a_x(f) \). If \( F(x) = S^a_x(f) \) is the signed area function of \( f \) based at \( a \), then
clearly \( F(a) = 0 \).

You can think of the graphs of all antiderivatives of \( f \) as being exactly the
same shape but shifted up or down with respect to the vertical axis. Two such
functions are either equal nowhere or equal everywhere. Said in another way,
if \( F(x) \) and \( G(x) \) are antiderivatives of \( f \) and \( F(a) = G(a) \) for some number
\( a \), then \( F(x) = G(x) \) for all values of \( x \) (for which the functions are defined).
In particular, any antiderivative \( G(x) \) which has \( G(a) = 0 \) must be exactly
the signed area function \( F(x) = S_x^a(f) \). This means that the set of all signed area functions for \( f \) is just the set of all antiderivatives \( G(x) \) that are zero for some value \( a \) (satisfy \( G(a) = 0 \) for some \( a \)). If \( G(a) = 0 \) then \( G(x) = S_x^a(f) \) for all values of \( x \). It might happen that \( G(b) = 0 \) also for some \( b \) different from \( a \). In this case \( G(x) = S_x^a(f) = S_x^b(f) \). It is easy to see that there are antiderivative functions which are not signed area functions. For example, \( x^2 + 2 \) is an antiderivative for \( 2x \) but not a signed area function for \( 2x \) since it does not vanish for any value of \( x \).

The common notation for the signed area function \( S_x^a(f) \) in calculus is \( \int_a^x f \), or \( \int_a^x f(t)dt \) if we wish to be explicit about the dependent variable of \( f \). It does not make any difference what we call this dependent variable: \( \int_a^y f(t)dt = \int_y^z f(t)dt = \int_z^a f(t)dt = \ldots \). Even \( \int_a^x f(t)dt \) is o.k. (but be careful!). They are all equal to \( S_x^a(f) \). The signed area functions \( \int_a^x f(t)dt \) are called the "definite integrals" of \( f \). Specifically, we say \( \int_a^x f(t)dt \) is the "definite integral of \( f \) from \( a \) to \( x \)." The signed area function \( \int_a^x f(t)dt \) viewed as a function of \( x \) is, as we have seen, nothing else than the antiderivative of \( f \) that vanishes at \( x = a \). If \( c \) is a number, then the value of the definite integral at \( c \) is \( \int_a^c f(t)dt \). This value is just a number. Thus \( \int_0^1 t^2 dt = 1/3 \).

The numbers such as \( a \) and \( x \) or \( 0 \) and \( 1 \) that appear at the top and bottom of the integral sign (\( \int_a^x \) or \( \int_0^1 \)) are called the "limits of integration." If the integral sign has no limits, such as \( \int f(x)dx \), then this stands for just any old antiderivative or integral of \( f \). Sometimes the phrase "indefinite integral \( \int f \)" is used to mean "integral of \( f \)" or "antiderivative of \( f \)." As we remarked before, the notation of calculus ranges from bad to horrible, but by working many examples, you will learn to tolerate it if not to love it!

If you want to compute a signed area function or definite integral \( \int_a^x f(t)dt \), you can do so by finding any antiderivative \( H \) for \( f \). Then we must have that \( G(x) = H(x) - H(a) \) is also an antiderivative. But \( G(a) = 0 \), so, by what we have said above, \( G(x) = \int_a^x f(t)dt \).

**The Riemann Sum**

As we have already remarked, in computing the signed area function of the function \( f \) of FIGURE 4.10 we used tedious numerical approximations to the areas bounded by the graph of \( f \) and small intervals on the horizontal axis. In the case of FIGURE 4.10, each subinterval was chosen to be of length one. The various areas that were computed for each subinterval are shown in FIGURE 4.10. It is very important to describe this process a little more carefully. Even in cases where we can find an antiderivative for \( f \), it is a useful intuitive guide to think about how the computation would be done geometrically.
Consider FIGURE 4.13. There we see the graph of a function $f$. We are interested in computing the definite integral (signed area) from $a$ to $b$. In other words, we want to compute $\int_a^b f(t) \, dt$. To approximate this integral, we have divided the interval $[a, b]$ into subintervals $[x_1, x_2]$, $[x_2, x_3]$, \ldots, $[x_{11}, x_{12}]$. In each subinterval $[x_i, x_{i+1}]$, we have chosen a number $t_i$. We then approximate the definite integral by

$$\int_a^b f(t) \, dt = \sum_{i=1}^{11} f(t_i)(x_{i+1} - x_i).$$

If we let $\Delta t_i = x_{i+1} - x_i$ this sum becomes

$$\int_a^b f(t) \, dt = \sum_{i=1}^{11} f(t_i)\Delta t_i.$$

A sum such as that used above to approximate a definite integral is called a "Riemann sum."

FIGURE 4.13 Riemann Sum

$$\sum_{i=1}^{n} f(t_i)\Delta t_i \text{ where } \Delta t_i = x_{i+1} - x_i \text{ and } n = 11$$

4.14 NOTATION (RIEMANN SUM) Let $f$ be a continuous function defined on the interval $[a, b]$. Let $a = x_1 < x_2 < \ldots < x_{n+1} = b$ be points in the interval $[a, b]$. Define $\Delta t_i = x_{i+1} - x_i$. Choose $t_i$ in $[x_i, x_{i+1}]$, $i = 1, \ldots, n$. The following sum is called the Riemann sum based on the points $x_i$ and $t_i$:

$$\sum_{i=1}^{n} f(t_i)\Delta t_i$$
Letting $n$ get larger while all $\Delta_i$ go to zero, the Riemann sum becomes a better and better approximation to $\int_a^b f(t)dt$:

$$\lim_{n \to \infty} \sum_{i=1}^n f(t_i)\Delta t_i = \int_a^b f(t)dt.$$  \hfill (RS)

The result (RS) is intuitively obvious, but a rigorous proof is usually reserved for a course in analysis. We define the definite integral $\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is any antiderivative of $f(x)$. It follows that

$$\int_a^b f(x)dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_a^b f(x)dx.$$  

**SUMMARY:** If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$ for some constant, $C$. The definite integral is $\int_a^b f(x)dx := F(b) - F(a)$ where $F(x)$ is any antiderivative of $f(x)$. We define the signed area function, $S_a^x(f)$ with base point $a$, graphically (FIGURE 4.10) and show that $F(x) = S_a^x(f)$ is an antiderivative of $f(x)$. Taking $F(x) = S_a^x(f)$ gives $\int_a^b f(x)dx = S_a^b(f)$. The Riemann sum (FIGURE 4.13) describes a “thought algorithm” for computing definite integrals and, hence, signed areas.

**EXERCISES 4.15** relate to these ideas and are a good time to practice using online resources for graphing, computing antiderivatives and definite integrals. Answers to variations 4.17, 4.18, 4.19 and 4.20 are given after each problem. Selected solutions to 4.20 through 4.23 are given in Appendix 1. The actual area between the graph of $f(x)$ and the $x$–axis refers to the area where regions both below and above the $x$–axis are positive. We use “actual area” and “area” as synonymous.

4.15 **EXERCISES** In these exercises “area” means actual area, not signed area.

(1) Find the area bounded by the graph of $y = x^3$, the lines $x = -1$, $x = +2$, and the horizontal axis.

(2) Find the area bounded by the graph of $y = x^3$, the lines $y = +8$, $y = -1$, and the vertical axis.
(3) Find the area of the bounded region between the curves \( y = -x^2 + 2 \) and \( y = 2^x \).

(4) Find the area enclosed by the curve \( \{(1 + 2\cos(t), 2 + 3\sin(t)) : 0 \leq t < 2\pi\} \).

(5) Using polar coordinates, find the area enclosed by the curve \( r(\phi) = 1 + \cos(\phi) \), \( 0 \leq \phi < 2\pi \).

4.16 SOLUTIONS TO EXERCISE 4.15

(1) A sketch of the situation looks as follows:

We are trying to find the area of the shaded region. The expression \( \int_{-1}^{2} x^3 \, dx \) would give the signed area between the graph of \( f(x) = x^3 \) and the interval \([-1, +2]\). The area between the curve and the interval \([-1, 0]\) would have negative weight. The problem as stated should be interpreted as finding the actual area. If we choose the base point for the signed area function to be \( a = 0 \), then the actual area is

\[
\int_{0}^{-1} x^3 \, dx + \int_{0}^{2} x^3 \, dx.
\]
An antiderivative of $x^3$ is $F(x) = x^4/4$. Thus the required area is $F(-1) - F(0) + F(+2) - F(0)$ or $1/4 + 4 = 17/4$. One standard way of writing $F(b) - F(a)$ is

$$\frac{x^4}{4} \bigg|_a^b.$$ 

Thus, we may write $(F(-1) - F(0)) + (F(2) - F(0))$ as

$$\frac{x^4}{4} \bigg|_0^{-1} + \frac{x^4}{4} \bigg|_0^2 = 1/4 + 4 = 17/4.$$

(2) We are trying to find the actual area of the dotted region in the following sketch:

The easiest way to work this problem is to view it as Sally is in the sketch above. She thinks of $y$ as the dependent variable and is looking at the curve $x = y^{1/3}$. She computes the signed area function based at $y = 0$ to get

$$\int_{0}^{8} y^{1/3} dy + \int_{0}^{8} y^{1/3} dy = \frac{y^{4/3}}{4/3} \bigg|_0^8 + \frac{y^{4/3}}{4/3} \bigg|_0^8 = 51/4.$$

Another way to evaluate this same area is by computing the following definite integrals (why?):

$$\int_{-1}^{0} (x^3 + 1) dx + \int_{0}^{8} (8 - x^3) dx.$$
(3) This is a typical calculus problem with an atypical twist. The graph of $-x^2 + 2$ is a parabola with the line $x = 0$ as axis of symmetry. If you've forgotten about parabolas, etc., it doesn't make much difference, since by graphing a few points you should discover the following picture:

Thus the required integral is $\int_a^b (-x^2 + 2 - 2^x) \, dx$. To evaluate this integral, we must know $a$ and $b$. In other words, we must know when $-x^2 + 2 - 2^x = 0$. Most calculus books set up these problems so that it is easy to find $a$ and $b$ (using, say, the quadratic formula). This problem is more realistic in that no such simple method works. Still, with a computer the problem is easy to solve. It is clear from the sketch that $-2^{1/2} < a < b < 2^{1/2}$ and $2^{1/2}$ is about 1.4. The following BASIC program gives us a closer look at the function $-x^2 + 2 - 2^x$ in the interval of interest.

```
10 FOR X = -1.4 TO +1.4 STEP .1
20 PRINT X, -X*X + 2 - 2^X
30 NEXT X
```

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<th>$x$</th>
<th>$-x^2 + 2 - 2^x$</th>
</tr>
</thead>
<tbody>
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<td>.9802461</td>
</tr>
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</table>
By inspecting the output of this program, we see that $-1.3 < a < -1.2$ and $0.6 < b < 0.7$. The next two BASIC programs give us some more accuracy.

(a)

10 FOR X = -1.3 TO -1.2 STEP .01
20 PRINT X, -X^2 + 2 - 2^X
30 NEXT X

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<th>$-x^2 + 2 - 2^x$</th>
</tr>
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</tr>
<tr>
<td>-1.22</td>
<td>8.231711E-02</td>
</tr>
<tr>
<td>-1.21</td>
<td>.1036312</td>
</tr>
<tr>
<td>-1.2</td>
<td>.1247246</td>
</tr>
</tbody>
</table>

(b)
10 FOR X = .6 TO .7 STEP .01
20 PRINT X, X^2 + 2 - 2^X
30 NEXT X

<table>
<thead>
<tr>
<th>x</th>
<th>(-x^2 + 2 - 2^x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.6</td>
<td>.1242836</td>
</tr>
<tr>
<td>.61</td>
<td>.1016408</td>
</tr>
<tr>
<td>.62</td>
<td>7.872498E-02</td>
</tr>
<tr>
<td>.63</td>
<td>.05555352</td>
</tr>
<tr>
<td>.64</td>
<td>.032071</td>
</tr>
<tr>
<td>.65</td>
<td>8.332015E-03</td>
</tr>
<tr>
<td>.66</td>
<td>-1.568234E-02</td>
</tr>
<tr>
<td>.67</td>
<td>-3.997278E-02</td>
</tr>
<tr>
<td>.68</td>
<td>-6.453955E-02</td>
</tr>
<tr>
<td>.69</td>
<td>-8.938336E-02</td>
</tr>
<tr>
<td>.7</td>
<td>-1.145045</td>
</tr>
</tbody>
</table>

It seems that \(a = -1.26\) and \(b = .65\) are reasonably good approximations. Thus we compute

\[
\int_{-1.26}^{.65} (-x^2 + 2x - 2^x)dx = -x^3/3 + 2x - 2^x \ln(2) \bigg|_{-1.26}^{.65}
\]

\[
= -1.06 - (-5.12) = 4.06
\]

(4) This is the parametric equation of an ellipse with center at (1, 2). The same ellipse with center at (0, 0) has parametric equation

\[
\{(2\cos(t), 3\sin(t)) : 0 < t < 2\pi\}
\]

This ellipse, which obviously has the same area as the original one, looks as shown at the top of page 174.

We compute the shaded area and multiply by 4 to get the total area. Thus we compute

\[
\int_{0}^{\pi/2} \sin^2(t) dt = \int_{0}^{\pi/2} (1 - \cos(2t)) dt = 6 \int_{0}^{\pi/2} \sin^2(t) dt.
\]

Using the trigonometric identity \(2\sin^2(t) = 1 - \cos(2t)\) we compute

\[
6 \int_{0}^{\pi/2} \sin^2(t) dt = 3 \int_{0}^{\pi/2} (1 - \cos(2t)) dt
\]

\[
= 3(t - (\sin(2t)) / 2) \bigg|_{0}^{\pi/2} = 3\pi/2.
\]

25
The total area is 4 times this or $6\pi$.

(5) This curve is called a “limacon” and looks as follows:

The small shaded triangle has area $dA = r(\phi)(r(\phi)d\phi)/2$ where $A(\phi)$ is the area as a function of $\phi$. Thus we compute

$$A(\phi) = \frac{1}{2} \int_{0}^{2\pi} (1 + \cos(\phi))^2 d\phi = \int_{0}^{\pi} (1 + \cos(\phi))^2 d\phi.$$

But $(1 + \cos(\phi))^2 = 1 + 2\cos(\phi) + \cos^2(\phi)$ and $\cos^2(\phi) = (\cos(2\phi) + 1)/2$. This gives
\[ \int_0^\pi (1 + \cos(\phi))^2 d\phi = \phi + 2\sin(\phi) + (1/4)\sin(2\phi) + \phi/2 \bigg|_0^\pi = 3\pi/2. \]

4.17 VARIATIONS ON EXERCISE 4.15

1. Find the area bounded by the graph \( y = x^2 \), the x-axis, and the lines \( x = 0 \) and \( x = 2 \). (Answer: 8/3)

2. Find the area bounded by the graph of \( y = x^2 \), the lines \( y = 4 \) and \( y = 9 \), and the y-axis. (Answer: 38/3)

3. Find the area between the curves \( y = x^2 \) and \( y = x^4 \). (Answer: 4/15)

4. Find the area enclosed by the curve \( \{(4\cos(2\pi t), 4\sin(2\pi t)): 0 < t < 1\} \).
   (Answer: 16\pi)

5. Using polar coordinates, find the area enclosed by the curve \( r(\phi) = \phi \) and the ray \( \phi = 0 \), \( 0 \leq \phi \leq \pi \). (Answer: \( \pi^2/6 \)).

4.18 VARIATIONS ON EXERCISE 4.15

1. Find the area bounded by the graph \( y = e^x \), the x-axis, and the lines \( x = 0 \) and \( x = 1 \). (Answer: \( e - 1 \))

2. Find the area bounded by the graph of \( y = e^x \), the lines \( y = e \) and \( y = e^2 \), and the y-axis. (Answer: \( e^2 \))

3. Find the area between the curves \( y = e^{2x} \) and \( y = e^x \), \( -2 < x < +2 \).

4. Find the area enclosed by the curve

   \[ \{(1 + \cos(t), 3 + 4\sin(t)): 0 < t < 2\pi/3\} \]

   and the straight line segments that join the endpoints of this curve to the point (1, 3).

5. Using polar coordinates, find the area enclosed by the curve \( r(\phi) = 1 + \sin(\phi) \), \( 0 < \phi < \pi \), and the ray \( \phi = 0 \).

4.19 VARIATIONS ON EXERCISE 4.15

1. Find the area bounded by the graph, \( y = x^{-1/2} \), the x-axis, the y-axis, and the line \( x = 1 \). (Answer: 2)

2. Find the area bounded by the graph of \( y = x^{-1/2} \), the line \( y = 1 \), and the y-axis. (Answer: 1)
(3) Find the area between the curves $y = \sin(x)$ and $y = 2x/\pi$. (Answer: $2(1 - \pi/4)$)

(4) Find the area under the curve $\{(t^2, \; t e^t): 0 < t < 1\}$. (Answer: $(2/3)(e - 1)$)

(5) Using polar coordinates, find the area enclosed by the curve $r(\phi) = \phi + 1$ and the rays $\phi = 0$ and $\phi = \pi/2$. (Answer: $1/6(1 + \pi/2)^3 - 1/6$)

4.20 VARIATIONS ON EXERCISE 4.15

(1) Find the area bounded by the graph $y = x^{-3}$, the $x$-axis, $x \geq 1$, and the line $x = 1$. Hint: Find the area under the curve $y = x^{-3}$ for $1 < x < \alpha$. Then, take the limit as $\alpha$ tends to infinity. (Answer: $1/2$)

(2) Find the area bounded by the graph of $y = x^{-3}$, the line $y = 1$, the $x$-axis, and the $y$-axis. Hint: Use the same trick used in problem (1). (Answer: $3/2$)

(3) Find the area between the curves $y = e^{-x}$ and $y = e^{-2x}$, $x > 0$. Hint: Once again, find the area between the curves for $0 < x < \alpha$ and let $\alpha$ tend to infinity. (Answer: $1/2$)

(4) Find the area between the curve $\{(t^{1/2}\sin(t), \; t^{1/2}): 0 < t < \pi/2\}$ and the $y$-axis. (Answer: $1/2$)

(5) Using polar coordinates, find the area enclosed by the curve $r(\phi) = 2\sin(\phi)$, $0 < \phi < \pi/2$, and the ray $\phi = \pi/2$. (Answer: $\pi/2$)

4.21 VARIATIONS ON EXERCISE 4.15 (set up the problem only)

(1) Find the area bounded by the graph of $y = 3^x$, the lines $x = 0$, $x = +1$, and the horizontal axis.

(2) Find the area bounded by the graph of $y = 3^x$, the lines $y = 3/2$, $y = +3$, and the vertical axis.

(3) Find the area of the bounded region between the curves $y = -x^2 + 3$ and $y = 3^x$.

(4) Find the area enclosed by the curve $\{(a + b\cos(t), \; c + d\sin(t)): 0 \leq t < 2\pi\}$ where $a$, $b > 0$, $c$ and $d > 0$ are real numbers.

(5) Using polar coordinates, find the area enclosed by the curve $r(\phi) = 1 + 2\cos(\phi)$, $0 \leq \phi < \pi/2$. 

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4.22 VARIATIONS ON EXERCISE 4.15 (set up the problem only)

(1) Find the area bounded by the graph of \( y = x^3 + 1 \), the line \( x = +2 \), and the horizontal axis. Explain the relationship between this problem and EXERCISE 4.15(1). In particular, how can you use the calculations of 4.15(1) to simplify the calculations of this problem.

(2) Find the area bounded by the graph of \( y = x^3 + 1 \), the lines \( y = +9 \) and \( x = -1 \).

(3) Find the area of the bounded region between the curves \( y = -x^4 + 16 \) and \( y = (1.5)^x \).

(4) Find the area enclosed by the curve \( \{(t - \sin(t), 1 - \cos(t)) : 0 \leq t < 2\pi\} \) and the horizontal axis.

(5) Using polar coordinates, find the area enclosed by the curve \( r(\phi) = \cos(2\phi) \), \( 0 \leq \phi < \pi/2 \).

4.23 VARIATIONS ON EXERCISE 4.15 (set up the problem only)

(1) Find the area bounded by the graph of \( y = x^{4/4} - 2x^{3/3} \), the line \( x = -1 \), the line \( x = 6/3 \), and the horizontal axis.

(2) Find the area bounded by the graph of \( y = x^{4/4} - 2x^{3/3} \), the line \( y = -4/3 \), and the vertical axis.

(3) Find the area of the bounded region between the curve \( y = -x^{4/4} - 2x^{3/3} + 4x^{2/2} - 2 \) and the horizontal axis. (Hint: \( y \) has two real roots, one near \(-10\), the other near \(-0.6\). Use your computer!)

(4) Find the area enclosed by the curve \( \{(1 + \sin(t), \cos(t)(1 + \sin(t))) : -\pi/2 \leq t \leq \pi/2\} \) and the horizontal axis.

(5) Using polar coordinates, find the area bounded by the line segment \( \{(x, \phi) : -1 < r < +1, \phi = 0\} \), the curve \( r_1(\phi) = e^{\phi/\pi} \), \( 0 \leq \phi \leq \pi/2 \), and the curve \( r_2(\phi) = e^{1 - \phi/\pi} \), \( \pi/2 < \phi \leq \pi \).

4.24 COMPUTING INTEGRALS Having had a glimpse of some of the applications of integration, it is now time to improve our technical ability to compute integrals. We first concentrate on three very important techniques: integration by parts, trigonometric identities, and trigonometric substitution. Mastering these techniques is almost entirely a matter of practice. The practice centers around EXERCISES 4.26, 4.27, and 4.28, and the VARIATIONS that follow. After learning these basic techniques of integration, we shall look
at a fourth method, integration by partial fractions. Most students find this method extremely boring. Fortunately, this is a class of integrals that algebraic symbol manipulation packages do well with. The ideas we shall learn are still necessary for communicating with such software.

**Integration By Parts—The Product Rule In Reverse**

The method of substitution, the theme of EXERCISE 4.5 and its VARIATIONS, is concerned with what we called the "chain rule in reverse." The method of "integration by parts" is the "product rule in reverse." The idea is simple. We know that $(fg)' = f'g + fg'$. Hence $f(fg)' = ff'g + fg'$. We write this as

$$fg = \int f'gdx + \int fg'dx.$$ 

If, by hook or by crook, we know any two of the terms in the above expression, we can then find the third. What usually happens is that we have some function $h(x)$, for example, $h(x) = x\cos(x)$, and we are asked to evaluate $\int x\cos(x)dx$. We are stuck. In our mind we split $h(x)$ into a product of two functions $h(x) = p(x)q(x)$ such that we know how to differentiate $p(x)$ and we know how to integrate $q(x)$. To dramatize this fact, we call $p(x) = f(x)$ and $q(x) = g'(x)$. Just how to split $h(x)$ in this way may not be clear at first glance. You may have to fool around a bit. In the case of $h(x) = x\cos(x)$, we try $f(x) = x$ and $g'(x) = \cos(x)$. Then $f'(x) = 1$ and $g(x) = \sin(x)$. Thus the identity

$$fg = \int f'gdx + \int fg'dx$$

becomes

$$x\sin(x) = \int (1)\sin(x)dx + \int x\cos(x)dx.$$ 

Solving for $\int x\cos(x)dx$ gives

$$\int x\cos(x)dx = x\sin(x) + \cos(x)$$

where we have used the simple fact that $\int \sin(x)dx = -\cos(x)$. This sequence of events represents the method of integration by parts working perfectly. We state the method as follows:

**4.25 INTEGRATION BY PARTS**

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$
For the method to be useful, we must know how to split up a given function \( h(x) \) into a product of two functions \( f(x)g'(x) \) such that \( f(x) \) is reasonable to differentiate to obtain \( f'(x) \), \( g'(x) \) is reasonable to integrate to obtain \( g(x) \), and (miracle of miracles) we can integrate \( f'(x)g(x) \). To see how we can foul up in this process, suppose we had thought of \( x\cos(x) \) as \( f(x)g'(x) \) with \( f(x) = \cos(x) \) and \( g'(x) = x \). We find easily that \( f'(x) = -\sin(x) \) and \( g(x) = x^2/2 \). In this case, \( f'(x)g(x) = (-1/2)x^2\sin(x) \) is not easy to integrate. You should note that the identity of 4.25 is still valid:

\[
\int x\cos(x)\,dx = \cos(x)x^2/2 + \int (1/2)x^2\sin(x)\,dx.
\]

The fact that we can’t evaluate \( \int (1/2)x^2\sin(x)\,dx \) as an explicit expression just means that we can’t use 4.13 (with this particular choice of \( f(x) \) and \( g'(x) \)) to evaluate \( \int x\cos(x)\,dx \).

It helps some students to remember the integration by parts rule in conjunction with the following table:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>( g'(x) )</td>
</tr>
</tbody>
</table>

\[
\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.
\]

In the above table, the integral of the product of the two functions along any diagonal is the product of the two functions in the top row minus the integral of the product of the functions along the other diagonal. Now try to work the problems in EXERCISE 4.26. EXERCISE 4.26(3) contains a useful trick. You apply integration by parts twice. After the second application, the original integral appears a second time to give an equation in which the original integral occurs in two places. Solve this equation for the original integral.

**4.26 EXERCISES** Work the following problems using integration by parts. In each case a HINT is given showing how to factor the integrand as a product \( fg' \). Other factorizations may work as well.

\[
(1) \int \arctan(x)\,dx = \quad \text{HINT:}
\]

<table>
<thead>
<tr>
<th>( \arctan(x) )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1 + x^2} )</td>
<td>1</td>
</tr>
</tbody>
</table>
**RELATED INTEGRALS**: \( \int dx \) and \( \int \frac{x}{1 + x^2} dx \)

(2) \( \int x \arctan(x) \, dx = \)

<table>
<thead>
<tr>
<th>( \arctan(x) )</th>
<th>( \frac{x^2}{2} )</th>
<th>HINT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1 + x^2} )</td>
<td>( x )</td>
<td></td>
</tr>
</tbody>
</table>

**RELATED INTEGRALS**: \( \int \frac{x^2}{(1 + x^2)^2} \, dx \)

(3) \( \int \sin(\ln(x)) \, dx = \)

<table>
<thead>
<tr>
<th>( \sin(\ln(x)) )</th>
<th>( x )</th>
<th>HINT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\cos(\ln(x))}{x} )</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \cos(\ln(x)) )</th>
<th>( x )</th>
<th>HINT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{\sin(\ln(x))}{x} )</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

(4) \( \int \frac{x}{(2 + 3x)^{1/2}} \, dx \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{2}{3} (2 + 3x)^{1/2} )</th>
<th>HINT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (2 + 3x)^{-1/2} )</td>
<td></td>
</tr>
</tbody>
</table>

**RELATED INTEGRALS**: \( \int (2 + 3x)^{-1/2} \, dx \) and \( \int (2 + 3x)^{1/2} \, dx \)

(5) \( \int \sec^3(x) \, dx = \)

<table>
<thead>
<tr>
<th>( \sec(x) )</th>
<th>( \tan(x) )</th>
<th>HINT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sec(x) \tan(x) )</td>
<td>( \sec^2(x) )</td>
<td></td>
</tr>
</tbody>
</table>

**RELATED INTEGRALS**: \( \int \sec(x) \, dx \) and \( \int \sec(x) \tan^2(x) \, dx \) (use \(-1 + \sec^2(x) = \tan^2(x))\)
4.27 NOTE: The method of “trigonometric integrals” is very general and refers vaguely to the clever use of trigonometric identities (available online and in the math tables at the end of this workbook). We illustrate this method by discussing exercise 4.27(3) below. Two identities you have encountered in trigonometry are

\[ \sin(2x) = 2 \sin(x) \cos(x) \quad \text{and} \quad \cos(2x) = \cos^2(x) - \sin^2(x). \]

The second identity gives \( \cos(2x) = 1 - 2 \sin^2(x) = 2 \cos^2(x) - 1 \) and, equivalently, \( \cos^2(x) = (1 + \cos(2x))/2 \) or \( \sin^2(x) = (1 - \cos(2x))/2 \).

From the latter identity we see that \( \int \sin^2(x)\,dx = x/2 - \sin(2x)/4 \). In 4.27(3) we are asked to find

\[ \int \sin^4(x) \cos^4(x)\,dx = \int \sin^2(x) \sin^2(x) \cos^2(x)\,dx = \int \frac{1 - \cos(2x)}{2} \cdot \frac{\sin^2(2x)}{4}\,dx = \frac{1}{8} \int \sin^2(2x)\,dx + \frac{1}{8} \int \sin^2(2x) \cos(2x)\,dx. \]

Using \( \int \sin^2(x)\,dx = x/2 - \sin(2x)/4 \) given above and techniques already discussed, we get

\[ \frac{1}{8} \int \sin^2(2x)\,dx = \frac{1}{16} \left( x - \frac{\sin(4x)}{4} \right) \]

\[ \frac{1}{8} \int \sin^2(2x) \cos(2x)\,dx = \frac{1}{16} \frac{\sin^3(2x)}{3}. \]

These results can be verified by routine differentiation. Explore the various online integral calculation programs at this point and apply them to the exercises that follow. You will see different expressions for the same antiderivatives.

**4.27 EXERCISES** Work the following trigonometric integrals:

1. \( \int \sec(x)\,dx = \) \quad HINT: \( \sec(x) = \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)} \)

2. \( \int \cos^3(x) \sin^2(x)\,dx = \) \quad HINT: \( \cos^2(x) = 1 - \sin^2(x) \)

3. \( \int \sin^4(x) \cos^2(x)\,dx = \)
(4) \[ \int \cos(5x)\cos(x)\,dx = \]  
HINT: \[ \cos(6x) = \cos(5x + x) \]  
\[ \cos(4x) = \cos(5x - x) \]  

(5) \[ \int \sin(3x)\cos(2x)\,dx \]  
HINT: \[ \sin(5x) = \sin(3x + 2x) \]  
\[ \sin(x) = \sin(3x - 2x) \]  

**Trigonometric Substitution**

The next class of integrals that we consider is related to the technique called "trigonometric substitution." Take a look at the five problems of EXERCISE 4.28. They all involve a number plus or minus the square of the variable. Look at the HINT provided for each case. The given triangle specifies a relation between the function to be integrated and a trigonometric function of the variable \( \tau \). In 4.28(1) we see that \( x = 4\tan(\tau) \) and thus \( dx = 4\sec^2(\tau)\,d\tau \). The integrand in 4.28(1) is evidently \( (1/4)\cos(\tau))^4 \). Making these substitutions transforms the integral of 4.28(1) into \( 4^{-3}\int \cos^2(\tau)\,d\tau \). Evaluating this integral gives \( 4^{-3}((1/4)\sin(2\tau) + \tau/2) \). We now express this in terms of the variable \( x \). Note first that \( \sin(2\tau) = 2\sin(\tau)\cos(\tau) \). Then, by looking at the triangle in the HINT, we see that \( \tau = \arcsin(x/(16 + x^2)^{1/2}) \) and also \( \tau = \arccos(4/(16 + x^2)^{1/2}) \). Substituting the first expression for \( \tau \) into \( \sin(\tau) \) and the second into \( \cos(\tau) \) gives

\[ \sin(2\tau) = 8x/(16 + x^2). \]

Using \( \tau = \arctan(x/4) \) gives

\[ \int (16 + x^2)^{-2}\,dx = 4^{-3}(2x/(16 + x^2) + (1/2)\arctan(x/4)). \]

You should check this answer by differentiation with respect to \( x \). Now try EXERCISE 4.28.

**4.28 Exercises** Work each of the following exercises by trigonometric substitution:

(1) \[ \int \frac{1}{(16 + x^2)^2}\,dx = \]  
HINT: \[ x \]

RELATED INTEGRALS: \( \int \sin^2(\tau)\,d\tau \) and \( \int \cos^2(\tau)\,d\tau \)
(2) \[ \int \frac{x^2}{(4 - x^2)^{3/2}} \, dx = \]

HINT: \( x \)

RELATED INTEGRALS: \( \int \tan^2(\tau) \sec^2(\tau) \, d\tau \) and \( \int \cot^2(\tau) \csc^2(\tau) \, d\tau \)

(3) \[ \int \frac{1}{(9 - x^2)^{3/2}} \, dx = \]

HINT: \( x \)

RELATED INTEGRALS: \( \int \sec^2(\tau) \, d\tau \) and \( \int \csc^2(\tau) \, d\tau \)

(4) \[ \int (x^2 + 25)^{1/2} \, dx = \]

HINT: \( x \)

RELATED INTEGRALS: \( \int \sec^3(\tau) \, d\tau \) and \( \int \csc^3(\tau) \, d\tau \)

(5) \[ \int \frac{1}{(9 + x^2)^2} \, dx = \]

HINT: \( x \)

RELATED INTEGRALS: \( \int \cos^4(\tau) \, d\tau \) and \( \int \sin^4(\tau) \, d\tau \)

We now give a series of VARIATIONS on EXERCISES 4.26, 4.27, and 4.28. Each VARIATION contains problems on integration by parts, trigonometric integrals, and trigonometric substitution. EXERCISES 4.29–4.31 and
EXERCISES 4.32–4.34 follow the pattern of EXERCISES 4.26, 4.27, and 4.28 but with fewer hints. In EXERCISES 4.35–4.37, we still classify the problems by techniques but no hints are given. In EXERCISE 4.38 we don’t tell you the technique to use, but the answers are given. Finally, in EXERCISE 4.39 you are entirely on your own. You’ll soon be an expert on techniques of integration. Good luck!

4.29 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Integration by parts:

(1) \[ \int \arcsin(x) \, dx = \]  
HINT: \hspace{1cm} \\

\[ \begin{array}{|c|} \hline \text{arcsinx} \\ \hline \end{array} \hspace{1cm} 1 \]

(2) \[ \int x^2 \arctan(x) \, dx = \]  
HINT: \hspace{1cm} \\

\[ \begin{array}{|c|} \hline \text{arctan(x)} \\ \hline \end{array} \hspace{1cm} x^2 \]

(3) \[ \int \cos(\ln x) \, dx = \]  
HINT: \hspace{1cm} \\

\[ \begin{array}{|c|} \hline \text{cos(lnx)} \\ \hline \end{array} \hspace{1cm} 1 \]

\[ \begin{array}{|c|} \hline \text{sin(lnx)} \\ \hline \end{array} \hspace{1cm} 1 \]

(4) \[ \int x \sqrt{\frac{1}{2} + \frac{3}{2}x} \, dx = \]  
HINT: \hspace{1cm} \\

\[ \begin{array}{|c|} \hline x \\ \hline \end{array} \hspace{1cm} (2 + 3x)^{1/2} \]

(5) \[ \int \csc^3(x) \, dx \]  
HINT: \hspace{1cm} \\

\[ \begin{array}{|c|} \hline \text{csc(x)} \\ \hline \end{array} \hspace{1cm} \csc^2(x) \]
(HINT: Use cot^2 x + 1 = csc^2 x. ∫ csc(x)dx is to be worked in Trigonometric Integrals 4.30(1). Check also TABLE OF INTEGRALS in Appendix 1.)

4.30 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric integrals:

(1) \[ \int \csc(x)dx = \]
(2) \[ \int \cos^3(x)\sin^3(x)dx = \]
(3) \[ \int \sin^2(x)\cos^2(x)dx = \]

(4) \[ \int \sin(5x)\sin xdx = \]
(5) \[ \int \sin(5x)\cos(2x) = \]

4.31 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric substitutions:

(1) \[ \int \frac{dx}{(25 + x^2)^2} \]

(2) \[ \int \frac{x^2dx}{(2 - x^2)^{3/2}} \]

(3) \[ \int \frac{dx}{(16 - x^2)^{5/2}} \]

(4) \[ \int (x^2 - 4)^{1/2}dx \]
(5) \[ \int \frac{dx}{(x^2 - 9)^2} \]

4.32 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28 Integration by parts:

(1) \[ \int \arccos(x)dx = \text{HINT:} \]

(2) \[ \int x \arcsin(x)dx = \text{HINT:} \]

(3) \[ \int \frac{x^2dx}{\sqrt{1 - x^2}} = \text{HINT:} \]

(Use an inverse trig substitution)

(4) \[ \int \frac{x^2dx}{(5 + 2x)^{3/2}} = \text{HINT:} \]

(5) \[ \int \cos^2(x)dx = \text{HINT:} \]

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RELATED INTEGRALS: \( \int \sec^2(x) \, dx \) and \( \int \sec(x) \tan^2(x) \, dx \) (Use \( \sec^2 x = \tan^2 x + 1 \)).

4.33 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric integrals:

1. \( \int \tan(x) \, dx = \) 
2. \( \int \cos^2(x) \sin^3(x) \, dx = \) 
3. \( \int \sin^2(x) \cos^4(x) \, dx = \)

4.34 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric substitutions:

1. \( \int \frac{dx}{36 + x^2} = \) 
2. \( \int \frac{x \, dx}{(5 - x^2)^{5/2}} = \) 
3. \( \int \frac{dx}{(16 - x^2)^{3/2}} = \)

4.35 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Integration by parts:

1. \( \int \text{arcsec}(x) \, dx = \) 
2. \( \int x^2 \arccos(x) \, dx = \) 
3. \( \int e^x \sin x \, dx = \)

4.36 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric integrals:

1. \( \int \cot(x) \, dx = \) 
2. \( \int (3x) \sin(6x) \, dx = \)
\[
\begin{align*}
(2) \quad \int \cos(x) \sin^2(x) \, dx &= \\
(3) \quad \int \sin^4(x) \cos^4(x) \, dx &= \\
(4) \quad \int (x^2 + 9)^{1/2} \, dx &= \\
(5) \quad \int \frac{dx}{(x^2 - 5)^{3/2}} =
\end{align*}
\]

4.37 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

Trigonometric substitutions:

\[
\begin{align*}
(1) \quad \int \frac{dx}{(36 + x^2)^{5/2}} &= \\
(2) \quad \int \frac{x^2 \, dx}{(36 - x^2)^{3/2}} &= \\
(3) \quad \int \frac{dx}{(36 - x^2)^{5/2}} &=
\end{align*}
\]

4.38 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28

We give you the answers but the problems are scrambled so you have to guess the method. Before working the problems, go through and mark them IP (for integration by parts), TI (for trigonometric identities), or TS (for trigonometric substitution). It will be fun to see how accurate your guesses as to method were.

\[
\begin{align*}
(1) \quad \int \sqrt{x^2 + 16} \, dx &= \\
&= \frac{1}{2} \left( \frac{\sqrt{x^2 + 16}}{4} \right) x + \frac{1}{2} \ln \left| \frac{\sqrt{x^2 + 16}}{4} + x \right| + c
\end{align*}
\]

\[
\begin{align*}
(2) \quad \int \cos^6(x) \, dx &= \frac{1}{4} \sin(2x) - \frac{1}{48} \sin^3(2x) + \frac{3x}{16} + \frac{3}{64} \sin(4x) + \frac{x}{8} + c
\end{align*}
\]

\[
\begin{align*}
(3) \quad \int x \sin x \, dx &= -x \cos x + \sin x + c
\end{align*}
\]

\[
\begin{align*}
(4) \quad \int \sin^4 x \cos^4 x \, dx &= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c
\end{align*}
\]
\( \int \sqrt{x^2 - 4} \, dx = \frac{1}{2} \left( \frac{\sqrt{x^2 - 4}}{2} \right) x - \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| + c \)

\( \int \sin x \cos 2x \, dx = \cos x - \frac{2}{3} \cos^3 x + c \)

\( \int \sin(\ln 2x) \, dx = \frac{x}{2} (\sin(\ln 2x) - \cos(\ln 2x)) + c \)

\( \int \arcsin(2x) \, dx = \arcsin(2x) + \frac{1}{2} (1 - 4x^2)^{1/2} + c \)

\( \int \frac{dx}{x^2 \sqrt{a - x^2}} = -\frac{1}{a} \frac{\sqrt{a - x^2}}{x} + c \)

\( \int \arctan(2x) \, dx = \frac{x^2}{2} \arctan(2x) - \frac{x}{4} + \frac{1}{8} \arctan(2x) + c \)

\( \int \frac{\sqrt{36 - x^2}}{x} \, dx = -6 \ln \left| \frac{6}{x} + \frac{\sqrt{36 - x^2}}{x} \right| + 6 \left( \frac{\sqrt{36 - x^2}}{x} \right) + c \)

\( \int \sin^2 x \cos^4 x \, dx = \frac{x}{16} - \frac{\sin(4x)}{64} + \frac{\sin^3(2x)}{48} + c \)

\( \int \sin^5 x \cos^2 x \, dx = \frac{2\cos^5 x}{5} - \frac{3\cos^3 x}{3} - \frac{\cos^7 x}{7} + c \)

\( \int \arctan(6x) \, dx = x \arctan(6x) - \frac{1}{12} \ln(1 + 36x^2) + c \)

\( \int \frac{dx}{(9 - x^2)^{3/2}} = \frac{1}{3} \frac{x}{\sqrt{9 - x^2}} + c \)
\[ (16) \quad \int \frac{x^2 \, dx}{(a^4 - x^2)^{3/2}} = \frac{x}{\sqrt{a^4 - x^2}} - \arcsin\left(\frac{x}{a^2}\right) + c \]

\[ (17) \quad \int \sin^3 x \cos^5 x \, dx = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c \]

\[ (18) \quad \int x^3 \sqrt{1 - x} \, dx = -\frac{2x^3}{3} (1 - x)^{3/2} - \frac{4x^2}{5} (1 - x)^{5/2} - \frac{16x}{35} (1 - x)^{7/2} - \frac{16}{35} \left(\frac{2}{9}\right) (1 - x)^{9/2} + c \]

\[ (19) \quad \int x^3 e^{-x} \, dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c \]

4.39 VARIATIONS ON EXERCISES 4.26, 4.27, 4.28  \hspace{1em} Now you must find both the method and the answers!

(1) \[ \int \cos^4(x) \, dx = \]

(2) \[ \int \frac{dx}{(16 - x^2)^2} = \]

(3) \[ \int \sec(x) \tan^2(x) \, dx = \]

(4) \[ \int \frac{x^2}{(6 + 2x)^{5/2}} \, dx = \]

(5) \[ \int \sec(x) \, dx = \]

(6) \[ \int \cos(2x) \cos(3x) \, dx = \]

(7) \[ \int \frac{x^3 \, dx}{(16 - x^2)^{3/2}} = \]

(8) \[ \int \arccsc(x) \, dx = \]

(9) \[ \int \cos^2(x) \sin^2(x) \, dx = \]

(10) \[ \int (x^2 - 16)^{1/2} \, dx = \]

(11) \[ \int x \ln x \, dx = \]

(12) \[ \int e^w \cos(3e^w) \sin(5e^w) \, dw = \]
You can go directly to 4.47 (applications) and skip the following on partial fractions. Return to it some day for reasons explained below!

**Partial Fractions**

Before returning to the applications of integration, we consider one more basic technique of integration, called “partial fractions.” This technique is concerned with computing integrals of rational functions. A rational function is a ratio of two polynomials. Thus, a rational function is an expression of the form \( f(x) = p(x)/q(x) \) where \( p(x) \) and \( q(x) \) are polynomials. In your precalculus course, you learned how to divide one polynomial by another. If the degree of \( p(x) \) is larger than the degree of \( q(x) \), then, by polynomial division, you can always write \( p(x)/q(x) = a(x) + r(x)/q(x) \) where the “remainder” \( r(x) \) has degree less than \( q(x) \). For example, if \( p(x) = 2x^4 - 3x^3 + x^2 + x - 2 \) and \( q(x) = x^2 - 3x + 2 \), then \( a(x) = 2x^2 + 3x + 6 \) and \( r(x) = 13x - 14 \). You should carry out the polynomial division for this example to make sure you haven’t forgotten how to do it. Polynomial division has obvious implications for the integration of rational functions. In the example just given we can easily integrate the polynomial \( a(x) \) to compute

\[
\int \frac{2x^4 - 3x^3 + x^2 + x - 2}{x^2 - 3x + 2} \, dx = \frac{2x^3}{3} + \frac{3x^2}{2} + 6x + \int \frac{13x - 14}{x^2 - 3x + 2} \, dx.
\]

Noting that \((13x - 14)/(x^2 - 3x + 2) = 1/(x - 1) + 12/(x - 2)\). We compute \(\int (13x - 14)/(x^2 - 3x + 2) \, dx = \ln|x - 1| + 12 \ln |x - 2|\) to get

\[
\int \frac{2x^4 - 3x^3 + x^2 + x - 2}{x^2 - 3x + 2} \, dx = \frac{2x^3}{3} + \frac{3x^2}{2} + 6x + \ln|x - 1| + 12 \ln |x - 2|.
\]

Replacing \(x^2 - 3x + 2\) by \(x^2 - 3x + 3\) and doing division gives

\[
\int \frac{2x^4 - 3x^3 + x^2 + x - 2}{x^2 - 3x + 3} \, dx = \frac{2x^3}{3} + \frac{3x^2}{2} + 4x + \int \frac{4x - 14}{x^2 - 3x + 3} \, dx.
\]

The polynomial \(x^2 - 3x + 3\) is irreducible. Using Appendix 1 or an online integral calculator gives \(\int (4x - 14)/(x^2 - 3x + 3) \, dx = 2 \ln(x^2 - 3x) - \frac{8}{3^{1/2}} \arctan\left(\frac{2x - 3}{3^{1/2}}\right)\).

The \(\ln\) and \(\arctan\) functions appear frequently in these problems. We need to know some of the theory to understand why.
All polynomials have real coefficients in what follows. Let $q(x) = \sum_{i=0}^{n} q_i x^i$ be a polynomial. A theorem in analysis states that $q(x)$ can be factored into powers of distinct factors

\[
P(x) = q(x) = r \prod_{i=1}^{m} L_i(x) \prod_{i=1}^{n} Q_i(x)
\]

where $L_i(x) = x - \alpha_i, i = 1, \ldots, m$, $Q_i(x) = x^2 + \alpha_i x + \beta_i, \alpha_i^2 - 4\beta_i < 0$, $i = 1, \ldots, n$, and $r$ is a real number. (Note: $Q$ is irreducible).

**4.40 DEFINITION** Partial fractions are functions of three types:

1. $ax^k$
2. $\frac{a}{(x - \alpha)^k}$
3. $\frac{a + bx}{(x^2 + \alpha x + \beta)^k}$ ($\alpha^2 - 4\beta < 0$)

$k = 0, 1, \ldots$ in type (1) and $k = 1, 2, \ldots$ in types (2) and (3).

**4.40 THEOREM (Partial Fractions Expansion)** Let $f(x) = p(x)/q(x)$ be a rational function with real coefficients. Then $f(x)$ is a sum of finitely many partial fractions. (We use this without proof.)

**4.40 EXAMPLE** (go to 4.41 for examples of PFE below)

Let $f(x) = \frac{p(x)}{q(x)} = \frac{p(x)}{(x - 1)^2(x^2 + x + 1)^2(x^2 + x + 2)^3},$ deg(p) < deg(q).

Define $L_1(x) = x - 1, Q_1 = x^2 + x + 1, Q_2 = x^2 + x + 2$. Then there are constants $C_i, A_i$ and $B_i$ such that $f(x) =$

\[
\frac{C_1(1)}{(x - 1)} + \frac{C_1(2)}{(x - 1)^2} + \frac{C_1(3)}{(x - 1)^3} + \frac{A_1(1) + B_1(1)x}{(x^2 + x + 1)} + \frac{A_1(2) + B_1(2)x}{(x^2 + x + 1)^2} + \\
\frac{A_2(1) + B_2(1)x}{(x^2 + x + 2)} + \frac{A_2(2) + B_2(2)x}{(x^2 + x + 2)^2} + \frac{A_2(3) + B_2(3)x}{(x^2 + x + 2)^3}.
\]

Using (PF) above, suppose $q(x) = \prod_{i=1}^{m} L_i(x) \prod_{i=1}^{n} Q_i(x), f(x) = p(x)/q(x), \deg(p) < \deg(q), L_i = (x - \alpha_i), Q_i = x^2 + \alpha_i x + \beta_i, \alpha_i^2 - 4\beta_i < 0$, then there are constants $C_i(t)$, $A_i(t)$ and $B_i(t)$ such that

\[
P(x) = \sum_{i=1}^{m} \sum_{j=1}^{\beta_i} C_i(t) + \sum_{i=1}^{n} \sum_{k=1}^{\beta_i} \frac{A_i(t) + B_i(t)x}{(x^2 + \alpha_i x + \beta_i)^k}.
\]
If we plan to integrate rational functions by reducing them to sums of partial fractions, we should be able to integrate partial fractions. Types (1) and (2) are easy to integrate:

**TYPE (1)** \[ \int ax^k \, dx = \frac{ax^{k+1}}{k+1}, \quad k \geq 0, \]

**TYPE (2)** \[ \int \frac{a}{(x - \alpha)^k} \, dx = \frac{-a}{k - 1} \frac{1}{(x - \alpha)^{k-1}}, \quad k \geq 2, \quad a \ln |x - \alpha|, \quad k = 1. \]

For **TYPE (3)** partial fractions, we will use integral tables or online integral calculators. It is important to understand how the results in these tables or calculators are derived. It suffices to look at the special case where \( a = b = 1, \alpha = 0 \) and \( \beta = 1 \) so the numerator \( a + bx \) is \( 1 + x \), and the denominator \((x^2 + ax + \beta)^k\) is \((x^2 + 1)^k\). We have

\[
\int \frac{1 + x}{(x^2 + 1)^k} \, dx = \int \frac{1}{(x^2 + 1)^k} \, dx + \int \frac{x}{(x^2 + 1)^k} \, dx, \quad k \geq 1.
\]

The last integral is easy, \( \int \frac{x \, dx}{(x^2 + 1)^k} = (1/2) \int \frac{d(x^2 + 1)}{(x^2 + 1)^k} \)

\[
\frac{1}{2} \int \frac{d(x^2 + 1)}{(x^2 + 1)^k} = \frac{\ln(x^2 + 1)}{2} \quad \text{if} \quad k = 1 \quad \text{or} \quad \frac{1}{2} \left( \frac{x^2 + 1}{-k + 1} \right)^{-k+1} \quad \text{if} \quad k \geq 2.
\]

We consider the more difficult integral, \( J_k := \int \frac{dx}{(x^2 + 1)^k} \). Note that

**J** \( J_{k+1} - J_k = \int \left( \frac{1}{(x^2 + 1)^{k+1}} - \frac{1}{(x^2 + 1)^k} \right) \, dx = - \int \frac{x^2}{(x^2 + 1)^{k+1}} \, dx. \)

To evaluate \( \int \frac{x^2}{(x^2 + 1)^{k+1}} \, dx \) we use 4.25 INTEGRATION BY PARTS

**P** \( \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx \)

with \( f(x) = x, \quad f'(x) = 1, \quad g'(x) = \frac{x}{(x^2 + 1)^{k+1}}. \) Thus, \( g(x) \) is given by (E) above \( (k + 1 \text{ substituted for } k) \): \( g(x) = \frac{-1}{2k} \left( \frac{1}{x^2 + 1} \right)^k \). From (J) and (P):

**D** \( J_{k+1} - J_k = - \int \frac{x^2}{(x^2 + 1)^{k+1}} \, dx = \frac{x}{2k(x^2 + 1)^k} - \frac{1}{2k} J_k. \)
Thus, for $J_k := \int \frac{dx}{(x^2 + 1)^k}$ where $J_1 = \arctan(x)$, we get the recursion

$$J_{k+1} = \left(1 - \frac{1}{2k}\right)J_k + \frac{x}{2k(x^2 + 1)^k}.$$  

Thus, $J_1 = \arctan(x), \ J_2 = \frac{1}{2} \arctan(x) + \frac{x}{2(x^2 + 1)}, \ldots$. We have

$$\text{TYPE(3)} \quad \int \frac{1 + x}{(x^2 + 1)^k} dx = J_k + \int \frac{x}{(x^2 + 1)^k} dx, \ k \geq 1, \text{ where}$$

$$\int \frac{xdx}{(x^2 + 1)^k} = \frac{\ln(x^2 + 1)}{2} \text{ if } k = 1 \text{ or } \frac{1}{2} \frac{(x^2 + 1)^{-k+1}}{-k+1} \text{ if } k \geq 2.$$

After several changes of variables, the integration of 4.40 DEFINITION type (3) functions reduces to our special case and spawns the same mix of rational functions, logarithms and arctangents.

### 4.41 EXAMPLES OF PARTIAL FRACTIONS

(1) What is the form of $\frac{2x^2 + 3}{(2x + 1)^3(x^2 + x + 1)}$ when expanded by partial fractions? To solve this, let $L_1 = 2x + 1$ and $Q_1 = 1 + x + x^2$. The numerator is $r(x) = 2x^2 + 3$ and this has degree less than the denominator (this is important!). We find, from THEOREM 4.40, that

$$\phi(L_1) = \frac{C_1(1)}{(2x + 1)} + \frac{C_1(2)}{(2x + 1)^2} + \frac{C_1(3)}{(2x + 1)^3}.$$  

Similarly, we find that

$$\phi(Q_1) = \frac{A_1(1) + B_1(1)x}{1 + x + x^2} + \frac{A_1(2) + B_1(2)x}{(1 + x + x^2)^2} + \frac{A_1(3) + B_1(3)x}{(1 + x + x^2)^3}.$$  

In this example, we asked for "the form of" the rational function when expanded by partial fractions. By this we meant that the actual calculation of the constants $C_1(i), B_1(i)$, and $A_1(i)$ was not required. For EXAMPLE 4.41(1), these constants could have been computed by multiplying both the original rational function and its partial fraction expansion by $(2x + 1)^3(x^2 + x + 1)^3$, equating coefficients, and solving for the constants, which would be quite tedious in this case.

*NOTE: In 4.41(1) the denominators of the partial fractions are not of the form $(x - \alpha)^k$ found in 4.40 DEFINITION (e.g., we use $(2x + 1)^3$ not $8(x - \alpha)^3, \alpha = -1/2$). This type of notational convenience is standard.*
(2) Find the partial fractions expansion of

\[
\frac{5x^2 - 3x - 4}{(3x - 2)(x^2 - 2x - 1)}
\]

This is a good time to point out that, although \(x^2 - 2x - 1\) is reducible, it can still be left as a quadratic in the partial fractions expansion. Thus, we could write

\[
\frac{5x^2 - 3x - 4}{(3x - 2)(x^2 - 2x - 1)} = \frac{A + Bx}{x^2 - 2x - 1} + \frac{C}{3x - 2}.
\]

Multiplying both sides by the expression \((3x - 2)(x^2 - 2x - 1)\) gives the identity

\[
5x^2 - 3x - 4 = (A + Bx)(3x - 2) + C(x^2 - 2x - 1)
\]

\[
= (3B + C)x^2 + (3A - 2B - 2C)x + (-2A - C).
\]

Solving the three equations \(3B + C = 5, 3A - 2B - 2C = -3, -2A - C = -4\), gives \(A = 1, B = 1,\) and \(C = 2\). The lesson to be learned in this example is that, even though a quadratic factor in the denominator of a rational function may be reducible, it may be preferable to leave it unfactored.

(3) Find the partial fraction expansion of \(\frac{x^4 + x^2 + 1}{(x^2 + 1)^3}\). This type of problem can be very embarrassing. The first thought is to write the form of the partial fraction expansion.

\[
\frac{x^4 + x^2 + 1}{(x^2 + 1)^3} = \frac{A + Bx}{(x^2 + 1)^1} + \frac{C + Dx}{(x^2 + 1)^2} + \frac{E + Fx}{(x^2 + 1)^3}
\]

and solve for the constants \(A, B, \ldots\). In this case, however, there is a useful trick for avoiding such a messy process. Let \(y = x^2 + 1\) so that \(x^2 = y - 1\). The rational function then becomes

\[
\frac{(y - 1)^2 + (y - 1) + 1}{y^3} = \frac{y^2 - y + 1}{y^3} = \frac{1}{y} - \frac{1}{y^2} + \frac{1}{y^3}.
\]

Thus we obtain the partial fraction expansion

\[
\frac{x^4 + x^2 + 1}{(x^2 + 1)^3} = \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} + \frac{1}{(x^2 + 1)^3}.
\]
(4) Evaluate the integral \( \int \frac{1 + x}{(1 + x + x^2)^2} \, dx \). This is one of the basic types of integrals that occur in partial fraction expansions. In these examples, and in the exercises and variations that follow, we keep separate the tedium of finding partial fraction expansions from the task of computing the resulting integrals. To evaluate these integrals, we shall make use of the integral tables in Appendix 1. Look there under the section "Expressions involving \((a + bx + cx^2)\)." The denominator of this particular integral is \( X = 1 + x + x^2 \), so \( a = b = c = 1 \) and \( q = 4ac - b^2 = 3 \). Our integral breaks up into two integrals, both of which we find in the table. The first is

\[
\int \frac{dx}{X^2} = \frac{2x + 1}{3X} + (2/3) \int \frac{dx}{X} \text{ where}
\]

\[
\int \frac{dx}{X} = \frac{2}{3^{1/2}} \arctan \left( \frac{2x + 1}{3^{1/2}} \right).
\]

The second integral to be evaluated is

\[
\int \frac{xdx}{X^2} = -\frac{x + 2}{3X} - (1/3) \int \frac{dx}{X} = -\frac{x + 2}{3X} - \frac{2}{3^{3/2}} \arctan \left( \frac{2x + 1}{3^{1/2}} \right).
\]

Putting these facts together, we obtain

\[
\int \frac{1 + x}{(1 + x + x^2)^2} \, dx = \frac{x - 1}{3(1 + x + x^2)} + \frac{2}{3^{3/2}} \arctan \left( \frac{2x + 1}{3^{1/2}} \right).
\]

4.42 EXERCISES

(1) What is the form of the partial fraction expansion of the following?

\[
\frac{2x^2 + 3}{(2x + 1)^5(x^2 + x + 1)^2}
\]

(2) Find the partial fraction expansion of

\[
\frac{3x^3 - 8x^2 + 2x + 3}{(3x - 2)(x^2 - 2x - 1)}
\]

(HINT: First multiply factors in denominator and divide into numerator.)

(3) Find the partial fraction expansion of

\[
\frac{x^3 + 1}{(x^3 - 1)^2}.
\]
If needed, use online partial fractions calculator for 4.43, 4.44

**4.43 VARIATIONS ON EXERCISE 4.42**

(1) What is the “form” of the partial fractions expansion of

\[
\frac{2x^2 + 3}{(2x + 1)^3(x^2 + x - 1)^2}
\]

(2) Find the partial fractions expansion of

\[
\frac{3x^3 - 8x^2 + 2x + 3}{(2x + 1)^3(x^2 - 2x + 1)}
\]

(3) Find the partial fractions expansion of

\[
\frac{x^2 + 3}{x^4 + 2x^2 + 1}
\]

**4.44 VARIATIONS ON EXERCISE 4.42**

(1) What is the “form” of the partial fractions expansion of

\[
\frac{2x^9 + 3x^3 + 2x^2 + 1}{(2x + 1)^3(x^2 + x - 1)^2} = \frac{1}{2^4} + \ldots
\]

(2) Find the partial fractions expansion of

\[
\frac{3x^3 - 8x^2 + 2x + 3}{(2x + 1)^3(x^2 - 2x + 2)}
\]

(3) Find the partial fractions expansion of

\[
\frac{(x + 1)^2}{(x^2 + 2x + 2)^2}
\]
If needed, use online partial fractions calculator for 4.45, 4.46

4.45 VARIATIONS ON EXERCISE 4.42

(1) What is the “form” of the partial fractions expansion of
\[
\frac{3x^8 + 5x^5 + 10x}{(x^2 + 1)^2(x^2 - 1)^2}
\]

(2) Find the partial fractions expansion of
\[
\frac{9}{(x^2 - 1)^2(x^2 + 1)}
\]

(3) Find the partial fractions expansion of
\[
\frac{1}{(x^2 - 1)(x^2 + 1)^2}
\]

4.46 VARIATIONS ON EXERCISE 4.42

(1) What is the “form” of the partial fractions expansion of
\[
\frac{x^9 + 2x^5 + x^3 + 2x^2 + 1}{(x + 1)^2(x - 1)^3(x^6 + 4)}
\]

(2) Find the partial fractions expansion of
\[
\frac{x^3}{(x^2 - 1)^2(x^2 + 1)}
\]

(3) Find the partial fractions expansion of
\[
\frac{x^4}{(x^2 - 1)(x^2 + 1)^2}
\]
4.47 GEOMETRIC APPLICATIONS OF INTEGRATION

By now, you should be pretty adept at calculating integrals. Thus, we return to the applications of integration, confident that you can handle any integral that may arise! We shall restrict our attention to the geometric rather than the scientific or engineering applications. The most basic scientific and engineering applications follow easily from the techniques we shall learn. More specialized applications to science and engineering are best done in courses devoted to these topics.

**Arclengths, Surface Areas, Volumes**

We want to compute functions, $F(x)$, of geometric interest (e.g., volumes, arclengths, surface areas). For these functions, we will have easy ways of computing $f(x) = F'(x)$. We then compute $F(x) = \int f(x)dx$. The basic theory is that of 4.11 THEOREM (Fundamental theorem). We define by example various techniques for thinking about $f(x)$ and its relation to $F(x)$ (e.g., volumes of revolution, surfaces areas of revolution, arclengths of curves). As usual, we use the pattern of exercises, solutions and variations. Put online resources for graphing, integration and definite integrals to use.

4.48 EXERCISES

(1) A solid object lies in three-dimensional space,

$$R^3 = \{(x, y, z): x, y, z \text{ real numbers}\}.$$

For each $x > 0$, a cross-sectional slice through the object and perpendicular to the x-axis has the shape of an ellipse

$$\frac{y^2}{a^2(1 - x)^2} + \frac{z^2}{b^2(1 + x)^2} = 1$$

where $a$ and $b$ are positive numbers. Find the volume of the object that lies between the planes $x = 0$ and $x = 1$.  

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(2) Consider the graph of the parabola \( y = x^2, \ x \geq 0 \). Imagine this graph revolved about the \( x \)-axis to create a surface in \( \mathbb{R}^3 \). This surface is called the "surface of revolution of the curve \( y = x^2, \ x \geq 0 \), about the \( x \)-axis." By using the method of discs, compute the volume, \( V(x) \), of the region bounded by this surface and the plane perpendicular to the \( x \)-axis at the point \( x \).

(3) Compute the volume of revolution of the parabola, \( V(x) \) of problem (2), by the method of cylinders.

(4) Compute the arclength of the curve \( y = x^2, \ x > 0 \), as a function of \( x \).

(5) Compute the area of the surface of revolution of the curve \( y = x^2, \ x \geq 0 \), about the \( x \)-axis that lies between the planes perpendicular to the \( x \)-axis at 0 and \( x \).

(6) Compute the arclength of the curve \( r(\phi) = e^{\phi/2}, \ 0 \leq \phi \leq \pi \).

(7) Compute the surface area obtained by revolving the curve \( r(\phi) = e^{\phi/2}, \ 0 \leq \phi \leq \pi \), about the ray \( \phi = 0 \).

(8) Compute the volume bounded by the surface of revolution of \( r(\phi) = e^{\phi/2}, \ 0 \leq \phi \leq \pi \), about the ray \( \phi = 0 \).

4.49 SOLUTIONS TO EXERCISE 4.48

(1) The object whose volume is to be found is shown in FIGURE 4.50 (with \( a = 1/2 \) and \( b = 1/4 \)). In 4.15(4), we computed the area of an ellipse with

\[
\frac{dV}{dx} = \pi ab(1 - x^2)
\]

\( a = 1/2 \) and \( b = 1/4 \)

\( \pi \) (semi-minor axis)(semi-major axis) is the area of an ellipse

FIGURE 4.50  Solid with Elliptical Cross-Sections
semi-major axis 3 and semi-minor axis 2. If we do the same computation with the constant A replacing 3 and the constant B replacing 2, we get \( \pi AB \) for the area of the ellipse. This means that the cross-sectional area of the slice through the object at the point \( x \) has area \( \pi ab(1 - x)(1 + x) = \pi ab(1 - x^2) \). If the slice has thickness \( \Delta x \) then the volume of the slice is \( \Delta V \approx \pi ab(1 - x^2)\Delta x \). This equality is approximate but gets better and better as \( \Delta x \) gets small. In other words, if we let \( V(x) \) denote the volume between the planes perpendicular to the \( x \)-axis at 0 and \( x \), then \( V'(x) = \pi ab(1 - x^2) \). Usually, when thinking about this type of problem intuitively, one imagines the cross-sectional slice at \( x \) to have thickness \( dx \) and volume \( dV = (\text{area of the slice})dx \). The area of the slice in this case is \( \pi ab(1 - x^2) \). In other words, \( \frac{dV}{dx} = (\text{area of slice at } x) \). We easily integrate the expression \( \pi ab(1 - x^2) \) to obtain \( V(x) = \pi ab(x - x^3/3) + c \), where the constant \( c \) must be set equal to 0 as \( V(0) = 0 \). Thus, \( V(1) = 2\pi ab/3 \).

(2) The solid object that has volume \( V(x) \) is shown in FIGURE 4.51. Imagine a very thin slice of thickness \( dx \) through this object perpendicular to the \( x \)-axis at the point \( (x, 0, 0) \). This slice is a disc of radius \( r = x^2 \). The

**FIGURE 4.51** The Method of Discs
disc has slanted sides, but this fact can be ignored in what follows if dx is imagined to be very small. The volume of this disc is $dV(x) = \pi (x^2)^2 \ dx$. In other words, as a function of $x$, $V'(x) = \pi x^4$. Hence, $V(x) = \pi x^4/5$.

If you are concerned about the "slanted sides," consider the Riemann sum, FIGURE 4.13 ($f(t_i)$ replaced by $y(t_i)$). Imagine rotating the rectangles of height $y(t_i)$ and thickness $\Delta t_i$ about the $x$-axis. The sums of the volumes of these cylinders approximate the volume we are computing. We are free to choose the position of the $t_i \in [x_i, x_{i+1}]$. Choosing the $t_i$ close to the $x_i$, the volumes of the cylinders are less than the corresponding slanted side “frustrums” of thickness $\Delta t_i$; to the right they are greater. Both choices converge to $V(x)$.

(3) Look at FIGURE 4.52, the volume of revolution of the parabola $y = x^2, x \in [0, a]$. Call this 3 dimensional object $V_a$. We imagine a sequence of expanding cylinders of variable radius $y, y \in [0, a^2]$. One such is shown in FIGURE 4.52. It has length $a$, radius $y$ (by definition), and has part of $V_a$ inside of it, part outside. Define $V_a(y)$ to be the volume of the material inside the cylinder of radius $y$. Clearly, $V_a(0) = 0$ and $V_a(a^2)$ is the total volume of the object $V_a$. It is easy to see that $\frac{dV_a(y)}{dy} = 2\pi y(a - y^{1/2})$ (see FIGURE 4.52). Thus, $V_a(y) = \pi ay^2 - (4/5)\pi y^{5/2}$ and $V_a(a^2) = \pi (a^5 - (4/5)a^5) = \pi a^5/5$ (see 4.49(2) above).

FIGURE 4.52  The Method of Cylinders

Cylinder, cut along length and rolled flat, forms a rectangle with dimensions $a - y^{1/2}, 2\pi y, dy$.

cylinder radius $y$, length $a - y^{1/2}$
Thus, \( V(x) = \pi x^5/5 \) in agreement with (2). This method seems harder in this example, but there are times when the method of cylinders is the easier of the two methods.

(4) The general idea is shown in FIGURE 4.53. As before, think of \( dx \) as a small number. In FIGURE 4.53, the portion of the curve over the interval from \( x \) to \( x + dx \) is shown. It's almost a straight line. If it were a straight line, it would be the hypotenuse of the small right triangle of sides \( dx \) and \( dy \). Calling the length of this hypotenuse \( dL \), we obtain \( (dL)^2 = (dx)^2 + (dy)^2 \).

Thus, \( \frac{dL}{dx} = (1 + \left( \frac{dy}{dx} \right)^2)^{1/2} \). With \( y = x^2 \), we obtain \( \frac{dL}{dx} = (1 + 4x^2)^{1/2} \).

Thus, \( 2L(x) = \int (1 + z^2)^{1/2} dz \) where \( z = 2x \). To integrate this function we can set \( z = \tan(\theta) \), \( (1 + z^2)^{1/2} = \sec(\theta) \), etc. as in 4.26(5). Let’s use instead the tables and an online integral calculator. We get \( \int (1 + z^2)^{1/2} dz = (1/2)(z^2 + 1)^{1/2} + \ln(z + (z^2 + 1)^{1/2}) \) according to Appendix 1, but \( \int (1 + z^2)^{1/2} dz = (1/2)(z + (1 + z^2)^{1/2} + \sinh^{-1}(z)) \) according to one of the online integral calculators. As we learned in the Workbook on derivatives (2.47 INVERSE HYPERBOLIC FUNCTIONS):

\[
\sinh^{-1}(z) \equiv \text{arcsinh}(z) = \ln(z + (1 + z^2)^{1/2}).
\]

(5) The geometric situation is shown in FIGURE 4.54. Imagine a knife of thickness \( dx \) cutting the surface perpendicular to the \( x \)-axis at the point \( x \). The shaded area in FIGURE 4.54 is what is removed. If this shaded area is cut and rolled out, it forms a rectangle of width \( dL \) and length \( 2\pi y \), as shown in FIGURE 4.54.

FIGURE 4.53  Arclength

\[
dL = ((dx)^2 + (dy)^2)^{1/2}
\]

\[
dL = (1 + \left( \frac{dy}{dx} \right)^2)^{1/2} dx
\]
The thin "rectangle" is not actually a rectangle; one side is slightly smaller than the other. The usual calculations show this doesn’t matter as \(dL\) gets very small. From solution 4.49(4) we have \(dL = (1 + (\frac{dy}{dx})^2)^{1/2} dx\). Thus, \(dS = 2\pi y dL = 2\pi x^2 dL\). Write \(S(x) = (\pi/4) \int z^2(1 + z^2) dz, \; z = 2x\) (we’ll choose \(S(0) = 0\)). From Appendix 1 integral tables we get

\[
(A1) \quad \int z^2(z^2+1)^{1/2} dz = \frac{z}{4}(z^2+1)^{3/2} - \frac{z}{8}(z^2+1)^{1/2} - \frac{1}{8}\ln(z+(z^2+1)^{1/2}).
\]

Going online and using an integral calculator, we get

\[
(IC) \quad \int z^2(z^2+1)^{1/2} dz = \frac{1}{8} (z^2 + 1)^{1/2}(2z^3 + z) - \sinh^{-1}(z).
\]

Checking with an online derivative calculator shows A1 correct. Using \(\sinh^{-1}(z) = \ln(z+(z^2+1)^{1/2})\) plus some simple algebra shows that A1 and IC are the same functions of \(z\). They are zero when \(z = 0\) so are correct expressions for \(S(x)\) when multiplied by \(\frac{\pi}{4}\) and \(z = 2x\).

**6** The basic idea here is the same as in problem (4) except that we are using polar coordinates. A sketch of the curve \(r(\phi) = e^{\phi/2}, \; 0 \leq \phi \leq \pi\), is shown in FIGURE 4.55. Imagine that the angle \(\phi\) changes by a small amount \(d\phi\) and the distance \(r(\phi)\) changes by an amount \(dr\). The arclength changes...
by an amount $dL$ represented as the hypotenuse of the small right triangle of sides $dr$ and $r d\phi$ shown in FIGURE 4.55. We thus obtain

$$L(\pi) = \int_0^\pi \left( r^2 + \left( \frac{dr}{d\phi} \right)^2 \right)^{1/2} d\phi = \int_0^\pi \left( 5/4 e^{2\phi/2} \right) d\phi = 5^{1/2} (e^{\pi/2} - 1).$$

(7) This problem involves the same idea as in problem (5), except we use polar coordinates. FIGURE 4.56 shows half of the surface of revolution. Note
the small increment of arclength denoted by \( dL \). As this increment of arclength revolves about the ray \( \phi = 0 \), it traces out the shaded area. Cutting out this shaded area (for the full revolution) and laying it flat, as in FIGURE 4.54, gives a rectangle of width \( dL \) and length \( 2\pi r \sin(\phi) \). If \( dS \) denotes the total shaded area, then \( dS = 2\pi r \sin(\phi) \, dL \) where \( dL \) is as in problem (6). If \( S \) denotes the surface area, then

\[
S = \int_0^\pi 2\pi e^{\phi^2/2} \sin(\phi)(5/4)^{1/2} e^{\phi^2/2} \, d\phi = 5^{1/2} \pi \int_0^\pi e^{\phi} \sin(\phi) \, d\phi.
\]

The integral \( \int e^{\phi} \sin(\phi) \, d\phi \) can be evaluated by two applications of integration by parts or by using the table in the appendix to get \( e^\phi(\sin(\phi) - \cos(\phi))/2 \). Thus, \( S = 5^{1/2}(e^\pi + 1)\pi/2 \) square units.

(8) Look at FIGURE 4.57. Imagine the small shaded area revolved about the ray \( \phi = 0 \). It traces out a thin circular wire with an almost rectangular cross-section. The circumference of this wire is \( 2\pi r \sin(\phi) \). If this thin wire is cut and unfolded, it forms a long, thin rectangular parallelepiped with dimensions \( dr, r \, d\phi \), and \( 2\pi r \sin(\phi) \). Let \( dV = 2\pi r \sin(\phi) \, r \, d\phi \, dr \) be the volume of this thin parallelepiped. If we were to fill the region bounded by the curve \( r(\phi) = e^{\phi^2}, 0 < \phi < \pi \), with millions of non-overlapping such little shaded regions and add up the volumes generated by all of them, we would obtain the volume bounded by the surface of revolution of this curve. Making this idea precise is the topic of "multiple integrals" studied in more advanced

---

**FIGURE 4.57** Volume in Polar Coordinates

\[
dV = 2\pi(r \sin(\phi)) \, dA = 2\pi r \sin(\phi) r \, d\phi \, dr
\]

\[
V = \int_0^\pi \int_0^{r(\phi)} 2\pi r^2 \sin(\phi) \, dr \, d\phi
\]

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courses. The basic ideas required to understand multiple integrals are the final
topic of this chapter. In our case, we express this idea as the following "double
integral":

\[ V = \int_0^\pi \int_0^{r(\phi)} 2\pi r^2 \sin(\phi) \, dr \, d\phi. \]

The limits on the outer integral are from 0 to \( \pi \) and refer to the variable \( \phi \).
The limits on the inner integral are from 0 to \( r(\phi) = e^{\phi/2} \) and refer to the
variable \( r \). To evaluate this integral, we first evaluate the inner integral with
respect to \( r \), treating \( \phi \) as a constant:

\[ \int_0^{r(\phi)} 2\pi r^2 \sin(\phi) \, dr = 2\pi \sin(\phi) (r(\phi))^{3/2} = (2\pi/3)e^{3\phi/2}\sin(\phi). \]

We now evaluate the outer integral

\[ \int_0^\pi \left(2\pi/3\right)e^{3\phi/2}\sin(\phi) \, d\phi = \left(2\pi/3\right)e^{3\phi/2}(3/2)(\sin(\phi) - \cos(\phi))(4/13) \right]_0^\pi. \]

Here, we used the integral \( \int e^{ax}\sin(bx) \, dx \) given in the integral tables in the
appendix to evaluate \( \int e^{3\phi/2}\sin(\phi) \, d\phi \). Thus the volume bounded by this
surface of revolution is \( (8\pi/39)(e^{3\pi/2} + 1) \) cubic units.

After having worked EXERCISE 4.48 and studied carefully the solutions
to these exercises, you should now be ready to try the VARIATIONS that
follow. In all cases involving arclengths, surface areas, and volumes, draw
sketches of the figures corresponding to your calculations.

4.58 VARIATIONS ON EXERCISE 4.48

(1) A solid object has the property that each cross-sectional slice perpen-
dicular to the \( x \)-axis is an isosceles triangle with base \( 3^x \) and height \( 2^x \). The
\( x \)-axis passes through the midpoint of the base and the vertex not on the base
has coordinates \((x, 0, 2^x)\). Draw a sketch of the solid. Compute the volume
of the solid bounded by the planes \( x = 0 \) and \( x = 2 \). What is the total volume
of the solid in the region \( x < 0 \)?

(2) The curve \( y = x^{3/2}, x \geq 0 \), is revolved about the \( x \)-axis. Using the
method of discs, compute the volume, \( V(x) \), of the region bounded by this
surface of revolution and the plane perpendicular to the \( x \)-axis at the point \( x \).

(3) Compute the volume \( V(x) \) of problem (2) using the method of cylinders.

(4) Compute the arclength of the curve \( y = x^{3/2}, x \geq 0 \), as a function
of \( x \).
(5) Compute the area of the surface of revolution of the curve \( y = x^{3/2}, \) \( x \geq 0, \) about the \( x \)-axis, as a function of \( x. \)

(6) Compute the arclength of the curve \( r(\phi) = 2^\phi, 0 \leq \phi \leq \pi. \)

(7) Compute the surface area obtained by revolving the curve \( r(\phi) = 2^\phi, 0 \leq \phi \leq \pi, \) about the ray \( \phi = 0. \)

(8) Compute the volume bounded by the surface of revolution of \( r(\phi) = 2^\phi, 0 \leq \phi \leq \pi, \) about the ray \( \phi = 0. \)

4.59 VARIATIONS ON EXERCISE 4.48

(1) A solid object has the property that each cross-sectional slice perpendicular to the \( x \)-axis is a rectangle with base \( \log_2(x) \) and height \( 2^x \) where \( x \geq 1. \) The \( x \)-axis passes through the midpoint of the base and the midpoint of the side opposite the base has coordinates \( (x, 0, 2^x). \) Draw a sketch of the solid. Compute the volume of the solid bounded by the planes \( x = 1 \) and \( x = 2. \)

(2) The curve \( y = \sin(x), 0 \leq x \leq \pi, \) is revolved about the \( x \)-axis. Using the method of discs, compute the volume, \( V(x), \) of the region bounded by this surface of revolution.

(3) The curve \( y = \sin(x), 0 \leq x \leq \pi/2, \) is revolved about the \( y \)-axis. Using the method of cylinders, compute the volume bounded by this surface and the plane \( y = 1. \)

(4) Compute the arclength of the curve \( y = \sin(x), 0 \leq x \leq \pi. \) If you can't evaluate the integral, try a Riemann sum approximation.

(5) Compute the area of the surface of revolution of the curve \( y = \sin(x), 0 \leq x \leq \pi, \) about the \( x \)-axis, as a function of \( x. \)

(6) Compute the arclength of the curve \( r(\phi) = \phi, 0 < \phi < \pi. \)

(7) Compute the surface area obtained by revolving the curve \( r(\phi) = \phi, 0 \leq \phi \leq \pi, \) about the ray \( \phi = 0. \) If you can't evaluate the integral, try a Riemann sum approximation.

(8) Compute the volume bounded by the surface of revolution of \( r(\phi) = \phi, 0 \leq \phi \leq \pi, \) about the ray \( \phi = 0. \)

NOTE: 4.60, 4.61, 4.62, extra VARIATIONS ON EXERCISE 4.48, are in the book Top-down Calculus but omitted here.
4.63 DOUBLE INTEGRALS

The general geometric idea is shown in FIGURE 4.64. A surface with equation \( z = f(x, y) \) is shown in three-dimensional space with rectangular coordinates. For example, \( f(x, y) = xy^2 \) might be one possibility. The "surface defined by \( z = f(x, y) \)" is the set of all points with coordinates \((x, y, f(x, y))\) where the pairs \((x, y)\) range over all values where \( f(x, y) \) is defined.

FIGURE 4.64 The Volume Under a Surface

\[
V(a) = \int_0^a \left( \int_0^{t(x)} f(x, y) \, dy \right) \, dx
\]

- \( f(x, y) \) defines region \( R \) in \((x,y)\) plane
- \( A(x) \) defines \( A(x) \), area of slice perpendicular to \( x \)-axis at \( x \)
- \( f(x, y) \) defines volume element
- \( \Delta V = f(x, y) \Delta x \Delta y \)
(i.e., the domain of \( f \)). In FIGURE 4.64, we are interested in computing
the volume of the solid above the region \( R \) and below the surface defined by \( z = f(x, y) \). Imagine a plane
perpendicular to the x-axis cutting through this solid at \( x \). The points of the solid on this plane are shown in FIGURE 4.64.
The area of this region of intersection is called \( A(x) \). Let \( V(x) \) denote the
volume of the portion of the solid that lies to the left of this region of
intersection. If you worked EXERCISE 4.48(1), you will see immediately
that \( \frac{dV}{dx} = A(x) \). Thus
\[
\int_0^a A(x) \, dx = V(a)
\]
In EXERCISE 4.48 and all of its corresponding VARIATIONS, we made it easy for you to compute \( A(x) \). Note that in FIGURE 4.64, the region \( R \),
which lies in the \((x, y)\) plane, is bounded by the y-axis, the x-axis, the line
\( x = a \), and the curve \( y = t(x) \). Thus,
\[
A(x) = \int_0^{t(x)} f(x, y) \, dy.
\]
In this integral, we treat \( x \) as a constant because it does not change in
computing the integral. We thus obtain
\[
V(a) = \int_0^a \left( \int_0^{t(x)} f(x, y) \, dy \right) \, dx.
\]
As an example, let \( z = f(x, y) = 4 - x \) be a plane. Let \( t(x) = (4 - x^2)^{1/2} \).
We shall find the area under the surface \( z = 4 - x \) and above the
region in the \( xy\)-plane bounded by the semicircle \( y = (4 - x^2)^{1/2} \) and
the interval \([-2, +2]\) on the x-axis. You should sketch the picture. We compute
\[
\int_{-2}^{+2} \int_0^{(4-x^2)^{1/2}} (4 - x) \, dy \, dx = \int_{-2}^{+2} \left[ (4 - x) y \right]_0^{(4-x^2)^{1/2}} \\
= \int_{-2}^{+2} (4 - x) \, (4 - x^2)^{1/2} \, dx.
\]
This integral is the sum of two integrals
\[
4 \int_{-2}^{+2} (4 - x^2)^{1/2} \, dx + (1/2) \int_{-2}^{+2} (4 - x^2)^{1/2} \, d(4 - x^2).
\]
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From the integral tables in the appendix we find that the first integral is

\[ \left[ 2x(4 - x^2)^{1/2} + 8 \arcsin(x/2) \right]_{-2}^{+2} = 8\pi. \]

The second integral is zero. Thus, the total volume under the plane and above the semicircular region is \(8\pi\) cubic units.

**The Two-Dimensional Riemann Sum**

There is another and equivalent way to think about computing the volume over a region and under a surface \(R\). Look at FIGURE 4.65. We are looking at the same region \(R\) of FIGURE 4.64, but now we are looking straight down onto the region from above (in a direction parallel to the z-axis). The region has been divided into many small squares or "area elements." If we imagine \((x_i, y_i)\) to be the coordinates of the center of the \(i\)th little square, then \(\Delta V = f(x_i, y_i) \Delta A\) is the volume between the little square of area \(\Delta A = \Delta x \Delta y\) and the surface \(z = f(x, y)\). This volume element is approximately a tall, thin parallelepiped as shown in FIGURE 4.64. If we write VOLUME to mean the volume under the surface and above the region \(R\), then we have the approximate equality

\[
\text{VOLUME} = \sum_{\text{all squares}} f(x_i, y_i) \Delta x \Delta y.
\]

**FIGURE 4.65** Partitioning the Region \(R\) into Lots of Little Squares

\(\Delta y \quad (x_i, y_i) \quad \Delta x \quad f(x_i, y_i) \Delta x \Delta y\)

\(\Delta x \quad \Delta A = \Delta x \Delta y = \text{area of square}\)

\(y = t(x)\)
In using the squares of FIGURE 4.65 to approximate VOLUME we are going to be off from the true VOLUME by a certain amount. If we take smaller squares, our intuition tells us correctly that the approximation gets better and better. We want to make sure that the entire region is covered by little squares and that no little square lies entirely outside of the region. This process of better and better approximating sums is what is used to define the double integral. Thus, we write

$$\iint_R f(x, y) \Delta A = \lim_{\text{squares \to 0}} \sum_{\text{all squares}} f(x_i, y_i) \Delta A \text{ where } \Delta A = \Delta x \Delta y.$$ 

There is an obvious similarity, at least in spirit, between approximating the double integral of the function $f(x, y)$ with finer and finer meshes of little squares, and the idea of a RIEMANN SUM (4.14). These approximating sums could be used to compute the double integral, just as Riemann sums can be used to compute integrals. The way calculus is used to compute double integrals is through the idea of the "iterated" multiple integral of FIGURE 4.64. The fact that double integrals can be computed in this way is really a theorem in more advanced courses. The key idea involved in proving this theorem is contained in the transition from FIGURE 4.64 to FIGURE 4.65.

The idea of thinking of double integration in terms of the volume under a surface $z = f(x, y)$ was just a way to give us geometric feeling for the process. The function $f(x, y)$ can be any continuous (''smoothly changing'') function and the process

$$\iint_{\text{REGION}} f(x, y) \Delta A = \lim_{\text{squares \to 0}} \sum_{\text{all squares}} f(x_i, y_i) \Delta A \text{ where } \Delta A = \Delta x \Delta y$$

can be used to define the double integral of $f(x, y)$ over a REGION. The process of making all of these ideas ''precise'' can be quite technical and is beyond the scope of this book. Anyone with the right intuition and a little mathematical sophistication can ''make things precise.''' If you have the wrong intuition, no amount of precision and attention to technical details will save you!

**Iterated Integrals**

One thing we need to do now is discuss a little more carefully the process of going from double integrals, thinking of them as a limit of approximating sums, and the iterated integrals of FIGURE 4.64. In symbols, we are concerned about the following sort of identities:
\[ \iint_{\text{REGION}} f(x, y) \, dA = \int_c^d \left( \int_{a(x)}^{b(x)} f(x, y) \, dy \right) \, dx \]

or perhaps

\[ \iint_{\text{REGION}} f(x, y) \, dA = \int_c^d \left( \int_{a(y)}^{b(y)} f(x, y) \, dx \right) \, dy \]

In each of the above two equalities, the left-hand side is conceptually a limit of approximating sums and the right-hand side is a technical process of evaluating two integrals. Taking the first of the above equalities as an example, the right-hand side involves computing the integral

\[ g(x) = \int_{a(x)}^{b(x)} f(x, y) \, dy \]

as if \( x \) were a constant. Next we treat \( x \) as a variable and compute

\[ \int_c^d g(x) \, dx. \]

For an example of this process, see the computation of the integral

\[ \int_{-2}^{2} \left( \int_0^{(4-x^2)^{1/2}} (4-x) \, dy \right) \, dx \]

done in connection with FIGURE 4.64.

**Determining The Limits Of Integration**

One difficulty for the beginner in all of this is in determining the limits of integration, such as \( a(x) \), \( b(x) \), \( c \), and \( d \) from the description of REGION. Take a look at FIGURE 4.66. To help describe what we see there, let's introduce some notation. Given a region \( R \) in the \( x,y \)-plane, let \( (x, -)_R \) denote the set \( \{ x: (x, y) \text{ is in } R \text{ for some } y \} \) and let \( (x, y)_R \) denote the set \( \{ y: (x, y) \text{ is in } R \text{ for that fixed } x \} \). Similarly, we let \( (-, y)_R \) denote the set \( \{ y: (x, y) \text{ is in } R \text{ for some } x \} \) and \( (x, y)_R = \{ x: (x, y) \text{ is in } R \text{ for that fixed } y \} \). In FIGURE 4.66,

\[ \int_{(x,-)_R} \left( \int_{(x,y)_R} f(x, y) \, dy \right) \, dx = \int_a^b \left( \int_{c(x)}^{d(x)} f(x, y) \, dy \right) \, dx \]

will be used for iterated integrals. Note how \( y \) is associated with \( dy \) and \( x \) with \( dx \) in this notation.
In 4.66(a), assume (going counter clockwise) the boundary from point 1 to 2 is the graph of a function \( c(x) \) and from 2 to 1 the graph of \( d(x) \). In 4.66(b), the boundary from 3 to 4 is \( s(x) \), from 4 to 5 is \( t(x) \), from 5 to 6 is \( u(x) \), and from 6 to 3 is \( v(x) \). In 4.66(a), \( (x, -)_{R_1} = \{ x \mid \text{there exists } y, (x, y) \in R_1 \} = [a, b] \), and \( (x, y)_{R_1} = \{ y \mid (x, y) \in R_1 \} = [c(x), d(x)] \). For 4.66(b), \( (x, -)_{R_2} = [q, r], \) and \( (x, y)_{R_2} \) breaks into cases: \( (x, y)_{R_2} = [s(x), v(x)], \) \( x \in [q, q'] \) and \( (x, y)_{R_2} = [s(x), t(x)] \cup [u(x), v(x)], \) \( x \in [q', r] \). The integral \( \int \int_R f(x, y)dA \) has different iterated integrals for \( R_1 \) and \( R_2 \):

1. \( \int \int_{R_1} f(x, y)dA = \int_{(x, -)_{R_1}} \left( \int_{(x, y)_{R_1}} f(x, y)dy \right) dx = \int_a^b \left( \int_{c(x)}^{d(x)} f(x, y)dy \right) dx. \)

For \( R = R_2 \) the right side of the above identity becomes

\[
\int_q^r \left( \int_{s(x)}^{v(x)} f(x, y)dy \right) dx + \int_q^r \left( \int_{s(x)}^{t(x)} f(x, y)dy \right) dx + \int_q^r \left( \int_{u(x)}^{v(x)} f(x, y)dy \right) dx.
\]

The case \( R_1 \) is common. In FIGURE 4.66(a), assume the counter clockwise path from \( S = (s_1, s_2) \) to \( T = (t_1, t_2) \) is the graph of a function \( \tau(y) \), and the path from \( T \) to \( S \) is the graph of a function \( \sigma(y) \). Then

2. \( \int \int_{R_1} f(x, y)dA = \int_{(-)_{R_1}} \left( \int_{(x, y)_{R_1}} f(x, y)dx \right) dy = \int_{s_2}^{t_2} \left( \int_{\tau(y)}^{\sigma(y)} f(x, y)dx \right) dy. \)

Equations 1. and 2. are referred to as changing the order of integration.
Here is an example of 4.66(a). Take \( c(x) = x + 2, d(x) = x^2, \sigma(y) = -y^{1/2}, y \in [0, 1], \tau(y) = y - 2, y \in [1, 4], \sigma(y) = y^{1/2}, y \in [0, 4]. \)

\[
\int \int_R f(x, y) \, dy \, dx = \int_{(x, \tau)} f(x, y) \, dx \, dy - \int_{(x, \tau)} f(x, y) \, dx \, dy.
\]

\[
\int \int_R f(x, y) \, dy \, dx = \int_{(\tau, \sigma)} f(x, y) \, dx \, dy - \int_{(\tau, \sigma)} f(x, y) \, dx \, dy.
\]

\[
\int \int_R f(x, y) \, dy \, dx = \int_{(\tau, \sigma)} f(x, y) \, dx \, dy - \int_{(\tau, \sigma)} f(x, y) \, dx \, dy.
\]

\[\int \int_R f(x, y) \, dy \, dx = \int_{(\tau, \sigma)} f(x, y) \, dx \, dy - \int_{(\tau, \sigma)} f(x, y) \, dx \, dy.\]

\[
\int \int_R f(x, y) \, dy \, dx = \int_{(\tau, \sigma)} f(x, y) \, dx \, dy - \int_{(\tau, \sigma)} f(x, y) \, dx \, dy.
\]

**Iterated Integration In Polar Coordinates**

The iterated integration used to evaluate the double integral over a region \( R \) can also be done in polar coordinates. In FIGURE 4.67, we are using polar coordinates to describe a region \( R = \{(r, \theta): 0 \leq r \leq e^{\theta/2}, 0 \leq \theta \leq \pi/2\}. \)

As in the case of rectangular coordinates, we define \((r, \theta)_R = \{r: (r, \theta) \text{ is in } R \text{ for some } \theta\},\)

\(-, \theta)_R = \{\theta: (r, \theta) \text{ is in } R \text{ for some } r\},\)

\((r, \theta)_R = \{\theta: (r, \theta) \text{ is in } R \text{ for that fixed } r\},\)

and \((r, \theta)_R = \{r: (r, \theta) \text{ is in } R \text{ for that fixed } \theta\}.\)

Expressed in terms of this notation, we have \((dA = r \, d\theta \, dr)\)

\[
\int \int_R f(r, \theta) \, dA = \int_{(r, \theta)_R} \left( \int_{(r, \theta)_R} \right) dr = \int_{(-, \theta)_R} \left( \int_{(r, \theta)_R} \right) \, d\theta.
\]

**FIGURE 4.67 A Region in Polar Coordinates**

\( \{\theta: 2 \ln(2) \leq \theta \leq \pi/2\} = [2 \ln(2), \pi/2] \)

\( (r, \theta) = (2, 2 \ln(2)) \)

\( \theta = 2 \ln(2) \)

\( r(\theta) = e^{\theta/2} \)

\( R = \{(r, \theta): 0 \leq r \leq e^{\theta/2}, 0 \leq \theta \leq \pi/2\} \)

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In the case of FIGURE 4.67, \((r, -\theta)_R = [0, e^{\pi/4}], (-, \theta)_R = [0, \pi/2], (r, \theta)_R = [0, e^{\theta/2}],\) and \((r, \theta)_R = [0, \pi/2]\) for \(0 \leq r \leq 1\) and \((r, \theta)_R = [2\ln(r), \pi/2]\) for \(1 \leq r \leq e^{\pi/4}\). The last set is illustrated in FIGURE 4.67 for \(r = 2\). These sets lead to the following iterated integrals:

\[
\int_R f(r, \theta) dA = \int_0^{\pi/2} \left( \int_0^{e^{\theta/2}} r f(r, \theta) dr \right) d\theta
\]

and

\[
\int_R f(r, \theta) dA = \int_0^1 \left( \int_0^{\pi/2} r f(r, \theta) d\theta \right) dr + \int_1^{e^{\pi/4}} \left( \int_{2\ln(r)}^{\pi/2} r f(r, \theta) d\theta \right) dr.
\]

**Integrals In Three Dimensions**

Integration over regions in three and higher dimensions is treated in a manner analogous to that of two dimensions. We now give a brief discussion of integration over regions in three-dimensional space. Our purpose is to give you some intuitive feeling for the situations that can arise. The idea of volume under a surface, used to give us an intuitive feeling for integrals over planar regions, will be replaced by the model of computing the mass or weight of a three-dimensional object from its mass density function.

Have you ever been to a county fair hobby show where some enthusiastic hobbyist has made a model of the cathedral Notre-Dame de Paris, or some other famous building, out of sugar cubes? If so, you’ve seen the basic idea of defining integrals in three-dimensional space. Imagine a simpler object than Notre-Dame de Paris, say a watermelon, made out of sugar cubes. Call the volume of each sugar cube \(\Delta V\) cubic centimeters. We have the obvious fact that

\[
\text{VOLUME OF WATERMELON} = \sum_{\text{all cubes}} \Delta V.
\]

Actually, it is the volume of the model of the watermelon that we get by the above expression. That’s not quite the same as the real volume of the watermelon, but it’s close. Suppose we have function \(f(x, y, z)\), which gives the density of the watermelon in grams per cubic centimeter at each point \((x, y, z)\) in the watermelon. We have put some coordinates in the watermelon. The weight of the \(i\)th sugar cube sized piece of the watermelon is \(f(x_i, y_i, z_i) \Delta V\) where \((x_i, y_i, z_i)\) are the coordinates of a point in that particular cube. Then the weight of the whole melon is given by
WEIGHT OF WATERMELON = \sum_{\text{all cubes}} f(x_i, y_i, z_i) \Delta V.

Again, this is only an approximation as the sugar cube watermelon, having a rather bumpy surface, is only an approximation to the real thing. But, just as with the Riemann sum or the integral over two-dimensional regions, we can imagine using smaller and smaller sugar cubes and thus getting a better and better approximation to the watermelon. We say,

$$\int \int \int_{\text{REGION}} f(x, y, z) dV = \lim_{\text{cubes} \to 0} \sum_{\text{all cubes}} f(x_i, y_i, z_i) \Delta V.$$ 

This "triple integral" is thus defined by approximating sums. Intuitively, when one thinks about a triple integral over some region R in three-dimensional space (corresponding to the watermelon above), one writes

$$\int \int \int_R f(x, y, z) dV$$

and speaks of dV as the "volume element" or "element of volume." As we shall see below, there are several common ways to describe the volume element, but in the standard x, y, z-rectangular coordinates, one usually writes dV = dx dy dz and imagines a very small cube with sides dx, dy, and dz. The intuition, of course, comes from the sum of small cubes of volume \( \Delta V = \Delta x \Delta y \Delta z \). Describing the region of integration and evaluating the integral over this region can be a bit trickier than in the case of regions in the plane. The basic idea, however, is the same. Although integrals over regions in three-dimensional space are conceptualized as limits of partitions of regions into smaller and smaller cubes, these integrals are computed by iterated integration. Our notation for describing the limits of integration in the two-dimensional case extends directly to the three-dimensional case.

**Limits Of Integration In Three Dimensions**

Let R be a region in three-dimensional space. We use the notation \((x, -, -)_R\) to denote the set \(\{x: (x, y, z) \text{ is in } R \text{ for some } y \text{ and } z\}\). Similarly define \((- , y, -)_R\) and \((- , - , z)_R\). Define \((x, y, -)_R\) to be \(\{y: (x, y, z) \text{ is in } R \text{ for that } x \text{ and some } z\}\). There are a total of six sets of this type, the others being \((x, y, -)_R\), \((- , y, z)_R\), \((- , y, z)_R\), \((- , z)_R\), \((- , z)_R\), and \((x, - , z)_R\). Finally, let \((x, y, z)_R\) be the set \(\{z: (x, y, z) \text{ is in } R \text{ for that } x \text{ and } y\}\). There are two other sets of this type, \((x, y, z)_R\) and \((x, y, z)_R\). If f(x, y, z)
is a continuous function defined on the region \( R \) in three-dimensional space, then the integral

\[
\iiint_R f(x, y, z)\,dV = \int \left( \int f(x, y, z)\,dz \right) dy \,dx.
\]

There are five other ways to express this integral based on the orders of integration in the iterated integral of \( dz\,dx\,dy \), \( dx\,dy\,dz \), \( dy\,dz\,dx \), and \( dy\,dx\,dz \).

As an example of setting up the limits of integration for a region \( R \) in three-dimensional space, consider the region bounded by the planes \( x = 0 \), \( y = 0 \), \( z = 0 \), and the plane passing through the three points \((3, 0, 0)\), \((0, 2, 0)\), and \((0, 0, 1)\). This latter plane has equation \( (x/3) + (y/2) + z = 1 \). These planes and the bounded region are shown in FIGURE 4.68. We shall set up the iterated integral according to the order of integration \( dz\,dx\,dy \).

The set \((x, - , -)R = [0, 3]\). The set \((x, y, -)R = [0, 2(1 - x/3)]\) because, for a fixed \( x \), the largest value of \( y \) still contained in the region \( R \) is \( 2(1 - x/3) \), as shown in FIGURE 4.68. Finally, \((x, y, z)R = [0, 1 - (x/3), 2 - (y/2)]\). These intervals are shown in FIGURE 4.68. Thus, we have

\[
\iiint_R f(x, y, z)\,dz\,dx\,dy = \int_0^3 \left( \int_0^{2(1-x/3)} \left( \int_0^{1-x/3-y/2} f(x, y, z)\,dz \right) dy \right) dx.
\]

**FIGURE 4.68** The Limits of Integration on a Three-Dimensional Region
Cylindrical And Spherical Coordinates

Depending on the shape of the region \( R \) in three-dimensional space, it may be better to use cylindrical or spherical coordinates rather than the standard rectangular coordinates to describe the boundaries of the region. In FIGURE 4.69(a), we see cylindrical coordinates. The plane perpendicular to the \( z \)-axis and passing through \( z = 0 \) has standard polar coordinates \((r, \theta)\). Every point in three-dimensional space can be assigned a triple \((r, \theta, z)\) where \((r, \theta)\) are the polar coordinates of the projection of the point onto the plane perpendicular to the \( z \)-axis. Because polar coordinates are not unique, neither are cylindrical coordinates. If it is desirable to have unique cylindrical coordinates, restrictions must be put on the values of \( r \) and \( \theta \). For example, \( r > 0, 0 \leq \theta < 2\pi \), results in unique cylindrical coordinates. The basic volume element, \( \Delta V = r \Delta r \Delta \theta \Delta z \), for cylindrical coordinates is shown in FIGURE 4.69(a).

In FIGURE 4.69(b), we see spherical coordinates. The lines \( \theta = 0, \theta = \pi/2 \) and \( \phi = 0 \) correspond to the lines \( \theta = 0, \theta = \pi/2 \), and the \( z \)-axis in FIGURE 4.69(a). These lines intersect at right angles. A point in three-dimensional space may be assigned a triple \((\rho, \theta, \phi)\) where \( \rho \) is the distance to the point, \( \theta \) is the same as in cylindrical coordinates, and \( \phi \) is the angle between the line joining the point to the origin and the line \( \phi = 0 \). To make spherical coordinates unique, we can make the restrictions \( \rho > 0, 0 \leq \theta < 2\pi, \) and \( 0 \leq \phi < \pi \). As shown in FIGURE 4.69(b), the volume element in spherical coordinates is \( \Delta V = \rho^2 \sin(\phi) \Delta \rho \Delta \theta \Delta \phi \).

Limits Of Integration In Cylindrical And Spherical Coordinates

In integrating a function \( f(r, \theta, z) \) expressed as function of cylindrical coordinates, or a function \( g(\rho, \theta, \phi) \) expressed in terms of spherical coordinates, over a region \( R \), we again must perform an iterated integration. We can adopt the same notational conventions for describing the limits of integration in the iterated integral. Thus, in spherical coordinates, \((-\infty, \theta, -\infty) = \{(\rho, \theta, \phi) \text{ is in } R \text{ for some } \rho \text{ and some } \phi \}, (-\infty, \theta, \infty) = \{(\rho, \theta, \phi) \text{ is in } R \text{ for that } \theta \text{ and some } \rho \}, \text{ etc.} \) To fix these ideas in your mind, we’ll set up the limits of integration for the same three-dimensional region in both cylindrical and spherical coordinates. Take a look at FIGURE 4.70. The region \( R \) will be the region above the shaded region and below the cone. In spherical coordinates, the shaded region is \( \{(\rho, \theta, \phi) : 0 \leq \rho \leq 2, 0 \leq \theta \leq \pi/2, \phi = \pi/2 \} \). The cone is the set \( \{(\rho, \theta, \phi) : \phi = \pi/4 \} \). In cylindrical coordinates, the shaded region is the set \( \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq \pi/2, z = 0 \} \). The cone, in cylindrical coordinates, is the set \( \{(r, \theta, z) : r = z \} \).
First, let’s set up the limits of integration for the iterated integrals that evaluate the integral of $f(r, \theta, z)$ over the region $R$ of FIGURE 4.70 using cylindrical coordinates. We have $(r, -, -)_R =$
[0, 2], \((r, \theta, -)_{R} = [0, \pi/2]\), and \((r, \theta, z)_{R} = [0, r]\). Thus the integral is

\[
\iiint_{R} f(r, \theta, z) dV = \int_{0}^{2} \left( \int_{0}^{\pi/2} \left( \int_{0}^{r} f(r, \theta, z) r dz \right) d\theta \right) dr.
\]

There are five other ways to set up this integral depending on the order of integration, corresponding to \(dzdrd\theta\), \(d\theta dz dr\), \(d\theta drdz\), \(drdzd\theta\), and \(drd\theta dz\). You should set up the limits of integration for at least two of these ways.

Finally, let’s set up the limits of integration for the region \(R\) of FIGURE 4.70 in spherical coordinates. First, \((-\rho, \theta, -\phi) = [0, \pi/2]\). Next, \((-\rho, \theta, \phi) = [\pi/4, \pi/2]\). Finally, \((\rho, \theta, \phi) = [0, 2\csc(\phi)]\). This gives

\[
\iiint_{R} g(\rho, \theta, \phi) dV = \int_{0}^{\pi/2} \int_{\pi/4}^{\pi/2} \int_{0}^{2\csc(\phi)} \rho^2 \sin(\phi) d\rho d\phi d\theta.
\]

Again, there are five other ways to set up this integral. You should describe these five ways and set up the limits of integration for at least two of them.

**Changing Coordinates**

Of course, any time you integrate the function \(f(x, y) = 1\) over a region in the plane, you obtain the area of the region, and any time you integrate
the function \( f(x, y, z) = 1 \) over a region in three-dimensional space, you obtain the volume of the region. Using FIGURE 4.69, it is easy to transform the coordinates of a point from rectangular to cylindrical or spherical coordinates and vice versa. For example, from rectangular to cylindrical we have the relations

\[
x = r\cos(\theta) \quad y = r\sin(\theta) \quad z = z.
\]

From rectangular to spherical coordinates, we have

\[
x = \rho\sin(\phi)\cos(\theta) \quad y = \rho\sin(\phi)\sin(\theta) \quad z = \rho\cos(\phi).
\]

If you are faced with a task of integrating a function over some region in three-dimensional space, you can choose the coordinate system that makes the work of computing the iterated integral the easiest. Once the coordinate system is chosen, you must express the limits of integration and the function to be integrated in terms of that coordinate system. If the function to be integrated is \( f(x, y, z) = x^2 + y^2 + z^2 \), for example, then in cylindrical coordinates the function becomes \( g(r, \theta, z) = r^2 + z^2 \), and in spherical coordinates, the function becomes \( h(\rho, \theta, \phi) = \rho^2 \).

### 4.71 EXERCISES

(1) In each of the following problems, change the order of integration. Choose one of the two iterated double integrals and evaluate it. Sketch the region of integration.

(a) \( \int_{0}^{2} \int_{x}^{\infty} dy \, dx \) (Answer: \(-2/3\))

(b) \( \int_{0}^{1} \int_{y}^{3y} (1 + y) \, dx \, dy \) (Answer: \(5/3\))

(c) \( \int_{0}^{\pi/2} \int_{0}^{\cos(\theta)} rsin(\theta) \, dr \, d\theta \) (Answer: \(1/6\))

(d) \( \int_{0}^{\pi/2} \int_{0}^{r^2\cos(\theta)} r^2 \cos(\theta) \, dr \, d\theta \) (Answer: \(8/3\))

(e) \( \int_{-1}^{2} \int_{x^2}^{x+2} 2x^2 \, dy \, dx \) (Answer: \(6.3\)) see also 4.71(3)(b)

(f) \( \int_{0}^{2\pi} \int_{0}^{1+\cos(\theta)} r^3 \cos^2(\theta) \, dr \, d\theta \) (Answer: \(35\pi/16\))

(HINT: Use online integral calculator.)
(2) Try to evaluate each of the following iterated double integrals. If you have trouble, change the order of integration and evaluate that iterated integral.

(a) \[ \int_0^1 \int_0^y e^{-x^2} \, dx \, dy \quad \left( \int_0^1 e^{-x^2} \, dx = \frac{1}{2} \right) \]

(b) \[ \int_0^1 \int_y^3 e^{x^2} \, dx \, dy \quad \left( \int_0^1 e^{x^2} \, dx = \frac{1}{2} \right) \]

(3) Each of the following iterated integrals results in a sum of two integrals when the order of integration is changed. Find the indicated constants a and b, the functions c and d, and sketch the region of integration for each of the following problems:

(a) \[ \int_{-1}^1 \int_{x^2}^{x+2} f(x, y) \, dy \, dx = \] (See FIGURE 4.66 discussion/example)

(b) \[ \int_0^1 \int_0^y f(x, y) \, dx \, dy + \int_0^b \int_{c(y)}^{d(y)} f(x, y) \, dx \, dy \]

(c) \[ \int_{-1}^1 \int_0^2 f(x, y) \, dy \, dx = \]}

(4) Evaluate each of the following iterated triple integrals:

(a) \[ \int_{-1}^1 \int_0^2 \int_0^1 (x^2 y + xy^2) \, dx \, dy \, dz \quad \text{ (Answer: 4)} \]

(b) \[ \int_0^1 \int_0^1 \int_0^1 y^2 \, dx \, dy \, dz \quad \text{ (Answer: 1/3)} \]

(c) \[ \int_0^{2\pi} \int_0^\pi \int_0^{a \cos (\phi)} \rho^2 \sin (\phi) \, d\rho \, d\phi \, d\theta \quad \text{ (Answer: } \frac{1}{8} a^3 \pi \text{)} \]

(d) \[ \int_0^1 \int_0^2 \int_0^3 (x^2 + yz) \, dx \, dy \, dz \quad \text{ (Answer: 21)} \]

(e) \[ \int_0^2 \int_0^2 \int_{x^2+y^2}^{4-x^2-y^2} (\frac{4-x^2-y^2}{2}) \, dx \, dy \, dz = \frac{2^{3/2}}{3} \int_0^2 (4-x^2)^{3/2} \, dx = 2^{3/2} \pi \text{ (table/online)} \]

(f) \[ \int_0^\pi \int_0^3 \int_0^r r^2 \, dr \, d\theta \quad \text{ (online or do it yourself: } \frac{31 \pi a^3}{4} \text{)} \]
SOLUTIONS TO SELECTED PROBLEMS

4.5 SOLUTIONS (Solutions to problem 4.5)

(3) \[ \int \frac{\sqrt{x}-1}{\sqrt{x}} \, dx = \int \left( 1 - \frac{1}{\sqrt{x}} \right) \, dx = \int dx - \int x^{-1/2} \, dx = x - 2 \sqrt{x} + C \]

(4) \[ \int \frac{8x}{1 + e^{2x}} \, dx = \int \left( \frac{8e^{-2}(1+e^{2x}) - 8e^{-2}}{1+e^{2x}} \right) \, dx = 8e^{-2} \int \frac{dx - 8e^{-2} \int (1+e^{2x})^{-1} \, dx}{1+e^{2x}} \]

\[ = 8e^{-2}x - 8e^{-2} \ln(1+e^{2x}) + C \]

(5) \[ \int \frac{x \cos \sqrt{5x^2+1}}{\sqrt{5x^2+1}} \, dx = \int \cos \sqrt{5x^2+1} \cdot \frac{1}{5} \cdot \frac{d}{dx} \sqrt{5x^2+1} = \frac{1}{5} \sin \sqrt{5x^2+1} + C \]

(6) \[ \int \sin^{3/4}(2x) \cos(2x) \, dx = \int \sin^{3/4}(2x) \cdot \frac{1}{2} \cdot d\sin(2x) = \frac{1}{2} \cdot \frac{4}{7} \sin^{7/4}(2x) + C \]

(7) \[ \int \tan^2(3t) \sec^2(3t) \, dt = \int \tan^2(3t) \cdot \frac{1}{3} \cdot d\tan(3t) = \frac{1}{9} \tan^3(3t) + C \]

(8) \[ \int e^x \sec^2(e^x) \, dx = \int \sec^2(e^x) \cdot d(e^x) = \tan(e^x) + C \]

(9) \[ \int \frac{2 \ln^3 x}{x} \, dx = 2 \int \ln^3 x \cdot d\ln x = \frac{1}{2} \ln^4 x + C \]

(10) \[ \int \frac{6x}{(x^2+9)^3} \, dx = \int \frac{3 \cdot d^2}{(x^2+9)^3} = 3 \int (x^2+9)^{-3} \, d(x^2+9) = - \frac{3}{2} (x^2+9)^{-2} + C \]

\[ \int \frac{\log x}{x} \, dx = \int \frac{e^{(\log x)^2}}{x} \, dx = \int \frac{x^{\log x}}{x} \, dx \]

\[ = \int x^{\log x - 1} \, dx = \frac{1}{\log e} x^{\log x} + C \quad \text{or} \]

\[ \int \frac{\log x}{x} \, dx = \ln(2) \int e^{\log x} = \ln(2) e^{\log x} = \ln(2) x^{\log x} = \ln(2) x^{\log e} \]

(12) \[ \int x \sec x \tan x \, dx = \int \sec x \tan x \cdot \frac{1}{2} \, dx^2 = \frac{1}{2} \sec x^2 + C \]

(13) \[ \int \frac{u}{(u^2+29)^{40}} \, du = \int \frac{1}{2} du^{29} \frac{1}{(u^2+29)^{40}} = \frac{1}{2} \int (u^2+29)^{-40} \, du^{29} \]

\[ = - \frac{1}{2 \times 39} (u^2+29)^{-39} + C \]
\[ \int \frac{\csc^2 x}{\sqrt{1+5\cot x}} \, dx = \int \frac{-d \cot x}{\sqrt{1+5 \cot x}} = -\frac{1}{5} \int (1+5 \cot x)^{-1/2} d(1+5 \cot x) \\
= -\frac{2}{5} (1+5 \cot x)^{1/2} + C \]

\[ \int \frac{(\arctan x)^{3/2}}{1+x^2} \, dx = \int (\arctan x)^{3/2} d(\arctan x) \\
= \frac{2}{5} (\arctan x)^{5/2} + C \]

4.9 PARTIAL SOLUTIONS

(8) \[ \int \frac{\cos \sqrt{x}}{2 \sqrt{x}} \, dx = \int \cos \sqrt{x} \, d \sqrt{x} = \sin \sqrt{x} + C \]

(9) \[ \int 5x^2(4x^3+1)^{4/5} \, dx = \frac{5}{3} \int (4x^3+1)^{4/5} \, dx^3 \\
= \frac{5}{12} \int (4x^3+1)^{4/5} d(4x^3+1) = \frac{5}{12} \cdot \frac{5}{9} (4x^3+1)^{9/5} + C \]

(10) \[ \int \frac{\ln^6 \sqrt{2x+1}}{2x+1} \, dx = \frac{1}{2} \int \ln^6 \sqrt{2x+1} \, d \ln(2x+1) \\
= \frac{1}{2} \cdot \left( \frac{1}{2} \right)^6 \int \ln^6 (2x+1) d \ln(2x+1) = \frac{1}{7} \left( \frac{1}{2} \right)^7 \ln^7 (2x+1) + C \]

(11) \[ \int \sqrt{\sin(e\theta)} \, \cos(e\theta) \, d\theta = \frac{1}{e} \int (\sin \theta)^{1/2} d \sin e\theta \\
= \frac{2}{3e} (\sin(e\theta))^{3/2} + C \]

(12) \[ \int (10 \sin t)^7 \cos t \, dt = \frac{1}{10} \int (10 \sin t)^7 d(10 \sin t) \\
= \frac{1}{10} \cdot \frac{1}{8} (10 \sin t)^8 + C \]

(13) \[ \int (\sin x + x \cos x)e^{\sin x} \, dx = \int e^{\sin x} d(\sin x) \\
= e^{\sin x} + C \]
\( (14) \int \sin(\sin \theta) \cos \theta \, d\theta = \int \sin(\sin \theta) \, d(\sin \theta) \)

\[ = -\cos(\sin \theta) + C \]

\( (15) \int \frac{\sin \theta}{(5-\cos \theta)^{10/9}} \, d\theta = \int (5-\cos \theta)^{-10/9} \, d(5-\cos \theta) \)

\[ = -\frac{1}{9} (5-\cos \theta)^{-9/9} + C. \]

### 4.21 Variations on Exercise 4.15

1. \[ A = \int_0^1 3^x \, dx = \int_0^1 e^{x \ln 3} \, dx = \frac{1}{\ln 3} e^{x \ln 3} \bigg|_0^1 = \frac{2}{\ln 3}. \]

2. \( y = 3^x, \ x = \log_3 y \)

\[ \therefore A = \int_{3/2}^3 \log_3 y \, dy = \frac{1}{\ln 3} \int_{3/2}^3 \ln y \, dy \]

\[ = \frac{1}{\ln 3} [\ln y]_{3/2}^3 = \frac{3}{\ln 3} (\ln 3 - 1) - \frac{3}{2 \ln 3} (\ln 3/2 - 1) \]

3. \( y = -x^2 + 3. \ y = 3^x. \) We have \( 3^x = -a^2 + 3, 3^b = -b^2 + 3. \)

\[ A = \int_a^b (-x^2 + 3) \, dx - \int_a^b 3^x \, dx \]

\[ = \left[ \frac{-x^3}{3} + 3x \right]_a^b - \frac{1}{\ln 3} \left[ 3^x \right]_a^b \]

\[ = \frac{1}{3} \left[ (b^3 - a^3) + 3(b-a) - \frac{1}{\ln 3} \left[ [-b^2 + 3] - (-a^2 + 3) \right] \right] \]

\[ = \frac{1}{3} \left[ (b^3 - a^3) + 3(b-a) + \frac{1}{\ln 3} (b^2 - a^2) \right], \]

where \( b = 0.7885, \ a = -1.586. \)

\[ \therefore A \approx 0.1277. \]
\[
(5) \quad A = \int_0^{\pi/2} \frac{1}{2} r^2(\phi) d\phi = \int_0^{\pi/2} \frac{1}{2} (1+2\cos^2\phi)^2 d\phi
\]
\[
= \frac{1}{2} \int_0^{\pi/2} (1+4\cos\phi+4\cos^2\phi) = \frac{1}{2} \int_0^{\pi/2} (1+4\cos\phi+2(1+\cos 2\phi)) d\phi\]

4.22 VARIATIONS ON EXERCISE 4.15

(1) \quad A = \int_{-1}^{2} (x^3+1) dx = 3 + \int_{-1}^{2} x^3 dx = 27/4

(3) \quad A = \int_{a}^{b} (-x^4+16-1.5x^5) dx
\]
\[
= 16(b-a) - \frac{b^5-a^5}{5} - (1.5^a - 1.5^b) / \ln 1.5.
\]

Note \(1.5^a = 16-a^4, \ 1.5^b = 16-b^4\).

\[ \therefore A = 16(b-a) - \frac{b^5-a^5}{5} + \frac{b^4-a^4}{\ln 1.5} \]

where \(a \approx -1.986, \ b \approx 1.928\).

\[ \therefore A \approx 52.820. \]

(4) \quad x = t - \sin t, \ y = 1 - \cos t.

\[ \therefore x' (t) = 1 - \cos t \geq 0, \ \therefore x(t) \text{ is an increasing function.} \]

\[ \therefore A = \int_0^{2\pi} (1-\cos t) dt = \int_0^{2\pi} (1-\cos t)^2 dt = \int_0^{2\pi} (1-2\cos t + \frac{1+\cos t}{2}) dt = \frac{3}{2} \cdot 2\pi = 3\pi. \]
4.23 VARIATIONS ON EXERCISE 4.15

(1) \[ A = \int_{-1}^{0} \left( \frac{x^4}{4} - \frac{2x^3}{3} \right) dx + \int_{0}^{\frac{8}{3}} \left( \frac{2x^3}{3} - \frac{x^4}{4} \right) dx. \]

(3) \[ y = -\frac{x^4}{4} - 2x^3 + 4x^2 - 2. \]

Compute \( x_1 \approx -0.649, \ x_2 = -0.623 \)

\[ \therefore A = \int_{x_1}^{x_2} \left( -\frac{x^4}{4} - 2x^3 + 4x^2 - 2 \right) dx \]

\[ = \left[ -\frac{x^5}{20} - \frac{x^4}{2} + \frac{4}{3} x^3 - 2x \right]_{x_1}^{x_2} \approx +1483.627. \]

(4) \( x = 1 + \sin t, \ y = \cos t (1+\sin t). \ -\pi/2 < t < \pi/2 \)

\[ A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t (1+\sin t) x'(t) dt \]

\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t (1+\sin t) \cos t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1+\cos 2t}{2} \right) dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \sin t dt \]

\[ = \frac{\pi}{2} + \frac{1}{4} \sin 2t \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos^2 t \sin t \ dt = \pi/2 \]
(5) \[ A = \int_0^{\frac{\pi}{2}} \frac{1}{2} e^{2\phi/\pi} \, d\phi + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} e^{2\phi/\pi} \, d\phi \]
\[ = \frac{\pi}{4} e^{\phi/\pi} \bigg|_{0}^{\frac{\pi}{2}} - \frac{\pi}{4} e^{2\phi/\pi} \bigg|_{\frac{\pi}{2}}^{\pi} \]
\[ = \frac{\pi}{4} (e-1) - \frac{\pi}{4} (1-e) = \frac{\pi}{2} (e-1). \]

4.26 PARTIAL SOLUTIONS

(1) \[ \int \arctan x \, dx = x \arctan x - \int x (\arctan x)' \, dx \]
\[ = x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C \]

(3) \[ \int \sin(\ln x) \, dx = x \sin(\ln x) - \int x (\sin(\ln x))' \, dx \]
\[ = x \sin(\ln x) - \int \cos(\ln x) \, dx \]
\[ = x \sin(\ln x) - [x \cos(\ln x) - \int x (\cos(\ln x))' \, dx] \]
\[ = x \sin(\ln x) - \cos(\ln x) - \int \sin(\ln x) \, dx \]
therefore \[ \int \sin(\ln x) \, dx = \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C. \]

(5) \[ \int \sec^2(x) \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx \]
\[ = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \]
\[ \therefore \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \]
\[ \therefore \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{4} [\ln(1+\sin x) - \ln(1-\sin x)] + C. \]
For \[ \int \sec x \, dx. \] See next exercise set or Appendix 1 (Table of Integrals).
4.27 PARTIAL SOLUTIONS

(2) \( \int \cos^3 x \sin^2 x \, dx = \int \cos^2 x \sin^2 x \, d \sin x = \int (1-\sin^2 x) \sin^2 x \, d \sin x \)
\[ = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C. \]

(4) \( \int \cos 5x \cos x \, dx = \int \frac{1}{2} [\cos 6x + \cos 4x] \, dx = \frac{1}{12} \sin(6x) + \frac{1}{8} \sin(4x) + C. \)

4.28 PARTIAL SOLUTIONS

(3) Let \( \theta = \arcsin \left( \frac{x}{3} \right) \) then \( \sin \theta = \frac{x}{3} \) and \( \, dx = 3 \cos \theta \, d \theta. \)

\[ \therefore \int \frac{dx}{(9-x^2)^{3/2}} = \int \frac{dx}{3^3(1-\left(\frac{x}{3}\right)^2)^{3/2}} = \int \frac{\cos \theta \, d \theta}{9(1-\sin^2 \theta)^{3/2}} = \int \frac{d \theta}{9 \cos^3 \theta} \]
\[ = \frac{1}{9} \int \sec^2 \theta \, d \theta = \frac{1}{9} \tan \theta + C = \frac{1}{9} \cdot \frac{x}{\sqrt{9-x^2}}. \]

(4) \( \int (x^2+25)^{1/2} \, dx = \int 5\left(1 + \left(\frac{x}{5}\right)^2\right)^{1/2} \, dx. \)

Let \( \theta = \arctan \frac{x}{5} \) then \( x = 5 \tan \theta \), and \( \, dx = 5 \sec^2 \theta \, d \theta, 1 + \left(\frac{x}{5}\right)^2 = \sec^2 \theta. \)

\[ \therefore \int (x^2+25)^{1/2} \, dx = \int 5 \sec \theta \cdot 5 \sec^2 \theta \, d \theta = 25 \int \sec^3 \theta \, d \theta \]
\[ = 25 \int \frac{d \theta}{\cos^3 \theta} = 25 \int \frac{d \sin \theta}{(1-\sin^2 \theta)^2} = \frac{25}{4} \int \left( \frac{1}{1-\sin^2 \theta} + \frac{1}{1+\sin^2 \theta} \right)^2 \, d \sin \theta \]
\[ = \frac{25}{4} \left[ \int \frac{d \sin \theta}{(1-\sin^2 \theta)^2} + \int \frac{d \sin \theta}{(1+\sin^2 \theta)^2} + 2 \int \frac{d \sin \theta}{(1-\sin^2 \theta)(1+\sin^2 \theta)} \right] \]

4.29 PARTIAL SOLUTIONS

(1) \( \int \arcsin x \, dx = x \arcsin x - \int x(\arcsin x)' \, dx \)
\[ = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x + \sqrt{1-x^2} + C. \]
(3) $\int \cos(\ln x)dx = x \cos(\ln x) - \int x(\cos(\ln x))' \, dx$

$= x \cos(\ln x) + \int \sin(\ln x)dx$

$= x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x)dx$

$\therefore \int \cos(\ln x)dx = \frac{x}{2} [\cos(\ln x)+\sin(\ln x)] + C.$

(5) $\int \csc^3 x \, dx = \int -\csc x \, d \cot x = - \csc x \cot x + \int \cot x \, (\csc x)' \, dx$

$= - \csc x \cot x - \int \csc x \cot^2 x \, dx$

$= - \csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx$

$\therefore \int \csc^3 x \, dx = - \frac{1}{2} \csc x \cot x + \frac{1}{2} \int \csc x \, dx.$

Therefore: $\int \csc^3 x \, dx = - \frac{1}{2} \csc x \cot x + \frac{1}{4} \ln \left(\frac{1-\cos x}{\cos x+1}\right) + C.$

### 4.30 PARTIAL SOLUTIONS

(2) $\int \cos^3 x \sin^3 x \, dx = \int 2^3 \sin^3 2x \, dx$

$= 2^3 \int -2^1 \sin^2 2x \, d \cos 2x = -2^4 \int (1-\cos^2 2x) d \cos 2x$

$= - \frac{1}{16} (\cos 2x - \frac{1}{3} \cos^3 2x) + C = \frac{1}{48} \cos^3 2x - \frac{1}{16} \cos 2x + C.$

(4) $\int \sin 5x \sin x \, dx = \int \frac{1}{2} (\cos 4x - \cos 6x) \, dx = \frac{1}{8} \sin 4x - \frac{1}{12} \sin 6x + C.$

### 4.31 PARTIAL SOLUTIONS

(1) Let $\theta = \arctan \frac{x}{5}$, then $x = 5 \tan \theta$, $dx = 5 \sec^2 \theta \, d\theta$

$25 + x^2 = 25 \left(1 + \left(\frac{x}{5}\right)^2\right) = 25 \sec^2 \theta$
\[ \int \frac{dx}{(25+x^2)^2} = \int \frac{5 \sec^2 \theta \, d\theta}{25^2 \, \sec^2 \theta} = \frac{1}{5^3} \int \cos^2 \theta \, d\theta \]

\[ = \frac{1}{5^3} \int \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{1}{125} \left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right] + C \]

\[ = \frac{1}{250} \left[ \theta + \tan \theta \right] + C = \frac{1}{250} \left[ \arctan \frac{x}{5} + \frac{x/5}{1+(x/5)^2} \right] + C. \]

(3) Let \( \theta = \arcsin \frac{x}{4} \), then \( x = 4 \sin \theta \), \( dx = 4 \cos \theta \, d\theta \) and

\[ 10(1 - (\frac{x}{4})^2) = 16 \cos^2 \theta \]

\[ \therefore \int \frac{dx}{(18-x^2)^{3/2}} = \int \frac{4 \cos \theta \, d\theta}{4 \sin \theta} = \frac{1}{4} \int \frac{d\theta}{\cos \theta} = 4 \int \sec^2 \theta \, d\theta \]

Note that: \( \int \sec^2 \theta \, d\theta = \int \sec^2 \theta \tan \theta \, d\theta \).

(5) Let \( \theta = \arccos \left( \frac{x}{3} \right) \). Then \( x = 3 \sec \theta \), \( dx = 3 \sec \theta \tan \theta \, d\theta \) and

\[ x^2 - 9 = 0 \left( \left( \frac{x}{3} \right)^2 - 1 \right) = 9 \tan^2 \theta \]

\[ \therefore \int \frac{dx}{(x^2-9)^2} = \int \frac{3 \sec \theta \tan \theta \, d\theta}{9 \tan^2 \theta} = \frac{1}{27} \int \frac{\cos^2 \theta \, d\theta}{\sin^2 \theta} \]

\[ = \frac{1}{27} \int \cot^2 \theta \csc \theta \, d\theta. \]

Use \( 1 + \cot^2 \theta = \csc^2 \theta \).

### 4.32 PARTIAL SOLUTIONS

(1) \( \int \arccos x \, dx = x \arccos x - \int x(\arccos x)^3 \, dx \)

\[ = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arccos x - \sqrt{1-x^2} + C. \]
(3) \[ \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C. \]

(5) \[ \int \cos^2 x \, dx = \int \cos x \, d\sin x = \cos x \sin x - \int \sin x \, d\cos x \]

\[ = \cos x \sin x + \int \sin^2 x \, dx = \cos x \sin x + x - \int \cos^2 x \, dx \]

\[ \therefore \int \cos^2 x \, dx = \frac{1}{2} \left[ \cos x \sin x + x \right] + C = \frac{1}{2} x + \frac{1}{4} \sin 2x + C. \]

4.33 PARTIAL SOLUTIONS

(1) \[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{d\cos x}{\cos x} = \ln |\cos x| + C. \]

(3) \[ \int \sin^2 x \cos^2 x \, dx = \int \frac{1}{4} \sin^2 2x \cos^2 x \, dx = \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) \, dx \]

\[ = \frac{1}{16} \int \sin^2 2x \, d(\sin 2x) + \frac{1}{8} \int \frac{1 - \cos 4x}{2} \, dx \]

\[ = \frac{1}{48} \sin^3 (2x) + \frac{1}{16} \left( x - \frac{1}{4} \sin 4x \right) + C. \]

(5) \[ 2 \int \sin x \cos 2x \, dx = 2 \int (\sin 3x - \sin x) \, dx = \cos x - \frac{1}{3} \cos 3x + C. \]

4.34 PARTIAL SOLUTIONS

(1) \[ \int \frac{dx}{3 + x^2} = \frac{1}{36} \int \frac{dx}{1 + \left( \frac{x}{3} \right)^2} = \frac{1}{6} \arctan \frac{x}{3} + C \]

(3) \[ \int \frac{dx}{(16 - x^2)^{3/2}} = \int \frac{dx}{4^3 \left[ 1 - \left( \frac{x}{4} \right)^2 \right]^{3/2}} x = 4 \sin \theta \int \frac{4 \cos \theta \, d\theta}{4^3 \cos^3 \theta} = \frac{1}{16} \int \sec^2 \theta \, d\theta \]

\[ = \frac{1}{16} \tan \theta + C. \]
\( \int \frac{dx}{(x^2-10)^3} = \int \frac{dx}{16^3 \left[ \left( \frac{x}{4} \right)^2 - 1 \right]^3} = \int \frac{4 \tan^2 \sec \theta \, d\theta}{16^3 \tan^6 \theta}, \) (let \( x = 4 \sec \theta \))

\[ = \frac{1}{4 \times 16^2} \int \cot^4 \theta \csc \theta \, d\theta = \frac{1}{4 \times 16^2} \int \cot^3 \theta \, d(-\csc \theta) \]

\[ = \frac{1}{4 \times 16^2} \left[ -\cot^3 \theta \csc \theta - \int 3 \cot^2 \theta \csc^3 \theta \, d\theta \right] \]

\[ = \frac{1}{4 \times 16^2} \left[ -\cot^3 \theta \csc \theta - 3 \int \csc^5 \theta \, d\theta + \int \csc^3 \theta \, d\theta \right] \]

\[ = -\frac{1}{2} \csc \theta \cot \theta + \frac{1}{4} \ln \left( \frac{\cos \theta - 1}{\cos \theta + 1} \right) + C. \]

Note \( \int \csc^5 \theta \, d\theta = \int \csc^3 \theta (-\cot \theta) = -\csc \theta \cot \theta - \int 3 \csc^3 \theta \, d\theta \]

\[ = -\csc \theta \cot \theta - 3 \int \csc^5 \theta \, d\theta + 3 \int \csc^3 \theta \, d\theta \]

\[ \therefore \int \csc^3 \theta \, d\theta = -\frac{1}{4} \csc \theta \cot \theta + \frac{3}{4} \int \csc^3 \theta \, d\theta. \]

### 4.35 PARTIAL SOLUTIONS

1. \( \int \arccos x \, dx = x \arccos x - \int x(\arccos x)' \, dx \)

\[ = x \arccos x - \int \frac{1}{\sqrt{x^2 - 1}} \, dx \]

Use Table of Integrals or trigonometric substitution on \( \int \frac{1}{\sqrt{x^2 - 1}} \, dx. \)

2. \( \int e^\sin x \, dx = \int \sin x \, e^x \, dx = e^\sin x - \int e^\cos x \, dx \)

\[ = e^\sin x - (e^\cos x + \int \sin x \, e^x \, dx) = -\int e^\sin x \, dx + e^x(\sin x - \cos x) \]

\[ \therefore \int e^\sin x \, dx = \frac{1}{2} e^x(\sin x - \cos x) + C. \]

3. \( \int \sin^2 x \, dx = \int -\sin x \, d\cos x = -\sin x \cos x + \int \cos^2 x \, dx \)
\[ = -\sin x \cos x + \int (1-\sin^2 x) \, dx = x - \sin x \cos x - \int \sin^2 x \, dx \]

\[ \therefore \int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x) + C. \]

4.36 PARTIAL SOLUTIONS

(1) \[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{d\sin x}{\sin x} = \ln(\sin x) + C \]

(3) \[ \int \sin^4 x \cos x \, dx = \int \frac{1}{16} \sin^4 2x \, dx = \frac{1}{16} \int \left(\frac{1-\cos 2x}{2}\right)^2 \, dx \]

\[ = \frac{1}{64} \int (1-2\cos 2x + \cos^2 2x) \, dx = \frac{1}{64} \left[ x - \sin 2x + \int \frac{1+\cos 4x}{2} \, dx \right] \]

\[ = \frac{1}{64} \left[ x - \sin 2x + \frac{1}{2} x + \frac{1}{8} \sin 4x \right] + C. \]

(5) \[ \int x \sin x^2 \cos 2x^2 \, dx = \int \frac{1}{2} \sin x^2 \cos 2x^2 \, dx^2 \]

\[ = \frac{1}{4} \int (\sin 3x^2 - \sin x^2) \, dx^2 = \frac{1}{4} \cos x^2 - \frac{1}{12} \cos 3x^2 + C. \]

4.37 PARTIAL SOLUTIONS

(1) Let \( x = 8\tan \theta \), i.e., \( \theta = \arctan \frac{x}{8} \), then \( dx = 8\sec^2 \theta \, d\theta \)

\[ \therefore \int \frac{dx}{\sqrt{x^2 + 36}} = \int \frac{8\sec^2 \theta \, d\theta}{\sqrt{64 + x^2}} = \frac{1}{8} \int \cos \theta \, d\theta \]

\[ = \frac{1}{8} \int \cos^2 \theta \, d\sin \theta = \frac{1}{8} \int (1 - \sin^2 \theta) \, d\sin \theta = \frac{1}{8} \left( \sin \theta - \frac{1}{3} \sin^3 \theta \right) + C \]

\[ = \frac{1}{8} \left( \frac{x}{\sqrt{x^2 + 36}} - \frac{1}{3} \frac{x^3}{(x^2 + 36)^{3/2}} \right) + C. \]
(3) Let $x = 6\sin\theta$, then $dx = 6\cos\theta d\theta$,

$$\int \frac{dx}{(36-x^2)^{3/2}} = \int \frac{6\cos\theta d\theta}{6^3 \cos^3\theta} = \frac{1}{6^4} \int \sec^4\theta d\theta$$

Note $d\tan^3\theta = 3\tan^2\theta \sec^2\theta d\theta$

$$= 3(\sec^2\theta - \sec^2\theta) d\theta$$

$$\therefore \int \sec^4\theta d\theta = \frac{1}{3} \int d\tan^2\theta + \int \sec^2\theta d\theta = \frac{1}{3} \tan^3\theta + \tan\theta + C$$

$$\therefore \int \frac{dx}{(36-x^2)^{3/2}} = \frac{1}{6^4} \left[ \frac{1}{3} \frac{x^3}{(36-x^2)^{3/2}} + \frac{x}{(36-x^2)^{1/2}} \right] + C.$$

(5) $\int \frac{dx}{(x^2-5)^2} = \int \left[ \frac{1}{(x+\sqrt{5})(x-\sqrt{5})} \right]^2 dx = \int \left[ \frac{1}{2\sqrt{5}} \left( \frac{1}{x-\sqrt{5}} - \frac{1}{x+\sqrt{5}} \right) \right]^2 dx$

$$= \frac{1}{20} \int \left[ \frac{1}{(x-\sqrt{5})^2} - \frac{1}{(x+\sqrt{5})^2} - 2 \frac{1}{(x-\sqrt{5})(x+\sqrt{5})} \right] dx$$

$$= \frac{1}{20} \left[ -\frac{1}{x-\sqrt{5}} - \frac{1}{x+\sqrt{5}} - \sqrt{5} \ln \frac{x-\sqrt{5}}{x+\sqrt{5}} \right] + C$$

$$= \frac{-1}{20} \left[ \frac{2x}{x^2-5} + \frac{1}{\sqrt{5}} \ln \frac{x-\sqrt{5}}{x+\sqrt{5}} \right] + C.$$

4.38 HINTS

(1) Set $x = 4\tan \theta$.

(2) Use $\cos^6 x = (\cos^2 x)^3 = \frac{1}{8} (1+\cos 2x)^3$.

(3) Integrate by parts.

(4) Use $\cos^5 x \, dx = \cos^4 x \, \sin x = (1-\sin^2 x)^2 \sin x$.

(5) Use $x = 2\sec \theta$.

(6) Use $\sin x \cos 2x = \frac{1}{2} [\sin 3x - \sin x]$ or $\sin x \cos 2x \, dx = (1-\cos^2 x) \, d\cos x$.

(7) Integrate by parts twice.
(8) Integrate by parts.
(9) Use $x = \sqrt{a} \sin \theta$.
(10) Integrate by parts.
(11) Use $x = 6 \sin \theta$.
(12) Use $\sin^2 x \cos^4 x = \frac{1}{4} \sin^2 2x \cos^2 x = \frac{1}{4} \left( \frac{1 - \cos 4x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right)$ and use $\cos 4x \cdot \cos 2x = \frac{1}{2} (\cos 6x + \cos 2x)$.
(13) Use $\sin^5 x \, dx = -\sin^5 x \, d\cos x = - (1 - \cos^2 x)^2 \, d\cos x$.
(14) Integrate by parts.
(15) Use $x = 3 \sin \theta$.
(16) Use $x = a^2 \sin \theta$.
(17) Use $\sin^3 x \, dx = (\cos^3 x - 1) \, d\cos x$.
(18) Let $y = 1 - x$ and expand $x^3 = (1 - y)^3 = 1 - 3y + 3y^2 - y^3$.
(19) Integrate by parts three times.

4.39 HINTS

(1) Use $\cos^4 x = \left( \frac{1 + \cos 2x}{2} \right)^2$, and $\cos^2 2x = \frac{1 + \cos 4x}{2}$
(2) Use $x = 4 \sin \theta$.
(3) Integrate by parts.
(4) Use $y = x + 3$, and expand $x^3 = (y - 3)^2 = y^3 - 6y + 9$.
(5) Use $\sec x \, dx = \frac{1}{\cos x} \, dx = \frac{\sin x}{1 - \sin^2 x}$.
(6) Use $\cos 2x \cos 3x = \frac{1}{2} (\cos 5x + \cos x)$.
(7) Use $x = 4 \sin \theta$.
(8) Integrate by parts.
(9) Use $\sin^5 x \, dx = - (1 - \cos^2 x)^2 \, d\cos x$.
(10) Use $x = 4 \sec \theta$.
(11) Integrate by parts.
(12) First use $\cos(3e^w) \sin(5e^w) = \frac{1}{2} [\sin(8e^w) + \sin(2e^w)]$ then note $e^w \, dw = de^w$. 

□
4.42 SOLUTIONS  (without online partial fractions calculator)

(1) \[
\frac{2x^3+3}{(2x+1)^3(x^2+x+1)^3} = \frac{A_1}{2x+1} + \frac{A_2}{(2x+1)^2} + \frac{A_3}{(2x+1)^3} + \frac{A_4}{(2x+1)^4} + \frac{A_5}{(2x+1)^5} \\
+ \frac{B_1x+C_1}{x^2+x+1} + \frac{B_2x+C_2}{(x^2+x+1)^2}
\]

(2) \[
\frac{3x^3-8x^2+2x+3}{(3x-2)(x^2-2x-1)} = 1 + \frac{A_1}{3x-2} + \frac{B_1x+C_1}{x^2-2x-1}
\]

In order to get \(A_1, B_1, C_1\), multiply by \((3x-1)(x^2-2x-1)\) to get
\[
3x^3-8x^2+2x+3 = (3x-2)(x^2-2x-1) + A_1(x^2-2x-1) + (3x-2)(B_1x+C_1) = 3x^3 + (-8+A_1+3B_1)x^2+(1-2A_1+3C_1-2B_1)x+(2-A_1-2C_1).
\]

Comparing the coefficients on both sides, we get \(A_1 + 3B_1 = 0, \ 1-2A_1+3C_1-2B_1 = 2, \ 2-A_1-2C_1 = 3.\)

Solving this linear system, \(A_1 = \frac{-15}{17}, \ B_1 = \frac{5}{17}, \ C_1 = \frac{-1}{17}.\)

so \[
\frac{3x^3-8x^2+2x+3}{(3x-2)(x^2-2x-1)} = 1 - \frac{15}{17} \cdot \frac{5x}{3x-2} - \frac{1}{17} \cdot \frac{x}{x^2-2x-1}.
\]

(3) Set \(y = x^3-1.\) Then \[
\frac{x^3+1}{(x^3-1)^2} = \frac{y+2}{y^2} = \frac{1}{y} + \frac{2}{y^2} = \frac{1}{x^3-1} + \frac{2}{(x^3-1)^2}.\] Write \(x^3-1 = (x-1)(x^2+x+1).\) We get
\[
\frac{x^3+1}{(x^3-1)^2} = \frac{-1/9}{x-1} + \frac{2/9}{(x-1)^2} + \frac{1/9}{x^2+x+1} + \frac{6/9}{(x^2+x+1)^2}.
\]

\(\square\)
4.43 HINTS

(1) \[
\frac{2x^2+3}{(2x+1)^3(x^2+x-1)^2} = \frac{A_1}{2x+1} + \frac{A_2}{(2x+1)^2} + \frac{A_3}{(2x+1)^3} + \frac{A_4}{(2x+1)^4} + \frac{A_5}{(2x+1)^5}
+ \frac{B_1x+C_1}{(x^2+x-1)} + \frac{B_2x+C_2}{(x^2+x-1)^2}.
\]

(2) \[
\frac{3x^3-8x^2+2x+3}{(3x-2)^2(x^2-2x-1)} = \frac{A_1}{3x-2} + \frac{A_2}{(3x-2)^2} + \frac{B_1x+C_1}{x^2-2x-1}.
\]

(3) \[
\frac{x^2+3}{(x^4+2x^2+1)^2} = \frac{x^2+3}{(x^2+1)^4} = \frac{1}{(x^2+1)^3} + \frac{2}{(x^2+1)^4}.
\]
4.44 HINTS

(1) \[ \frac{2x^2+3x^5+2x^2+1}{(2x+1)^2(x^2+x-1)^2} = \frac{1}{2^4} + \frac{A_1}{2x+1} + \frac{A_2}{(2x+1)^2} + \frac{A_3}{(2x+1)^3} + \frac{A_4}{(2x+1)^4} + \frac{A_5}{(2x+1)}. \]

\[ + \frac{B_1x+C_1}{x^2+x-1} + \frac{B_2x+C_2}{(x^2+x-1)^2}. \]

(2) \[ \frac{3x^2-8x^3+2x+3}{(x^2+1)(x^2-2x+2)} = \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{x^2-2x+2}. \]

(3) \[ \frac{(x+1)^2}{(x^2+2x+2)^2} = \frac{B_1x+C_1}{x^2+2x+2} + \frac{B_2x+C_2}{(x^2+2x+2)^2}. \]
4.58 SOLUTIONS

(1) Solid with cross section base $3^x$, height $2^x$.

\[ V = \int_0^2 (1/2)3^x2^x = (1/2) \int_0^2 6^x = (1/2)(1/ \ln 6)6^x \bigg|_0^2 = \frac{35}{2 \ln 6}. \]

The total volume of the solid for $x < 0$ is

\[ \int_{-\infty}^0 (1/2)6^xdx = (1/2)(1/ \ln 6)[6^0 - \lim_{x \to -\infty} 6^x] = \frac{1}{2 \ln 6}. \]

(2) $V(x) = \int_0^x \pi(t^{3/2})^2 dt = [\frac{t^4}{4}]_0^x = \frac{x^4}{4}$

(3) Use method of cylinders, each with radius $y$ and length (height) $x - y^{2/3}$:

\[ V(x) = \int_0^{x^{3/2}} 2\pi y(x - y^{2/3})dy = 2\pi [x(y^{2/2}) - y^{8/3}(3/8)]^{3/2} = \frac{\pi}{4}. \]

(4) Let $y(t) = t^{3/2}$, then

\[ L(x) = \int_0^x \left(1 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} \ dt = \int_0^x \left(1 + \frac{9}{4}t\right)^{1/2} \ dt \]

We have $\int (1 + \frac{9}{4}t)^{1/2} dt = \frac{4}{9} \int (1 + \frac{9}{4}t)^{1/2} dt(1 + \frac{9}{4}t) = \frac{8}{27}(1 + \frac{9}{4}t)^{3/2}$. Thus,

\[ L(x) = \int_0^x \left(1 + \frac{9}{4}t\right)^{1/2} \ dt = \frac{8}{27}(1 + \frac{9}{4}x)^{3/2} - 1]. \]
4.58 SOLUTIONS (continued)

(5) Area $S(x) = \int_0^x 2\pi y(t)(1+(y'(t))^2)^{1/2} dt = \int_0^x 2\pi t^{3/2}(1+(9/4)t)^{1/2} dt$

$$= \frac{2\pi}{486} \left( 3x^{1/2}(9x + 4)^{1/2}(27x^2 + 3x - 2) + 8 \ln \left( \frac{1}{2} (3x^{1/2} + (9x + 4)^{1/2}) \right) \right).$$

This answer was obtained by giving an online integrator the indefinite integral $\int x^{3/2}(1 + (9/4)x)^{1/2} dx$, multiplying the answer by $2\pi$, and checking that this answer is zero when $x = 0$.

(6) Let $r(\phi) = 2^{\phi/2}$, $0 \leq \phi \leq \pi$: $r^2(\phi) = 2^\phi$, $r'(\phi)^2 = \frac{\ln^2(2)}{4} 2^\phi$.

$$L = \int_0^\pi (r^2 + r'(\phi)^2)^{1/2} d\phi = \int_0^\pi (2^\phi + \frac{\ln^2(2)}{4} 2^\phi)^{1/2} d\phi =$$

$$\left(1 + \frac{\ln^2(2)}{4}\right)^{1/2} \int_0^\pi 2^{\phi/2} d\phi = \left(1 + \frac{\ln^2(2)}{4}\right)^{1/2} 2 \frac{\ln(2)}{\ln(2)} [2^{\pi/2} - 1].$$

(7) $dS = 2\pi 2^{\phi/2} \sin(\phi) dL$ with (see 6) $dL = (1 + \frac{\ln^2(2)}{4})^{1/2} 2^{\phi/2} d\phi$. Thus,

$$S = 2\pi \left(1 + \frac{\ln^2(2)}{4}\right)^{1/2} \int_0^\pi 2^{\phi} \sin(\phi) d\phi$$

$$\int 2^{\phi} \sin(\phi) d\phi = 2^{\phi} [\ln(2) \sin(\phi) - \cos(\phi)]/(1 + \ln^2(2))$$

is gotten from an online integrator (or a double integration by parts).

(8) $V = \int_0^\pi \int_0^{r(\phi)} 2\pi r \sin(\phi) dA = \int_0^\pi \int_0^{r(\phi)} 2\pi r \sin(\phi) dr d\phi$ which becomes $\int_0^\pi 2\pi (r^3/3) \sin(\phi) d\phi$. With $r = 2^{\phi/2}$ we get

$$\int 2^{\phi} \sin(\phi) d\phi = \frac{2\pi}{3} \left( \frac{4(1 + \frac{\ln^2(2)}{4})}{4 + 9 \ln^2(2)} \right).$$

We used an online integrator.
4.59 SOLUTIONS

(1) $V = \int_1^2 2^x \log_2(x) dx$ gives volume. Graphing (online software) and estimating area under the $2^x \log_2(x)$ gives 1.7. An online integrator gives $V = \int_1^2 2^x \log_2(x) dx = (\text{li}(2) - \text{li}(4) + \ln(16))/\ln^2(2) = 1.769$. The function $\text{li}(x) := \int_0^x (1/\ln(t)) dt$ is the logarithmic integral. A calculator for this function is given online. Using integration by parts,

$$\int 2^x \log_2(x) dx = \log_2(x) \frac{2^x}{\ln(2)} - \frac{1}{\ln^2(2)} \int \frac{2^x}{x} dx.$$ 

If $u = 2^x$ then $du = \ln(2)u dx$ so $\int \frac{u}{x} dx = \int \frac{u}{\log_2(u) \ln^2(u)} du = \int \frac{du}{\ln(u)}$.

Recall, $\ln(2) \log_2(u) = \ln(u)$. This computation shows how the logarithmic integral probably arises in the online integrator.

(2) $\int_0^1 \pi \sin^2(t) dt = (\pi/2) \int_0^1 (1 - \cos(2t))/2 = \pi(x/2 - \sin(2x)/4)$.

(3) Using the terminology of Figure 4.52, we consider cylindrical shells of radius $x$, height (or length) $1 - \sin(x)$, circumference $2\pi x$ and thickness $dx$. The volume, $V = \int_0^{\pi/2} 2\pi x(1 - \sin(x)) dx = \pi/4(\pi^2 - 8)$ (using an online integrator). Checking this, $\int_0^{\pi/2} 2\pi x(1 - \sin(x)) dx = \int_0^{\pi/2} 2\pi x dx - \int_0^{\pi/2} 2\pi x \sin(x) dx = [2\pi x^2/2]_{\pi/2}^{\pi/2} - 2\pi [-x \cos(x) + \sin(x)]_{\pi/2}^{\pi/2} = \pi^2/4 - 2\pi$. The evaluation $\int x \sin(x) dx = -x \cos(x) + \sin(x)$ can be done by parts.
(4) \[ \int_0^\pi (1 + \cos^2(x))^{1/2} \, dx \approx 3.8 \] is gotten from graphing the function and estimating the area under the graph. Riemann sums give more precision. An online integrator gives

\[
\int_0^\pi (1 + \cos^2(x))^{1/2} \, dx = 2(2)^{1/2} E \left( \frac{1}{2} \right) = 3.8202.
\]

For \(0 < m < 1\), \(E(\phi|m) := \int_0^\phi (1 - m \sin^2(x))^{1/2} \, dx\), is the elliptic integral of the second kind with parameter \(m\), and \(E(m) := E(\frac{\pi}{2}|m)\) is the complete elliptic integral of the second kind. Thus, \(2^{1/2} E \left( \frac{1}{2} \right) = 2^{1/2} \int_0^{\pi/2} (1 - (1/2) \sin^2(x))^{1/2} \, dx = \int_0^{\pi/2} (1 + \cos^2(x))^{1/2} \, dx\).

(5) An online integrator gives \(\int 2\pi \sin(x)(1 + \cos^2(x))^{1/2} \, dx = (2\pi) \left( -\frac{1}{2} (1 + \cos^2(x))^{1/2} \cos(x) - \frac{1}{2} \ln((1 + \cos^2(x))^{1/2} + \cos(x)) \right) + C\).

The arclength \(L(x) = \int_0^x 2\pi \sin(t)(1 + \cos^2(t))^{1/2} \, dt\) equals the above expression with \(C = 0\).

(6) We have \(r(\phi) = \phi\) so \(L(\phi) = \int_0^\pi (r^2(\phi) + (r'(\phi))^2)^{1/2} \, d\phi = \int_0^\pi (\phi^2 + 1)^{1/2} \, d\phi\). Tables, online integrator or trig substitution plus parts gives

\[
\left. \int_0^\pi (\phi^2 + 1)^{1/2} \, d\phi \right]_0^\pi = \frac{1}{2} \left[ \phi(\phi^2 + 1)^{1/2} + \ln(\phi + (\phi^2 + 1)^{1/2}) \right]_0^\pi.
\]

(7) \(A = \int_0^\pi 2\pi r \sin(\phi)(\frac{dr}{d\phi})^2 + r^2)^{1/2} \, d\phi = \int_0^\pi 2\pi \phi \sin(\phi)(1 + \phi^2)^{1/2} \, d\phi\) when \(r(\phi) = \phi\). Graphing the integrand and computing area gives 42 and the online integral calculator gives 42.32.

(8) Compute the volume of revolution of the curve of problem (7). We have, \(V = \int_0^\phi \int_0^\phi 2\pi r \sin(\phi) r \, dr \, d\phi = (2\pi/3) \int_0^\pi \phi^3 \sin(\phi) \, d\phi = (2\pi^2/3)(\pi^2 - 6)\).

We used an online integrator.
Appendix 1

MATH TABLES

DERIVATIVES

\[
\frac{d}{dx}(uv) = \frac{du}{dx} + \frac{dv}{dx}
\]

\[
\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}
\]

\[
\frac{d}{dx}(u/v) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

\[
\frac{d}{dx}(f(u)) = \frac{df}{du} \cdot \frac{du}{dx}
\]

\[
\frac{d^2}{dx^2}(f(u)) = \frac{df}{du} \cdot \frac{du}{dx} \frac{du}{dx} + \frac{d^2f}{du^2} \cdot \frac{du}{dx} \frac{du}{dx}
\]

\[
\frac{dx^n}{dx} = nx^{n-1}
\]

\[
\frac{de^x}{dx} = e^x
\]

\[
\frac{d}{dx}(a^x \cdot u \cdot \log_a x) = a^x \cdot \frac{du}{dx} \cdot \log_a x
\]

\[
\frac{dx^x}{dx} = x^x(1 + \log x).
\]

\[
\frac{d}{dx}(\log_a x) = \frac{\log_a e}{x \cdot \log_a e}.
\]

\[
\frac{d}{dx} \sin x = \cos x.
\]

\[
\frac{d}{dx} \cos x = -\sin x.
\]

\[
\frac{d}{dx} \tan x = \sec^2 x.
\]

\[
\frac{d}{dx} \cot x = -\csc^2 x.
\]

\[
\frac{d}{dx} \sec x = \tan x \cdot \sec x.
\]

\[
\frac{d}{dx} \csc x = -\cot x \cdot \csc x.
\]

\[
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.
\]
\[ \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \]
\[ \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \]
\[ \frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \]
\[ \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \]
\[ \frac{d}{dx} \sinh x = \cosh x \]
\[ \frac{d}{dx} \cosh x = \sinh x \]
\[ \frac{d}{dx} \tanh x = \text{sech}^2 x \]
\[ \frac{d}{dx} \text{ctnh} x = -\text{csch}^2 x \]
\[ \frac{d}{dx} \text{sech} x = -\text{tanh} x \]

**SERIES AND PRODUCTS**

[The expression in brackets attached to an infinite series shows values of the variable which lie within the interval of convergence. If a series is convergent for all finite values of \(x\), the expression \([x^2 < \infty]\) is used.]

\( (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n!}{(n-k)!k!}a^{n-k}b^k + \cdots \) \([b^2 < a^2]\)

\( (a - bx)^{-1} = a \left[ 1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \cdots \right] \) \([b^2x^2 < a^2]\)

\( (1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \)

\([x^2 < 1]\)
\[(1 \pm x)^{-n} = 1 = nx + \frac{n(n + 1)}{2!} x^2 + \frac{n(n + 1)(n + 2)}{3!} x^3 + \cdots (n - k - 1)! (n - k)! k! \cdot x^k + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)^{\frac{1}{2}} = 1 = \frac{1}{2} x + \frac{1 \cdot 3}{2 \cdot 4} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)^{-1} = 1 = \frac{1}{1} x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)\frac{1}{2} = 1 = \frac{1}{2} x + \frac{1 \cdot 2}{3 \cdot 6} x^2 + \frac{1 \cdot 2 \cdot 5}{3 \cdot 6 \cdot 9} x^3 + \cdots. \quad [x^3 < 1.0]
\]

**SERIES**

\[(1 \pm x)^{-\frac{1}{2}} = 1 = \frac{1}{1} x + \frac{1 \cdot 4}{3 \cdot 6} x^2 + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} x^3 + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)^{-1} = 1 = \frac{1}{1} x + \frac{1}{2 \cdot 4} x^2 + \frac{1 \cdot 3 x^3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)^{-1} = 1 = \frac{1}{1} x + \frac{1}{2 \cdot 4} x^2 + \frac{1 \cdot 3 x^3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots. \quad [x^3 < 1.0]
\]
\[(1 \pm x)^{-1} = 1 = \frac{1}{1} x + \frac{1}{2 \cdot 4} x^2 + \frac{3 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 + \frac{3 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 + \cdots. \quad [x^3 < 1.0]
\]

\[(1 \pm x)^{-1} = 1 = \frac{1}{1} x + \frac{3 \cdot 5}{2 \cdot 4} x^2 + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} x^3 + \cdots. \quad [x^3 < 1.0]
\]
\((1 \pm x)^{-s} = 1 = 2x + 3x^2 = 4x^3 + 5x^4 = 6x^5 + \cdots. \quad [x^5 < 1.]
\)

\(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \quad [x^3 < \infty.]\)

\(a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots. \quad [x^3 < \infty.]\)

\(\frac{1}{2} (e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \quad [x^3 < \infty.]\)

\(\frac{1}{2} (e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \quad [x^3 < \infty.]\)

\(e^{-x} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots. \quad [x^3 < \infty.]\)

\[\log x = 2 \left[ \frac{x - 1}{x + 1} + \frac{1}{2} \left( \frac{x - 1}{x + 1} \right)^2 + \frac{1}{3} \left( \frac{x - 1}{x + 1} \right)^3 + \cdots \right]. \quad [x > 0.]
\]

\[\log (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots. \quad [x^3 < 1.]
\]

\[\log \left( \frac{1 + x}{1 - x} \right) = 2 \left[ x + \frac{1}{2} x^3 + \frac{1}{4} x^5 + \frac{1}{6} x^7 + \cdots \right]. \quad [x^3 < 1.]
\]

\[\log \left( \frac{x + 1}{x - 1} \right) = 2 \left[ \frac{1}{x} + \frac{1}{2} \left( \frac{1}{x} \right)^3 + \frac{1}{3} \left( \frac{1}{x} \right)^5 + \cdots \right]. \quad [x^3 > 1.]
\]

\[\log (x + \sqrt{1 + x^2}) = x - \frac{x^3}{6} + \frac{1 \cdot 3 x^4}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots. \quad [x^3 < 1.]
\]

Series for denary and other logarithms can be obtained from the foregoing developments by aid of the equations,

\[\log_x a = \log x \cdot \log_x e, \quad \log_x e = \log x \cdot \log e,
\]

\[\log_x (-x) = (2n + 1) \pi i + \log x.
\]

\[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \quad [x^7 < \infty.]\]

\[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \text{versin } x. \quad [x^6 < \infty.]
\]

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TABLE OF INTEGRALS

Fundamental Forms

\[ \int a \, dx = ax. \]
\[ \int a f(x) \, dx = a \int f(x) \, dx. \]
\[ \int \frac{dx}{x} = \log x. \quad [\log x = \log(-x) + (2k + 1)i] \]
\[ \int x^n \, dx = \frac{x^{n+1}}{m+1}, \text{ when } m \text{ is different from } -1. \]
\[ \int e^x \, dx = e^x. \]
\[ \int a^x \log a \, dx = a^x. \]
\[ \int \frac{dx}{1 + x^2} = \tan^{-1} x, \text{ or } -\cot^{-1} x. \]
\[ \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x, \text{ or } -\cos^{-1} x \]
\[ \int \frac{dx}{x \sqrt{x^2 - 1}} = \sec^{-1} x, \text{ or } -\csc^{-1} x. \]
\[ \int \frac{dx}{\sqrt{2x - x^2}} = \text{versin}^{-1} x, \text{ or } -\text{coversin}^{-1} x. \]
\[ \int \cos x \, dx = \sin x, \text{ or } -\text{coversin} x. \]
\[ \int \sin x \, dx = -\cos x, \text{ or } \text{versin} x. \]
\[ \int \cot x \, dx = \log \sin x. \]
\[ \int \tan x \, dx = -\log \cos x. \]
\[ \int \tan x \sec x \, dx = \sec x. \]
\[ \int \sec^4 x \, dx = \tan x. \]
\[ \int \csc^4 x \, dx = -\cot x. \]

In the following formulas, \( u, v, w, \) and \( y \) represent any functions of \( x \):

\[ \int (u + v + w + \text{etc.}) \, dx = \int u \, dx + \int v \, dx + \int w \, dx + \text{etc.} \]
\[ \int u \, dv = uv - \int v \, du. \]
\[ \int \frac{u}{dx} \, dx = uv - \int \frac{v}{dx} \, dx. \]
\[ \int f(y) \, dx = \int \frac{f(y)}{dy} \, \frac{dy}{dx}. \]

**Rational Algebraic Functions**

**Expressions Involving \( (a + bx) \).**

The substitution of \( y \) or \( z \) for \( x \), where \( y \equiv a + bx, \)
\( z \equiv (a + bx) / x, \) gives

\[ \int (a + bx)^n \, dx = \frac{1}{b} \int y^n \, dy. \]
\[ \int x (a + bx)^n \, dx = \frac{1}{b} \int y^n (y - a) \, dy. \]
\[ \int x^n (a + bx)^n \, dx = \frac{1}{b^{n+1}} \int y^n (y - a)^n \, dy. \]
\[ \int x^n \frac{dx}{(a + bx)^n} = \frac{1}{b^{n+1}} \int \frac{(y - a)^n}{y^n} \, dy. \]
\[ \int \frac{dx}{x^n (a + bx)^n} = -\frac{1}{a^{n+1}} \int \frac{(z - b)^{n+1-n}}{z^n} \, dz. \]

Whence

\[ \int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx). \]
\[ \int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}. \]

\[ \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}. \]

\[ \int \frac{x\,dx}{a + bx} = \frac{1}{b^2}[a + bx - a\,\log(a + bx)]. \]

\[ \int \frac{x\,dx}{(a + bx)^2} = \frac{1}{b^3}\left[\log(a + bx) + \frac{a}{a + bx}\right]. \]

**Expressions Involving \((a + bx^n)\).**

\[ \int \frac{dx}{x^2 + c^2} = \frac{1}{c}\tan^{-1}\left(\frac{x}{c}\right) = \frac{1}{c}\sin^{-1}\left(\frac{x}{\sqrt{x^2 + c^2}}\right). \]

\[ \int \frac{dx}{x^2 - c^2} = \frac{1}{2c}\log\frac{c + x}{c - x} \quad \text{or} \quad \int \frac{dx}{x^2 - c^2} = \frac{1}{2c}\log\frac{x - c}{x + c}. \]

\[ \int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\frac{x}{\sqrt{b/a}}\right) \quad \text{or} \quad \int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{-ab}}\tanh^{-1}\left(\frac{x}{\sqrt{-b/a}}\right). \]

\[ \int \frac{dx}{a + bx^n} = \frac{1}{\sqrt[2n]{a^n - b^n}}\log\frac{a + x\sqrt[n]{b^n}}{a - x\sqrt[n]{b^n}}, \text{ if } a > 0, b < 0. \]

\[ \int \frac{dx}{x^2 + a} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}. \]

\[ \int \frac{dx}{(a + bx^n)^{m+1}} = \frac{x}{2ma(a + bx^m)^m} + \frac{m - 1}{2ma} \int \frac{dx}{(a + bx^n)^m}. \]

\[ \int \frac{x\,dx}{a + bx^2} = \frac{1}{b}\log\left(x^2 + \frac{a}{b}\right). \]

\[ \int \frac{x\,dx}{(a + bx^2)^{m+1}}, \text{ where } z = ax. \]

\[ \int \frac{dx}{x(a + bx^2)} = \frac{1}{2a}\log\frac{x^2}{a + bx^2}. \]

\[ \int \frac{x^2\,dx}{a + bx^2} = \frac{1}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}. \]

\[ \int \frac{dx}{x^2(a + bx^2)} = \frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}. \]
\[
\int \frac{x^3}{(a + bx^3)^{m+1}} = -\frac{x}{2mb(a + bx^m)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^m)^m}.
\]
\[
\int \frac{dx}{x^2(a + bx^3)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^m)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^m)^{m+1}}.
\]
\[
\int_{a^\gamma}^{b^\gamma} \frac{dx}{c^\gamma - x^\gamma} = \frac{1}{c} \tanh^{-1} \left( \frac{c}{b} \right); \int_{a^\gamma}^{b^\gamma} \frac{dx}{x^\gamma} = -\frac{1}{c} \text{th}^{-1} \left( \frac{c}{b} \right).
\]

**Expressions Involving** \((a + bx + cx^3)\).

Let \(X = a + bx + cx^3\) and \(q = 4ac - b^3\), then
\[
\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \log \frac{2x + b - \sqrt{-q}}{2x + b + \sqrt{-q}}, \text{ or } -\frac{2}{\sqrt{-q}} \tanh^{-1} \frac{2x + b}{\sqrt{-q}}.
\]
\[
\int \frac{dx}{X^3} = \frac{2x + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}.
\]
\[
\int \frac{dx}{X^4} = \frac{2x + b}{q} \left( \frac{1}{2X^3} + \frac{3c}{qX} \right) + \frac{6c}{q^2} \int \frac{dx}{X}.
\]
\[
\int \frac{dx}{X^{n+1}} = \frac{2x + b}{nqX^n} + \frac{2(2n - 1)c}{qn} \int \frac{dx}{X^n}.
\]
\[
\int \frac{dx}{X^2} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.
\]
\[
\int \frac{dx}{X^3} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.
\]
\[
\int \frac{dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n - 1)}{nq} \int \frac{dx}{X^n}.
\]
\[
\int \frac{x^2}{X} dx = \frac{x^2 - b^2}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}.
\]
\[
\int \frac{x^2}{X^2} dx = \frac{(b^2 - 2ac)x + ab}{eqX} + \frac{2a}{q} \int \frac{dx}{X}.
\]
\[
\int \frac{x^n}{X^{n+1}} = \frac{x^{n-1}}{(2n - m + 1)cX^n} - \frac{n - m + 1}{2n - m + 1} \frac{b}{c} \int \frac{x^{n-1} dx}{X^{n+1}} + \frac{m - 1}{2n - m + 1} \frac{a}{c} \int \frac{x^{n-2} dx}{X^{n+1}}.
\]
Irrational Algebraic Functions

Expressions Involving $\sqrt{a + bx}$.

The substitution of a new variable of integration, $y = \sqrt{a + bx}$, gives

$$\int \sqrt{a + bx} \, dx = \frac{2}{3b} \sqrt{(a + bx)^3}.$$

$$\int x \sqrt{a + bx} \, dx = \frac{-2(2a - 3bx)}{15b^3} \sqrt{(a + bx)^3}.$$

$$\int x^2 \sqrt{a + bx} \, dx = \frac{2(8a^2 - 12abx + 15b^2x^2)}{105b^3} \sqrt{(a + bx)^3}.$$

$$\int \frac{\sqrt{a + bx}}{x} \, dx = 2\sqrt{a + bx} + a \int \frac{dx}{x \sqrt{a + bx}}.$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}.$$

$$\int \frac{x \, dx}{\sqrt{a + bx}} = \frac{-2(2a - bx)}{3b^3} \sqrt{a + bx}.$$

$$\int \frac{x^2 \, dx}{\sqrt{a + bx}} = \frac{2(8a^2 - 4abx + 3b^2x^2)}{15b^3} \sqrt{a + bx}.$$

$$\int \frac{dx}{x \sqrt{a + bx}} = \frac{1}{\sqrt{a}} \log \left( \frac{\sqrt{a + bx} - a}{\sqrt{a + bx} + a} \right), \text{ for } a > 0.$$

$$\int \frac{dx}{x \sqrt{a + bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a + bx}{a - a}}, \text{ or } \frac{-2}{\sqrt{a}} \tanh^{-1} \sqrt{\frac{a + bx}{a}}.$$

$$\int \frac{x^m \, dx}{\sqrt{a + bx}} = \frac{2x^m \sqrt{a + bx}}{(2m + 1)b} - \frac{2ma}{(2m + 1)b} \int \frac{x^{m-1} \, dx}{\sqrt{a + bx}}.$$

$$\int \frac{dx}{x^n \sqrt{a + bx}} = -\frac{\sqrt{a + bx}}{(n-1)a x^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1} \sqrt{a + bx}}.$$
Expressions Involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$.

\[
\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{2} \left[ x \sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2}) \right].
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).
\]

\[
\int \frac{dx}{x \sqrt{a^2 - x^2}} = \frac{1}{a} \cos^{-1} \frac{x}{a}, \text{ or } -\frac{1}{a} \sec^{-1} \frac{x}{a}.
\]

\[
\int \frac{dx}{x \sqrt{x^2 \pm a^2}} = -\frac{1}{a} \log \left( \frac{a + \sqrt{a^2 \pm x^2}}{x} \right).
\]

\[
\frac{\sqrt{a^2 \pm x^2}}{x} dx = \sqrt{a^2 \pm x^2} - a \log\left( x + \sqrt{a^2 \pm x^2} \right).
\]

\[
\frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.
\]

\[
\frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2}.
\]

\[
\frac{x dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.
\]

\[
\frac{x \sqrt{a^2 - x^2}}{x} dx = \frac{1}{3} \sqrt{(a^2 - x^2)^3}.
\]

\[
\frac{x \sqrt{a^2 - x^2}}{x} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}.
\]

\[
\begin{align*}
\log \left( \frac{x + \sqrt{x^2 + a^2}}{a} \right) &= \sinh^{-1} \left( \frac{x}{a} \right); \\
\log \left( \frac{x + \sqrt{a^2 - x^2}}{a} \right) &= \cosh^{-1} \left( \frac{x}{a} \right); \\
\log \left( \frac{a + \sqrt{a^2 - x^2}}{x} \right) &= \sech^{-1} \left( \frac{x}{a} \right); \\
\log \left( \frac{a + \sqrt{a^2 + x^2}}{x} \right) &= \csch^{-1} \left( \frac{x}{a} \right).
\end{align*}
\]

\[
\int \sqrt{(x^2 + a^2)^3} dx = \frac{1}{3} \left[ x \sqrt{(x^2 + a^2)^3} \pm \frac{3a^2 x^2}{2} \sqrt{x^2 \pm a^2} \pm \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right].
\]
\[ \int \frac{dx}{x^3 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2} a^3 \sec^{-1} \left( \frac{x}{a} \right). \]

\[ \int \frac{x^3 dx}{\sqrt{x^3 \pm a^3}} = \frac{x}{2} \sqrt{x^3 \pm a^3} + \frac{a^3}{2} \log (x + \sqrt{x^3 \pm a^3}). \]

\[ \int \frac{x^3 dx}{\sqrt{a^3 - x^3}} = - \frac{\sqrt{a^3 - x^3}}{a^3x}. \]

\[ \int \frac{dx}{x^4 \sqrt{x^2 \pm a^2}} = \frac{\sqrt{x^2 \pm a^2}}{a^2x}. \]

\[ \int \frac{dx}{x^3 \sqrt{a^3 - x^3}} = - \frac{\sqrt{a^3 - x^3}}{a^3x}. \]

\[ \int \frac{\sqrt{x^3 \pm a^3} dx}{x^4} = - \frac{\sqrt{x^3 \pm a^3}}{x} + \log (x + \sqrt{x^3 \pm a^3}). \]

\[ \int \frac{\sqrt{a^3 - x^3} dx}{x^4} = - \frac{\sqrt{a^3 - x^3}}{x} - \sin^{-1} \frac{x}{a}. \]

\[ \int \frac{x^3 dx}{\sqrt{(x^3 \pm a^3)^3}} = \frac{-x}{\sqrt{x^3 \pm a^3}} + \log (x + \sqrt{x^3 \pm a^3}). \]

\[ \int \frac{x^3 dx}{\sqrt{(a^3 - x^3)^3}} = \frac{x}{\sqrt{a^3 - x^3}} - \sin^{-1} \frac{x}{a}. \]

**Expressions Involving $\sqrt{a + bx + cx^3}$**

Let $X = a + bx + cx^3$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a + bx + cx^3})$ we may put $\sqrt{a + bx + cx^3} = \sqrt{\pm c} \sqrt{X + Bx \pm x^3}$, according as $c$ is positive or negative, and then substitute for $x$ a new variable $z$, such that

\[ z = \sqrt{A + Bx + x^3 \pm x}, \text{ if } c > 0. \]

\[ z = \frac{\sqrt{A + Bx - x^3} - \sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0. \]

\[ z = \frac{\sqrt{x - \beta}}{a - x}, \text{ where } \alpha \text{ and } \beta \text{ are the roots of the equation } \]

\[ A + Bx - x^3 = 0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0. \]
By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

\[
\int \frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{c}} \log \left( \sqrt{x} + x \sqrt{c} + \frac{b}{2\sqrt{c}} \right), \text{ if } c > 0.
\]

\[
\int \frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \left( \frac{2cx + b}{\sqrt{-q}} \right), \text{ or } \frac{1}{\sqrt{c}} \sinh^{-1} \left( \frac{2cx + b}{\sqrt{q}} \right).
\]

\[
\int \frac{dx}{x^4 + a^4} = \frac{1}{4a^2 \sqrt{2}} \left\{ \log \left( \frac{x^4 + ax \sqrt{2} + a^2}{x^4 - ax \sqrt{2} + a^2} \right) + 2 \tan^{-1} \left( \frac{ax \sqrt{2}}{a^2 - x^2} \right) \right\}.
\]

\[
\int \frac{dx}{x^4 - a^4} = \frac{1}{4a^2} \left\{ \log \left( \frac{x - a}{x + a} \right) - 2 \tan^{-1} \left( \frac{x}{a} \right) \right\}.
\]

**Transcendental Functions**

\[
\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin x + \frac{1}{n} x = \frac{1}{n} x - \frac{1}{n} \sin 2x.
\]

\[
\int \sin^3 x \, dx = -\frac{1}{3} \cos x (\sin^2 x + 2).
\]

\[
\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.
\]

\[
\int \cos x \, dx = \sin x.
\]

\[
\int \cos^4 x \, dx = \frac{1}{4} \sin x \cos x + \frac{1}{4} x = \frac{1}{4} x + \frac{1}{4} \sin 2 x.
\]

\[
\int \cos^3 x \, dx = \frac{1}{3} \sin x (\cos^2 x + 2).
\]

\[
\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.
\]

\[
\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x.
\]

\[
\int \sin^4 x \cos^3 x \, dx = -\frac{1}{4} (\frac{1}{4} \sin 4x - x).
\]

\[
\int \sin x \cos^m x \, dx = -\frac{\cos^{m+1} x}{m+1}.
\]
\[
\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x}{m+1}. \\
\int \cos^m x \sin^n x \, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} \\
+ \frac{m-1}{m+n} \int \cos^m x \sin^n x \, dx. \\
\int \cos^m x \sin^n x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} \\
+ \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx.
\]

\[
\int \frac{dx}{\cos^a x} = \frac{1}{n-1} \sin \frac{x}{n-1} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.
\]

\[
\int \tan x \, dx = -\log \cos x.
\]

\[
\int \tan^2 x \, dx = \tan x - x.
\]

\[
\int \tan^4 x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.
\]

\[
\int \cot x \, dx = \log \sin x.
\]

\[
\int \cot^2 x \, dx = -\cot x - x.
\]

\[
\int \cot^4 x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx.
\]

\[
\int \sec x \, dx = \log \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) = \frac{1}{2} \log \frac{1 + \sin x}{1 - \sin x}.
\]

\[
\int \sec^2 x \, dx = \tan x.
\]

\[
\int \sec^n x \, dx = \int \frac{dx}{\cos^a x} = \frac{\sin x}{(n-1) \cos^{a-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{a-2} x}
\]

\[
= \frac{\sin x}{(n-1) \cos^{a-1} x} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.
\]
\[
\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx.
\]
\[
\int x^n \sin x \, dx = -\frac{x^n \cos x}{m - 1} + m - 1 \int x^{m-1} \sin x \, dx.
\]
\[
\int x^n \cos x \, dx = -\frac{\cos x}{m - 1} + m - 1 \int x^{m-1} \cos x \, dx.
\]
\[
\int x^n \sin x \, dx = -\frac{x^n \cos x}{m - 1} + m - 1 \int x^{m-1} \sin x \, dx.
\]
\[
\int x^n \cos x \, dx = -\frac{x^n \cos x}{m - 1} + m - 1 \int x^{m-1} \sin x \, dx.
\]
\[
\int x^n \sin x \, dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots.
\]
\[
\int x^n \sin x \, dx = \log x - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \frac{x^8}{8 \cdot 8!} + \cdots.
\]
\[
\int x^n \sin x \, dx = x + \frac{x^3}{3 \cdot 3!} + \frac{7 x^5}{3 \cdot 5!} + \frac{31 x^7}{3 \cdot 7!} + \frac{127 x^9}{3 \cdot 9!} + \cdots.
\]
\[
\int x^n \sin x \, dx = x - \frac{x^3}{2 \cdot 4!} + \frac{5 x^5}{4 \cdot 6!} - \frac{61 x^7}{8 \cdot 6!} + \frac{1385 x^9}{10 \cdot 8!} + \cdots.
\]
\[
\int x^n \sin x \, dx = -x \cot x + \log x.
\]
\[
\int x^n \sin x \, dx = x \tan x + \log \cos x.
\]
\[
\int x^n \sin^a x \, dx = x^{m-1} \sin^a x (m \sin x - nx \cos x) + n (n - 1) \int x^{m-1} \sin^{a-2} x dx - m (m - 1) \int x^{m-1} \sin^a x dx.
\]
\[
\int x^n \sin x \, dx = x^{m-1} \cos^a x (m \cos x + nx \sin x) + n (n - 1) \int x^{m-1} \cos^{a-2} x dx - m (m - 1) \int x^{m-1} \cos^a x dx.
\]
\[
\int \sin^n x \, dx = \frac{1}{n - m} \left( -\frac{\sin^{n-1} x}{\cos^{m-1} x} + (n - 1) \int \sin^{n-2} x \, dx \right)
\]
\[
\int \sin^n x \, dx = \frac{1}{m - 1} \left( \frac{\sin^{n+1} x}{\cos^{m-1} x} - (n - m + 2) \int \sin^n x \, dx \right)
\]
\[
\int \sin^n x \, dx = \frac{1}{m - 1} \left( \frac{\sin^{n-1} x}{\cos^{m-1} x} - (n - 1) \int \sin^{n-2} x \, dx \right).
\]
\[ \int \sin mx \sin nx \, dx = \frac{\sin (m - n)x}{2(m - n)} - \frac{\sin (m + n)x}{2(m + n)}. \]
\[ \int \sin mx \cos nx \, dx = -\frac{\cos (m - n)x}{2(m - n)} - \frac{\cos (m + n)x}{2(m + n)}. \]
\[ \int \cos mx \cos nx \, dx = \frac{\sin (m - n)x}{2(m - n)} + \frac{\sin (m + n)x}{2(m + n)}. \]
\[ \int \sin^2 mx \, dx = \frac{1}{2m} (mx - \sin mx \cos mx). \]
\[ \int \cos^2 mx \, dx = \frac{1}{2m} (mx + \sin mx \cos mx). \]
\[ \int \sin mx \cos mx \, dx = -\frac{1}{4m} \cos 2mx. \]
\[ \int \sin nx \sin^n x \, dx = \frac{1}{m + n} \left[ -\cos nx \sin^n x 
+ m \int \cos (n - 1)x \cdot \sin^{n-1} x \, dx \right]. \]
\[ \int \sec^{-1} x \, dx = x \sec^{-1} x - \log (x + \sqrt{x^2 - 1}). \]
\[ \int \csc^{-1} x \, dx = x \csc^{-1} x + \log (x + \sqrt{x^2 - 1}). \]
\[ \int \versin^{-1} x \, dx = (x - 1) \versin^{-1} x + \sqrt{2} x - x^2. \]
\[ \int (\sin^{-1} x)^2 \, dx = x (\sin^{-1} x)^2 - 2x + 2 \sqrt{1 - x^2} \sin^{-1} x. \]
\[ \int (\cos^{-1} x)^2 \, dx = x (\cos^{-1} x)^2 - 2x - 2 \sqrt{1 - x^2} \cos^{-1} x. \]
\[ \int x \sin^{-1} x \, dx = \frac{1}{4} [(2x^2 - 1) \sin^{-1} x + x \sqrt{1 - x^2}]. \]
\[ \int x \cos^{-1} x \, dx = \frac{1}{4} [(2x^2 - 1) \cos^{-1} x - x \sqrt{1 - x^2}]. \]
\[ \int x \tan^{-1} x \, dx = \frac{1}{4} [(x^2 + 1) \tan^{-1} x - x]. \]
\[ \int \frac{x^m}{\log x} \, dx = \int \frac{e^{-y}}{y} \, dy, \text{ where } y = -(m + 1) \log x. \]

**MISCELLANEOUS DEFINITE INTEGRALS**

\[ \int_{a}^{\infty} \frac{a}{a^2 + x^2} \, dx = \frac{\pi}{2}, \text{ if } a > 0; \quad 0, \text{ if } a = 0; \quad -\frac{\pi}{2}, \text{ if } a < 0. \]

\[ \int_{0}^{\infty} x^{n-1} e^{-x} \, dx = \int_{0}^{1} \left[ \log \frac{1}{x} \right]^{n-1} \, dx \equiv \Gamma(n). \]

\[ \Gamma(s + 1) = z \cdot \Gamma(z), \text{ if } z > 0. \]

\[ \Gamma(y) \cdot \Gamma(1 - y) = \frac{\pi}{\sin \pi y}, \text{ if } 1 > y > 0. \quad \Gamma(2) = \Gamma(1) = 1. \]

\[ \Gamma(n + 1) = n!, \text{ if } n \text{ is an integer.} \quad \Gamma(z) = \Pi(z - 1). \]

\[ \Gamma(\frac{1}{2}) = \sqrt{\pi}. \quad \Gamma(y) = \mathcal{L}[\log \Gamma(y)]. \quad \Gamma(1) = -0.577216.\]

\[ \int_{0}^{1} x^{n-1} (1 - x)^{n-1} \, dx = \int_{0}^{1} \frac{x^{n-1} \, dx}{(1 + x)^{n+1}} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m + n)}. \]

\[ \int_{0}^{\frac{\pi}{2}} \sin^n x \, dx = \int_{0}^{\frac{\pi}{2}} \cos^n x \, dx \]

\[ = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots (n)} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer,} \]

\[ = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}, \text{ if } n \text{ is an odd integer,} \]

\[ = \frac{1}{4} \sqrt{\pi} \frac{\Gamma \left( n + \frac{1}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)}, \text{ for any value of } n \text{ greater than } -1. \]

\[ \int_{0}^{\pi} \sin mx \, dx = \frac{\pi}{2}, \text{ if } m > 0; \quad 0, \text{ if } m = 0; \quad -\frac{\pi}{2}, \text{ if } m < 0. \]

\[ \int_{0}^{1} \frac{1 + x}{1 - x} \cdot \frac{dx}{x} = \frac{\pi^2}{4}. \]

\[ \int_{0}^{1} \log \frac{x}{1 - x^2} \, dx = -\frac{\pi}{2} \log 2. \]
\[
\int_0^\infty e^{-x^2} \, dx = \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right).
\]
\[
\int_0^\infty x^n e^{-ax} \, dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.
\]
\[
\int_0^\infty x^2 e^{-ax^2} \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}.
\]
\[
\int_0^\infty e^{-x^2} \, dx = \frac{e^{\frac{a^2}{2}} \sqrt{\pi}}{2}.
\]
\[
\int_0^\infty e^{-x^2} \sqrt{x} \, dx = \frac{1}{2n} \sqrt{\frac{\pi}{n}}.
\]
\[
\int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{n}}.
\]

**TRIGONOMETRIC FUNCTIONS**

<table>
<thead>
<tr>
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<th>30°</th>
<th>45°</th>
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<td>1</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
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<tr>
<td>cos</td>
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<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(\frac{\sqrt{3}}{-2})</td>
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<td>(\frac{-\sqrt{3}}{2})</td>
<td>(\frac{-1}{2})</td>
</tr>
<tr>
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<td>(-\sqrt{3})</td>
<td>(-\frac{1}{\sqrt{2}})</td>
<td>(-\sqrt{3})</td>
<td>(\infty)</td>
</tr>
<tr>
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<td>(\infty)</td>
<td>(\sqrt{3})</td>
<td>1</td>
<td>(\frac{1}{\sqrt{3}})</td>
<td>0</td>
<td>(\frac{-1}{\sqrt{3}})</td>
<td>(-\frac{1}{\sqrt{2}})</td>
<td>(-\sqrt{3})</td>
<td>(\infty)</td>
</tr>
<tr>
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<td>(\sqrt{2})</td>
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<td>(\infty)</td>
<td>(-2)</td>
<td>(-\sqrt{2})</td>
<td>(\frac{2}{\sqrt{3}})</td>
<td>(-1)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>csc</td>
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<td>(\frac{\sqrt{2}}{\sqrt{3}})</td>
<td>(\frac{2}{\sqrt{3}})</td>
<td>(\frac{\sqrt{2}}{\sqrt{3}})</td>
<td>(\frac{1}{\sqrt{3}})</td>
<td>(\frac{2}{\sqrt{3}})</td>
<td>(\frac{2}{\sqrt{3}})</td>
<td>(\frac{\sqrt{2}}{\sqrt{3}})</td>
</tr>
</tbody>
</table>

\[
\sin \frac{a}{2} = \sqrt{\frac{1}{2} (1 - \cos a)}.
\]
\[
\cos \frac{a}{2} = \sqrt{\frac{1}{2} (1 + \cos a)}.
\]
\[
\tan \frac{a}{4} = \sqrt{\frac{1 - \cos a}{1 + \cos a}} = \frac{1 - \cos a}{\sin a} = \frac{\sin a}{1 + \cos a}.
\]
\[
\sin 2a = 2 \sin a \cos a.
\]
\[
\sin 3a = 3 \sin a - 4 \sin^3 a.
\]
\[
\sin 4a = 8 \cos^4 a \cdot \sin a - 4 \cos a \cdot \sin a.
\]
\[ \sin 5a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a. \]

\[ \sin 6a = 32 \cos^2 a \sin a - 32 \cos^4 a \sin a + 6 \cos a \sin a. \]

\[ \cos 2a = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1. \]

\[ \cos 3a = 4 \cos^3 a - 3 \cos a. \]

\[ \cos 4a = 8 \cos^4 a - 8 \cos^2 a + 1. \]

\[ \cos 5a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a. \]

\[ \cos 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1. \]

\[ \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}. \]

\[ \operatorname{ctn} 2a = \frac{\operatorname{ctn}^2 a - 1}{2 \operatorname{ctn} a}. \]

\[ \sin (a \pm \beta) = \sin a \cdot \cos \beta \pm \cos a \cdot \sin \beta. \]

\[ \cos (a \pm \beta) = \cos a \cdot \cos \beta \mp \sin a \cdot \sin \beta. \]

\[ \tan (a \pm \beta) = \frac{\tan a \pm \tan \beta}{1 \mp \tan a \cdot \tan \beta}. \]

\[ \operatorname{ctn} (a \pm \beta) = \frac{\operatorname{ctn} a \cdot \operatorname{ctn} \beta \mp 1}{\operatorname{ctn} a \pm \operatorname{ctn} \beta}. \]

\[ \sin a \pm \sin \beta = 2 \sin \frac{1}{2} (a \pm \beta) \cdot \cos \frac{1}{2} (a \mp \beta). \]

\[ \cos a \pm \cos \beta = 2 \cos \frac{1}{2} (a + \beta) \cdot \cos \frac{1}{2} (a - \beta). \]

\[ \cos a - \cos \beta = - 2 \sin \frac{1}{2} (a + \beta) \cdot \sin \frac{1}{2} (a - \beta). \]

\[ \tan a \pm \tan \beta = \frac{\sin (a \pm \beta)}{\cos a \cdot \cos \beta}. \]

\[ \operatorname{ctn} a \pm \operatorname{ctn} \beta = \pm \frac{\sin (a \pm \beta)}{\sin a \cdot \sin \beta}. \]

\[ \sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)]. \]

\[ \cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)]. \]

\[ \sin \alpha \sin \beta = \frac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)]. \]
HYPERBOLIC FUNCTIONS

\[
\sinh x = \frac{1}{2} (e^x - e^{-x}) = -\sinh (-x) = -i \sin (ix) \\
= (\cosh x)^{-1} = 2 \tanh \frac{1}{2} x + (1 - \tanh^2 \frac{1}{2} x).
\]

\[
cosh x = \frac{1}{2} (e^x + e^{-x}) = \cosh (-x) = \cos (ix) = (\text{sech } x)^{-1} \\
= (1 + \tanh^2 \frac{1}{2} x) + (1 - \tanh^2 \frac{1}{2} x).
\]

\[
tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})} = -\tanh (-x) \\
= -i \tan (ix) = (\text{ctanh } x)^{-1} = \sinh x + \cosh x.
\]

\[
cosh x i = \cos x.
\]

\[
sinh x i = i \sin x.
\]

\[
cosh^2 x - \sinh^2 x = 1.
\]

\[
1 - \tanh^2 x = \text{sech}^2 x.
\]

\[
1 - \text{ctanh}^2 x = -\csc^2 x.
\]

\[
\sinh (x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.
\]

\[
\cosh (x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.
\]

\[
tanh (x \pm y) = (\tanh x \pm \tanh y) + (1 \pm \tanh x \cdot \tanh y).
\]

\[
\sinh (2x) = 2 \sinh x \cosh x.
\]

\[
cosh (2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x.
\]

\[
tanh (2x) = 2 \tanh x + (1 + \tanh^2 x).
\]

\[
\sinh \left( \frac{1}{2} x \right) = \sqrt{\frac{1}{2} (\cosh x - 1)}.
\]

\[
cosh \left( \frac{1}{2} x \right) = \sqrt{\frac{1}{2} (\cosh x + 1)}.
\]

\[
tanh \left( \frac{1}{2} x \right) = (\cosh x - 1) + \sinh x = \sinh x + (\cosh x + 1).
\]

\[
\sinh x + \sinh y = 2 \sinh \frac{1}{2} (x + y) \cdot \cosh \frac{1}{2} (x - y).
\]

\[
\sinh x - \sinh y = 2 \cosh \frac{1}{2} (x + y) \cdot \sinh \frac{1}{2} (x - y).
\]

\[
\sin a \pm \sin B = \frac{\sin \frac{1}{2} (a \pm B)}{\cos \frac{1}{2} (a \pm B)}.
\]

\[
\cos a \pm \cos B = \frac{\cos \frac{1}{2} (a \pm B)}{\sin \frac{1}{2} (a \pm B)}.
\]

\[
\sin a \pm \sin B = \frac{\sin \frac{1}{2} (a \pm B)}{\cos \frac{1}{2} (a \pm B)}.
\]

\[
\sin a - \sin B = \tan \frac{1}{2} (a - B).
\]
\[ \sin^2 a - \sin^2 \beta = \sin (a + \beta) \cdot \sin (a - \beta). \]
\[ \cos^2 a - \cos^2 \beta = -\sin (a + \beta) \cdot \sin (a - \beta). \]
\[ \cos^2 a - \sin^2 \beta = \cos (a + \beta) \cdot \cos (a - \beta). \]
\[ \sin x i = \frac{1}{2} \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \sinh x. \]
\[ \cos x i = \frac{1}{2} \frac{(e^x + e^{-x})}{e^x + e^{-x}} = \cosh x. \]
\[ \tan x i = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \tanh x. \]
\[ e^{x + yi} = e^x \cos y + ie^x \sin y. \]
\[ a^{x + yi} = a^x \cos (y \cdot \log a) + ia^x \sin (y \cdot \log a). \]
\[ (\cos \theta \pm i \cdot \sin \theta)^n = \cos n\theta \pm i \cdot \sin n\theta. \]
\[ \sin x = -\frac{1}{2} i(e^{x i} - e^{-x i}). \]
\[ \cos x = \frac{1}{2} (e^{x i} + e^{-x i}). \]
\[ \tan x = -\frac{i(e^{x i} - 1)}{e^{x i} + 1}. \]
\[ \sin(x \pm yi) = \sin x \cos yi \pm \cos x \sin yi \]
\[ = \sin x \cosh y \pm i \cos x \sinh y. \]
\[ \cos(x \pm yi) = \cos x \cos y \mp \sin x \sin y \]
\[ = \cos x \cosh y \mp i \sin x \sinh y. \]
\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \]
\[ a^2 = b^2 + c^2 - 2bc \cos A. \]
\[ \frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} C}{\tan \frac{1}{2} (A - B)} \]
\[ \sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}, \text{ where } 2s = a + b + c. \]
\[ \cos \frac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}. \]
\[ \tan \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \]
Area = \frac{1}{2} bc \sin A = \sqrt{s(s - a)(s - b)(s - c)}. \]
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