Top-down Calculus Workbook

Derivatives Intuition Practice

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Preface


Reviewing calculus is much more fun than learning it for the first time. You can go over the material in any order, including backwards. Make one pass through the material fairly quickly, focusing on concepts and choosing carefully which problems to work. Additional problems can be worked later to improve technique. Excellent online software is available for calculus (differentiation and integration) and graphing functions.

To facilitate navigation, important elements of each chapter (e.g., definition, lemma, theorem, exercise, figure, etc.) are labeled in order of appearance using a shared numbering system. For example, 2.13 is a figure and 2.14 a definition. A list of all such labeled elements is in the index.

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Chapter 1

LINEAR FUNCTIONS AND DERIVATIVES

Linear Functions Are The Foundation Of Calculus

A “linear function” is a function whose graph is a straight line. Almost every calculus book has a section or two on linear functions. We shall describe such functions in this chapter and leave it to the reader to supplement our discussion by reading in his or her precalculus textbook. Every linear function has an equation of the form \( f(x) = ax + b \). In such an equation, \( x \) is called the “independent variable” or simply the “variable.” For example, \( f(x) = 2x + 1 \) is a linear function. The graph of this function and the graphs of two other linear functions are shown in FIGURE 1.1. The graph of the function \( f(x) = 2x + 1 \) is, by definition, the set of all points in the plane of the form \( (x, 2x + 1) \). Thus \( (1, 3) \) and \( (2, 5) \) are points on the graph of \( 2x + 1 \) but \( (1, 2) \) and \( (2, 4) \) are not on the graph of \( 2x + 1 \). In the linear function \( f(x) = ax + b \), the number \( a \) is called the “slope of the function \( f \)” and the number \( b \) is called the “intercept of the function \( f \)” or the “vertical intercept of the function \( f \).” A discussion of these terms and the alternative ways of describing linear functions will be found in your algebra or precalculus textbook.

The Notation Of Calculus Demands Critical Attention

One of the major problems in learning calculus is the notation, which cynics say “ranges from bad to horrible.” We should begin now to think about
notation in a critical way. Take a look at FIGURE 1.1 again. The graph of the function \( f(x) = 2x + 1 \) is the "set of all points in the plane of the form \((x, 2x + 1)\)." If you look in the previous sentence at the statement in quotes, there is no mention of the symbol \( f \). Thus, if you had been asked to graph \( p(x) = 2x + 1 \) or \( \beta(x) = 2x + 1 \) you would have produced exactly the same graph! The only difference would be in how you would refer to the graph that you produced. If you have graphed \( f(x) = 2x + 1 \) you might say "'Bill, put your finger on the graph of \( f \)." If you graphed \( p(x) = 2x + 1 \) or \( \beta(x) = 2x + 1 \) instead, then you would use \( p \) or \( \beta \) in making such a statement in place of \( f \). One standard way of specifying a linear function is to use "'y'" rather than "'f'" as we have done. Of course, it makes no difference which letter you use except in referring to the linear function (as we have just noted). Thus, instead of \( f(x) = 2x + 1 \), one might write \( y(x) = 2x + 1 \). The notation "'f(x)'" tells us that \( f \) is the name of the function and \( x \) is the name of the variable. One could also write \( f = 2x + 1 \) or \( y = 2x + 1 \) and the function and variable names would be equally well specified. Oddly enough, one rarely sees \( f = 2x + 1 \), but \( y = 2x + 1 \) is very common. This sort of arbitrary notational tradition is quite common in calculus and is a source of confusion to the beginner.
Playing The Envelope Game

There is a little game that the beginning calculus student can play that will help keep the question of notation in perspective. This game is called THE ENVELOPE GAME. We shall play it from time to time. For linear functions the game goes like this: Imagine you have a linear function in an envelope. Before you open up the envelope try and describe what it is that you will see on the inside. The object of the game is to list the various ways that a linear function might be described. One possibility is that the envelope contains a piece of graph paper with the graph of the linear function (a straight line, of course) on it. Another possibility is that one sees an equation of the form \( y = ax + b \). Can you think of other possible contents of the envelope? What you should begin to understand from THE ENVELOPE GAME is the distinction between a mathematical object or concept and the manner used to specify or describe a particular instance of this object or concept. In the case at hand, the concept is that of a linear function which can be described in many different ways. A computer scientist might refer to these various ways of describing the same basic object as “different data structures.”

Locally, Linear Functions Approximate Nonlinear Functions

THE PROPERTIES OF LINEAR FUNCTIONS ARE THE FOUNDATION OF ALL OF CALCULUS. There is a simple intuitive reason for this, which is shown in FIGURE 1.2. In FIGURE 1.2 we see the graph of a nonlinear function \( f \). The function is nonlinear because its graph is not a straight line. Imagine that we choose three points on the graph of \( f \) and look at the curve under a microscope at these points. The three points we have selected are called \( A' \), \( B' \), and \( C' \) and are shown in FIGURE 1.2(a). The circles centered at these points represent the field of view of the microscope. In FIGURE 1.2(b) we see the view under the microscope centered at point \( B' \). What we see is (essentially!) a straight line segment. This straight line segment defines a straight line as indicated by the dotted line of FIGURE 1.2(b). Like any straight line, it is the graph of a linear function of the form \( y = ax + b \). By inspection, it appears that \( a = -1/3 \) in this case. What do you think \( b \) is for this straight line (a rough guess will do)? Actually, from the point of view of calculus, the value of \( b \) is not too important. It is the value of \( a \) that gets all of the attention. The value of \( a \) (in this case \( -1/3 \)) is called the derivative of \( f \) at \( B' \). Thus, to compute the derivative of a function \( f \) at a point \( B' \) on the graph of \( f \) one focuses a microscope at the point \( B' \),
ups the magnification until the portion of the graph under the microscope looks like a straight line segment, and computes the slope of this line. The resulting number is the derivative of \( f \) at \( B' \).

Let’s play THE ENVELOPE GAME. Imagine that you have an envelope and inside it is the derivative of a function \( f \) at a point \( B' \) on the graph of
the function. Before opening up the envelope, describe what it is that you are going to see (Answer: a number). These ideas are summarized in SLOPPY DEFINITION 1.3.

**The Intuitive Definition Of The Derivative**

1.3 **SLOPPY DEFINITION** Let f be a function and let B' be a point on the graph of f. Focus a microscope on the point B' and increase the magnification until the portion of the graph in the field of view of the microscope looks like a straight line segment. The slope of this straight line segment is the derivative of f at B'.

**A Function May Not Have A Derivative At Certain Points**

One problem with SLOPPY DEFINITION 1.3 is shown at the point C' in FIGURE 1.2(a). At this point, the graph of f has a sharp "spike" or "cusp." No matter how much we increase the magnification of the microscope, we never see a straight line segment! Thus, SLOPPY DEFINITION 1.3 doesn't work for such a point. We say, in this case, that "f does not have a derivative at C'". The functions that are studied in calculus have very few bad points such as C' where the derivative does not exist. Some functions that naturally occur do have such points (for example, f(x) = |x| has such a point at x = 0 and f(x) = |x| + |x - 1| has two such points). Notice that the function shown in FIGURE 1.2 has infinitely many "good points" where the derivative exists but only one point C' where the derivative does not exist (there may be more points not shown in FIGURE 1.2, but you get the idea!).

**Compute The Derivative At Lots Of Points To Get The Derivative Function**

We have been discussing the derivative of a function f at a point on the graph of f. Look now at FIGURE 1.4. There we see a function f(x). At each point on the graph of f(x) we have attempted to compute the derivative of f using SLOPPY DEFINITION 1.3. At the point (-4, +3.6), which is on the graph of f, we thought the derivative was about -0.4. At the point (-1, +1) on the graph of f we thought that the derivative was about -1.0. At (-7, +4.5) the derivative was 0, etc. Of course, there are infinitely many such points and we can't compute derivatives for all of them. We computed derivatives for a number of such points and then drew a smooth curve through them to obtain the graph of f'(x) shown in FIGURE 1.4. This new function is called the derivative function of f.
1.5 **DEFINITION**  Let $f$ be a function. For each point $(x, f(x))$ on the graph of $f$ that has a derivative, compute that derivative and call it $f'(x)$. The function $f'(x)$ is called *the derivative function of $f$*.

**Computing Derivative Functions Graphically**

1.6 **EXERCISES**

(1) Draw the graph of the derivative functions $f'$, $g'$, and $h'$ for each of the three linear functions of FIGURE 1.1.

(2) Draw the graph of the derivative function $f'$ for the function $f$ of FIGURE 1.2. The point $C'$ is a problem as there is no derivative at that point. Draw carefully what the graph of $f'$ looks like for points of the graph of $f$ to the left of $C'$ (values of $x < 4.35$) and to the right of $C'$ (values of $x > 4.35$).

1.7 **IMPORTANT PROPERTIES OF LINEAR FUNCTIONS**  We now take a look at some simple but extremely important properties of linear functions as they relate to calculus. To emphasize when we are talking about a *linear* function we shall use a "tilde" over the symbol representing that function. Thus $\tilde{f}$ will be a linear function.
Suppose that $\tilde{f}(x) = ax + b$ and $\tilde{g}(x) = cx + d$ are two linear functions. Define $\tilde{s}(x) = \tilde{f}(x) + \tilde{g}(x)$ to be the sum of $\tilde{f}$ and $\tilde{g}$. Thus $\tilde{s}(x) = (a + c)x + (b + d)$. The important thing to notice here is that THE SLOPE OF THE SUM OF TWO LINEAR FUNCTIONS IS THE SUM OF THEIR SLOPES. It is evident from the calculation that we just did that the sum of two linear functions is again a linear function. This is not the case for the product of linear functions!

If a linear function is multiplied by any real number $r$, then we again obtain a linear function. For example, $r\tilde{f}(x) = rax + rb$. Note that the slope of the new linear function is $r$ times the slope of the function $\tilde{f}$. Thus we have that THE SLOPE OF A REAL NUMBER TIMES A LINEAR FUNCTION IS THAT REAL NUMBER TIMES ITS SLOPE.

For Linear Functions, The Slope Of A Composition Is The Product Of The Slopes

The above two properties of linear functions are really pretty simple. We now consider a much deeper property of linear functions. Understanding this property well will be the key to a good grade in calculus!

Adding linear functions produces another linear function whose slope is the sum of the slopes of the two original linear functions. What sort of operation on linear functions produces another linear function whose slope is the product of the slopes of the two original linear functions? Taking products of linear functions won’t do the job. For example, $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$, which is not even linear (it’s quadratic). It turns out that the operation that we need is composition of functions. The operation of composition of functions is not restricted to linear functions but we shall start with that case as it is particularly easy to follow. Let $\tilde{f}$ and $\tilde{g}$ be as in the previous paragraph. The composition of $\tilde{f}$ and $\tilde{g}$ is the function $\tilde{f}(\tilde{g}(x))$ obtained by replacing each occurrence of the variable $x$ in $\tilde{f}(x)$ by the entire expression $\tilde{g}(x)$. Thus, if $\tilde{f}(x) = ax + b$ and $\tilde{g}(x) = cx + d$ then $\tilde{f}(\tilde{g}(x)) = a(cx + d) + b = acx + (ad + b)$. From this calculation we see that the composition of two linear functions is again a linear function and THE SLOPE OF THE COMPOSITION OF TWO LINEAR FUNCTIONS IS THE PRODUCT OF THEIR SLOPES. Let us summarize these important properties of linear functions:

1. If a real number is multiplied times a linear function, then the new function is linear and its slope is the product of the real number and the slope of the original function.
(2) THE SUM OF TWO LINEAR FUNCTIONS IS A LINEAR FUNCTION. THE SLOPE OF THE NEW FUNCTION IS THE SUM OF THE SLOPES OF THE ORIGINAL FUNCTIONS.

(3) THE COMPOSITION OF TWO LINEAR FUNCTIONS IS A LINEAR FUNCTION. THE SLOPE OF THE NEW FUNCTION IS THE PRODUCT OF THE SLOPES OF THE ORIGINAL FUNCTIONS.

We now must take a look at the meaning of composition of functions \( f \) and \( g \) where \( f \) and \( g \) may not be linear. Consider DEFINITION 1.8.

But We Can Compose Nonlinear Functions Too

1.8 DEFINITION  Let \( f \) and \( g \) be functions (real valued). Define a function \( h \) whose value \( h(x) \) at any real number \( x \) is gotten by first evaluating \( g \) at \( x \) to obtain the real number \( g(x) \) and then evaluating \( f \) at \( g(x) \) to obtain \( f(g(x)) \). The function \( h \) is called the composition of \( f \) and \( g \). We write \( h(x) = f(g(x)) \).

In calculus we deal primarily with functions whose "domain" and "range" are real numbers. These functions are called real valued and can be graphed using the standard horizontal and vertical real number lines (as in FIGURES 1.1 and 1.2, for example). Thus, when we form the composition of two real valued functions we obtain another real valued function.

It is now time to play THE ENVELOPE GAME with DEFINITION 1.8. Suppose we have an envelope and inside it are two functions \( f \) and \( g \) and their composition \( h(x) = f(g(x)) \). What are we going to see when we open the envelope? One possibility is that \( f \) and \( g \) and \( h \) are described by "formulas" or "closed expressions." Another possibility is that \( f \) and \( g \) and hence \( h \) are given as graphs. As an example of the formula type description, we might have \( f(x) = 2x^3 + 5x^2 - 3x + 2 \) and \( g(x) = \sin(x) + \sqrt{x} \). Then we find that

\[
f(g(x)) = 2(\sin(x) + \sqrt{x})^3 + 5(\sin(x) + \sqrt{x})^2 - 3(\sin(x) + \sqrt{x}) + 2.
\]

As the above expression shows, the composition of two functions \( f \) and \( g \) given as formulas or closed expressions can be done just as with linear functions by replacing each occurrence of \( x \) in the expression for \( f(x) \) by the whole expression for \( g(x) \). There is a problem that might occur here. Suppose \( f(x) = \sqrt{x} \) and \( g(x) = x^3 \). Let \( h \) be the composition of \( f \) and \( g \). If we try to evaluate \( h(-1) \) then we first form \( g(-1) = -1 \) and then try to evaluate \( f(-1) \), which is the square root of \(-1\). Thus, \( h(-1) \) is not defined as a real number. In general, this sort of thing happens frequently.
1.8 NOTE: Intuitively, a function is a rule that assigns to each element of a set \( X \) (called the domain) a unique element of a set \( Y \) (called the codomain or range). In calculus, we usually take \( Y = \mathbb{R} \) (the real numbers) and \( D \subseteq \mathbb{R} \) (a subset, possibly equal, of \( \mathbb{R} \)). Common choices of \( D \) are open intervals \((a, b) := \{ x \mid a < x < b \}\), closed intervals \([a, b] := \{ x \mid a \leq x \leq b \}\) and intervals of the form \([a, b)\) and \((a, b]\) with the obvious definitions. Either \( a, b \) or both may be replaced by \( \infty \) (the infinity symbol): \((a, \infty) = \{ x : a < x \}, (\infty, \infty) = \mathbb{R} \).

**Composing Functions Graphically**

In FIGURE 1.9 we see three functions \( f, g, \) and \( h \) given graphically. The function \( h \) is the composition of \( f \) and \( g \): \( h(x) = f(g(x)) \). Computing the composition of functions given in graphical form can be a rather tedious process. In principle, one must compute \( f(g(x)) \) at each point \( x \). Thus, for \( x = 3 \), we first compute \( g(3) \) by looking at the graph of \( g \). It seems that \( g(3) = 2 \). We now compute \( f(g(3)) = f(2) \), which is about \(-1 \). Thus, by definition, \( h(3) = -1 \). If we do that for enough points then we can draw the graph of \( h \) as has been done in FIGURE 1.9(c). In FIGURE 1.9(a) a microscope has been placed at the point \( A' = (3, g(3)) = (3, 2) \) on the graph of \( g \). The straight line segment (or nearly so) that appears in the field of view (indicated by the circle) has been extended to obtain a straight line \( \hat{g}(x) = (1/3)x + 1 \). This means, by SLOPPY DEFINITION 1.3, that slope(\( \hat{g} \)) = 1/3 is \( g'(3) \). Similarly, in FIGURE 1.9(c) a microscope has been put at the point \( B' = (g(3), f(g(3))) = (2, -1) \). The line segment in the field of view at \( B' \) has been extended to a straight line \( \hat{f}(x) = (-3/2)x + 2 \).

**Composing Linear Approximations To Functions**

Suppose that two students, Larry (for Linear) and Nancy (for Nonlinear) are given the task of composing functions from FIGURE 1.9. Larry is going to compose \( f \) and \( \hat{g} \) to obtain the function \( h \) of FIGURE 1.9(c). Nancy is going to compose \( f \) and \( g \) to obtain the function \( h \) of FIGURE 1.9(c). Of course, their work is going to be quite different for most values of \( x \). Note, however, that the graphs of \( \hat{g} \) and \( g \) coincide (essentially) in the circle about \( A' \) and the graphs of \( \hat{f} \) and \( f \) coincide (essentially) in the circle about \( B' \). Thus when Larry and Nancy are dealing with values in these circled regions they will essentially get the same answer for the composition as is shown in the circled region about \( C' \) in FIGURE 1.9(c).

Larry, being a good student, has learned IMPORTANT PROPERTIES 1.7 (in particular number 3). Thus, he knows that the slope of \( \hat{f} \) times the slope
Nancy worked hard to graphically compose $f$ and $g$ to get $h$. Having done this, she can graphically compute $h'(x)$ for any $x$. Lazy Larry, will only compose linear functions. If Nancy asks him to compose $f$ and $g$ to get $h$ he will unwittingly compute $h'(3)$ for her. Forget Larry! She can just go along the graph of $h$ computing slopes to get $h'(x)$ for all $x$. Thinking about this, Nancy sees that $x = 3$ is not special and $[f(g(x))]' = f'(g(x))g'(x)$ for all $x$. She has discovered the chain rule. This is how good math is done!
of \( \tilde{g} \) is the slope of \( \tilde{h} \) (in this case \(-3/2 \) times \(1/3 \) equals \(-1/2 \)). Nancy, also a good student, has learned SLOPPY DEFINITION 1.3, which she applies to the points \( A', B', \) and \( C' \). She concludes correctly that the slope of \( \tilde{g} \) is the derivative \( g'(3) \) of \( g \) at the point \( A' \), the slope of \( \tilde{f} \) is the derivative \( f'(g(3)) = f'(2) \) of \( f \) at the point \( B' \), and the slope of \( \tilde{h} \) is the derivative \( h'(3) \) of \( h \) at the point \( C' \). Putting their observations together, Larry and Nancy conclude that \( h'(3) = f'(g(3))g'(3) \).

The same process would have worked for any \( x \), not just \( x = 3 \) (the functions would have to be defined at \( x \) and \( g(x) \), of course, and have derivatives). Thus we would discover that \( h'(x) = f'(g(x))g'(x) \). This observation is called the chain rule or the composite function rule and is without doubt the single most important fact that must be learned and thoroughly understood by the beginning calculus student. We state this rule in IMPORTANT THEOREM 1.10.

The Most Important Concept To Master . . .

1.10 IMPORTANT THEOREM (THE CHAIN RULE) Let \( f(x) \) and \( g(x) \) be functions and let \( h(x) = f(g(x)) \) be the composition of \( f \) and \( g \). Then, the derivative \( h'(x) \) is equal to \( f'(g(x))g'(x) \).

Many beginning calculus students would be much happier if IMPORTANT THEOREM 1.10 stated \( h'(x) = f'(x)g'(x) \) but, alas, this is not true in general. It is not true for \( x = 3 \) in FIGURE 1.9, as the reader should verify.

Linearity Of The Derivative: 

\[
(rf(x) + sg(x))' = rf'(x) + sg'(x)
\]

As we have just seen, IMPORTANT PROPERTY 1.7(3) of linear functions gives rise to IMPORTANT THEOREM 1.10 on derivatives. What rules of derivatives follow from IMPORTANT PROPERTIES 1.7(1) and (2)? Fortunately, these rules are so simple that most beginning calculus students apply them automatically. There are times, however, when these rules need to be articulated clearly in solving a problem so we shall now discuss them briefly.

Take a look at FIGURE 1.11. There we see the graphs of functions \( f(x) \), \( g(x) \), and \( h(x) \). These functions have been constructed so that \( h(x) = f(x) + g(x) \). As with FIGURE 1.9, microscopes have been placed on these graphs at points \( A' \), \( B' \), and \( C' \). These points all correspond to \( x = 1 \). Thus \( A' = (1, f(1)) = (1, 3) \), \( B' = (1, g(1)) = (1, -2) \), and \( C' = (1, h(1)) = (1, 1) \). In the microscope, each curve looks like a straight line segment. These straight line segments are extended to give the lines \( \tilde{f} \), \( \tilde{g} \), and \( \tilde{h} \) as shown in
FIGURE 1.11 Derivative of a Sum Is the Sum of the Derivatives

\[ \tilde{h}(x) = \tilde{f}(x) + \tilde{g}(x) \]
\[ h(x) = f(x) + g(x) \]
\[ h'(x) = f'(x) + g'(x) \]

FIGURE 1.11. Clearly, \( \tilde{h} = \tilde{f} + \tilde{g} \). By IMPORTANT PROPERTY 1.7(2), slope(\( h \)) = slope(\( f \)) + slope(\( g \)) and hence by SLOPPY DEFINITION 1.3, \( h'(1) = f'(1) + g'(1) \), as can be seen in the example of FIGURE 1.11. Instead of using \( x = 1 \) we could have used any \( x \) in this argument (provided that the relevant derivatives were defined) and we would obtain \( h'(x) = f'(x) + g'(x) \). This rule states that \textit{the derivative of a sum} (in this case \( h'(x) \)) is \textit{the sum of the derivatives} (in this case \( f'(x) + g'(x) \)). The rule corresponding to IMPORTANT PROPERTY 1.7(1) is even easier: if \( h(x) = rf(x) \) where \( h \) and \( f \) are functions and \( r \) is a real number, then \( h'(x) = rf'(x) \). The reader should explain this rule graphically as was done for the other two rules in FIGURE 1.9 and FIGURE 1.11 respectively. We state these three rules in RULES FOR DERIVATIVES 1.12.
1.12 RULES FOR DERIVATIVES

(1) CONSTANT MULTIPLE RULE: If \( f(x) \) is a function and \( r \) is a constant (i.e., a real number) and \( h(x) = rf(x) \) then \( h'(x) = rf'(x) \).

(2) SUM RULE: If \( f(x) \) and \( g(x) \) are functions and \( h(x) = f(x) + g(x) \) then \( h'(x) = f'(x) + g'(x) \).

(3) CHAIN RULE: If \( f(x) \) and \( g(x) \) are functions and \( h(x) = f(g(x)) \) then \( h'(x) = f'(g(x))g'(x) \).

The Envelope Game Again...

"Understanding the RULES FOR DERIVATIVES 1.12 has been the basic object of this chapter. The next chapter will be devoted to the problem of "computing derivatives." To begin our thinking about this problem, let's play THE ENVELOPE GAME once again. Suppose we are given a function \( f(x) \) graphically, as in FIGURE 1.4. We are given an envelope and inside the envelope is the derivative function \( f'(x) \). What is it that we are going to see when we open the envelope? Most likely it will be another graph, just as in FIGURE 1.4. If this were all that was involved in the study of derivatives, calculus would be a trivial subject. Given any function \( f(x) \), we would draw its graph and compute \( f'(x) \) as in FIGURE 1.4. This would work for any function. For example, it would work for the function \( f(x) = x^2 \). You may have already had enough experience with calculus to know, however, that if your calculus instructor asked you to find the derivative of \( f(x) = x^2 \) and your answer was a graph, he or she would probably "flip out." The expected answer would be "\( f'(x) = 2x \)." In other words, given a function \( f(x) \) as a formula or "closed expression" and an envelope containing \( f'(x) \), one would expect to open the envelope and find another formula or "closed expression." So given \( f(x) = x^2 \) the envelope should contain \( f'(x) = 2x \).

If \( f(x) = x^2 \) Then \( f'(x) = 2x \). Here's Why...

Every calculus book proves that if \( f(x) = x^2 \) then \( f'(x) = 2x \). To understand how this interesting fact relates to SLOPPY DEFINITION 1.3, take a look at FIGURE 1.13. In FIGURE 1.13(a) we see a portion of the graph of \( f(x) = x^2 \). In the spirit of SLOPPY DEFINITION 1.3, we have put our microscope at the point \( A' = (1, 1) \) on the graph of \( f \). The slope of the straight line segment in the field of view of the microscope seems to be about 2. If \( f'(x) = 2x \) is the correct formula then this is as it should be as \( f'(1) = 2 \).
The Square Of A Small Number Is Even Smaller

In FIGURE 1.13(b) we are taking a more careful look at the field of view of the microscope at the point A'. To be a little more general, we are looking at A' = (x, x^2) (instead of just x = 1 as shown in FIGURE 1.13(a)) where x is some value near 1. In FIGURE 1.13(b) the line joining A' to C' is a straight line segment of slope 2x. The distance from A' to B' is Δx. The symbol "Δx" is used in calculus to stand for a "small number which is to be added to x to change it slightly." The line joining A' to D' is a part of the graph of f(x) = x^2 and is not, of course, exactly a straight line segment.

To see how far off from being straight this latter line is, let's compute the distance the point D' is above the point C'. The second coordinate of C' is x^2 + 2xΔx and the second coordinate of D' is (x + Δx)^2 = x^2 + 2xΔx + (Δx)^2. The difference between these two coordinates is (Δx)^2, which is the height of D' above C'. We have in mind that Δx is small, say .001. In this case, (Δx)^2 = .000001, is much smaller still. The ratio of the second of these two quantities to the first is Δx = .001. Referring to FIGURE 1.13(b),
we have shown that the ratio of the distance between $D'$ and $C'$ to the distance between $A'$ and $B'$ is $\Delta x$. Thus, this ratio goes to zero as $\Delta x$ goes to zero. This is the analytic statement of the fact that the smaller the field of view (which is roughly $\Delta x$) the more the part of the graph in the field of view appears to be a straight line.

**The Difference Quotient:** $\Delta f = f(x + \Delta x) - f(x)$ Over $\Delta x$

Look again at FIGURE 1.13(b). If the line from $A'$ to $D'$ was a straight line segment then its slope would be $\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x$. The quantity $f(x + \Delta x) - f(x)$ is called the ‘‘difference quotient of $f$ with respect to $\Delta x’’. As we have just seen, the difference quotient is almost $f'(x)$ and the smaller $\Delta x$ is the closer this difference quotient is to $f'(x)$. Mathematicians would express this fact by saying that ‘‘the limit of $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ as $\Delta x$ tends to zero is $f'(x)$.’’ This statement assumes that $f$ has a derivative at $x$. This important idea is summarized in ERUDITE OBSERVATION 1.14.

**1.14 ERUDITE OBSERVATION** If $f$ is any function that has a derivative at $x$ then the difference quotient $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is very close to $f'(x)$ for small enough values of $\Delta x$. In other words, the limit of $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ as $\Delta x$ tends to zero is $f'(x)$. In symbols we may write

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Another common notation for ERUDITE OBSERVATION 1.14 is to define $\Delta f = f(x + \Delta x) - f(x)$. We then may write $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x)$. In the case where $f(x) = x^2$ we found that $\frac{\Delta f}{\Delta x} = 2x + \Delta x$ and thus, $f'(x) = 2x$.

In the next chapter we shall be concerned with techniques for computing derivatives. Most of what the beginning calculus student has to learn is precisely “techniques of computation.” The major difficulty in this task is the profusion of notation brought about by the many different subjects to
which calculus has been applied. We close this chapter by reviewing the ideas we have discussed thus far through the study of a number of specific examples.

1.15 EXAMPLES OF RULES FOR DERIVATIVES

(1) We now know how to compute the derivative function of \( f(x) = x^2 \). The answer is \( f'(x) = 2x \). Another standard notation for the derivative function \( f'(x) \) is \( \frac{df}{dx} \). Thus if \( f(x) = x^2 \) then \( \frac{df}{dx} = 2x \). The notation \( f'(x) \) tells us that \( f' \) is a function of \( x \). In the notation \( \frac{df}{dx} \) it is the \( x \) in \( dx \) that tells us that the variable is \( x \). Sometimes one sees \( \frac{df}{dx} \) or \( \frac{df}{dx} \) if one wants to emphasize that the variable is \( x \). Another way of saying that the derivative of \( x^2 \) is \( 2x \) is to write \( (x^2)' = 2x \) or \( \frac{d}{dx} x^2 = 2x \). Unfortunately, one must get used to all of these notations!

Constant Functions Have Zero Derivative Functions

(2) A function \( f(x) = r \) where \( r \) is a number is called a "constant function." Thus \( f(x) = 2 \) is a constant function. The graph of \( f(x) = 2 \) is a line parallel to and two units above the horizontal axis. The function \( g(x) \) of FIGURE 1.1 is a constant function with value 3/2.

Beginning students sometimes confuse constant functions with constants (i.e., numbers). There is an important difference. The function \( f(x) = 2 \) is a rule that assigns to every real number \( x \) the number 2 in the same sense that \( f(x) = x^2 \) assigns to every \( x \) the value \( x^2 \). The graph of \( f(x) = 2 \) is the set of all pairs \((x, 2)\). The graph of \( g(x) = 3/2 \) of FIGURE 1.1 is the set of all pairs \((x, 3/2)\) as shown. The numbers 2 or 3/2 are not the same as the functions \( f(x) = 2 \) or \( g(x) = 3/2 \). Any constant function, such as \( f(x) = 2 \), has slope zero at every point of its graph. Thus \( f'(x) = 0 \) for such a function.

In practice, the possibility of being seriously confused by the difference between constants and constant functions is very small. If one sees a statement such as \( \frac{d}{dx}(2) = 0 \) then one knows that the constant function with value 2 is being differentiated (i.e., having its derivative taken) and its derivative func-
tion is the constant function with value 0. Another way of saying the same thing is \((2)' = 0\). No matter how large the value of the constant function, its derivative function is still the zero function. Thus, \((1,000,000)' = 0\).

**The Derivative Of A Linear Function Is Its Slope**

(3) In EXERCISE 1.6, the reader was asked to compute the derivative functions of the three linear functions of FIGURE 1.1. The answers are \(f'(x) = 2\), \(g'(x) = 0\), and \(h'(x) = -1/2\). In general, if \(f(x) = ax + b\) is a linear function then \(f'(x) = a\) is the constant function with value equal to the slope of \(f(x)\). Thus, \(\frac{d}{dx}(23x + 45) = 23\), \((-12x - 124)' = -12\), and \(\frac{d}{dx}x = 1\). This latter result can also be written \((x)' = 1\) and, probably because it is so simple, is sometimes a source of confusion to the beginner.

**The Derivative Of A Sum Is...**

(4) We now take a look at RULES FOR DERIVATIVES 1.12. Applying 1.12(1), we compute \((2x^2)' = 2(x^2)' = 2(2x) = 4x\). In our alternative notation, \(\frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2(2x) = 4x\). In a similar fashion, \((45x^3)' = 90x\), \((-4x^2)' = -8x\). Applying 1.12(2) we can compute \((2x^2 + 4x)' = (2x^2)' + (4x)'\) which, by 1.12(1), is \(2(x^2)' + 4(x)' = 4x + 4\). In our alternative notation we would have \(\frac{d}{dx}(2x^2 + 4x) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(4x) = 4x + 4\). As stated, RULE 1.12 applies to the sum of two functions, but it is obviously valid for 3, 4, or any finite sum of functions. Thus \((e(x) + f(x) + g(x))' = e'(x) + (f(x) + g(x))' = e'(x) + f'(x) + g'(x)\). As an example, \((5x^2 + 6x + 9)' = (5x^2)' + (6x)' + (9)' = 10x + 6 + 0 = 10x + 6\).

**Using The Chain Rule Requires Some Guesswork**

(5) Rules 1.12(1) and 1.12(2) illustrated in example (4) above are easily mastered by the beginning student and usually applied correctly with little thought involved. The CHAIN RULE, 1.12(3), is a different matter! As we have already stated, it is the most important rule of calculus, at least in the beginning. The CHAIN RULE concerns functions of the form \(f(g(x))\) which are compositions of two functions \(f(x)\) and \(g(x)\). The CHAIN RULE is then used to compute the derivative \((f(g(x)))'\) of this function. The CHAIN RULE requires that you compute first \(f'(x)\) and then compose this with \(g(x)\) to get
f'(g(x)). This is then multiplied times g'(x) to get f'(g(x)) g'(x), which is the
correct answer.

That sounds easy enough. For example, let f(x) = x^2 and let g(x) = 9x^2
- 6x + 4. Then f(g(x)) = (9x^2 - 6x + 4)^2. We compute that f'(x) =
2x and g'(x) = 18x - 6 and hence f'(g(x))g'(x) = 2(9x^2 - 6x + 4) (18x
- 6). In practice, there is an additional complication: f(g(x)) may be given
but not f(x) and g(x). Suppose we are asked to differentiate (3x^2 + 3x
+ 3)\(^{2/3}\). We think we would like to use the CHAIN RULE, but what are f(x)
and g(x)? The answer is that we must guess what they are! In this case we
could take f(x) = x^2 and g(x) = (3x^2 + 3x + 3)^{1/3} or we could take f(x)
= x^{2/3} and g(x) = 3x^2 + 3x + 3. Both work to give the expression (3x^2
+ 3x + 3)^{2/3} for f(g(x)). In terms of applying the CHAIN RULE, the latter
is a much better choice. To apply the CHAIN RULE in this case we need to
know that (x^r)' = rx^{r-1} for any real number r (not just r = 2 as we have
already shown). This formula will be derived in the next chapter and can
just be accepted for now. Using this formula, we see that f'(x) =
(2/3)x^{-1/3}. Obviously by now, g'(x) = 6x + 3 and so

f'(g(x))g'(x) = (2/3) (3x^2 + 3x + 3)^{-1/3} (6x + 3),

which is the correct derivative. We shall continue worrying about the CHAIN
RULE in the next example.

The Chain Rule In Differential Notation

(6) Suppose we are going to apply the CHAIN RULE to f(g(x)) where
f(x) = x^3. We would first find f'(x) = 3x^2 and then substitute g(x) for x to
obtain f'(g(x)) = 3(g(x))^2, which we may write more simply as f'(g) =
3g^2. If instead of writing f(x) = x^3 we had written f(g) = g^3 and differentiated
with respect to g to get \( \frac{df}{dg} = 3g^2 \) we would have obtained the same result
directly. For this reason, f'(g(x))g'(x) may be written \( \frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx} \). The CHAIN
RULE may now be stated \( \frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx} \).

But We Still Must Guess A Lot...

(7) In applying the CHAIN RULE in the form \( \frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx} \) we are faced
with the same difficulties as in example (5) above. Suppose we are asked to
find the derivative of (2x^2 + 1)^3 + 8(2x^2 + 1)^2 + 6(2x^2 + 1) + 3.
We would like to think of this expression as \( f(g(x)) \) for some \( f \) and \( g \), but what are \( f \) and \( g \)? Again we must guess. In this case it looks like a natural choice for \( g \) is the function \( 2x^2 + 1 \) which occurs throughout this expression. With this choice for \( g \), \( f = g^3 + 8g^2 + 6g + 3 \). We compute
\[
\frac{df}{dg} = 3g^2 + 16g + 6 \quad \text{and} \quad \frac{dg}{dx} = 4x. \quad \text{Thus,} \quad \frac{df}{dx} = (3g^2 + 16g + 6)4x = (3(2x^2 + 1)^2 + 16(2x^2 + 1) + 6)4x. \quad \text{Of course, we could use any two distinct symbols for} \ f \ \text{and} \ g. \ \text{If we had used} \ v \ \text{and} \ w \ \text{instead the formula would be} \ \frac{dv}{dx} = \frac{dv}{dw} \ \frac{dw}{dx}.
\]

\[
\frac{d}{dx} g^r = rg^{r-1} \frac{dg}{dx}
\]

(8) As we remarked in example (5) above, \((x^r)' = rx^{r-1}\) for any real number \( r \). We shall derive this result in the next chapter. This means that for any function \( g(x) \), \((g(g(x)))' = r(g(x))^{r-1}g'(x)\). This follows from the CHAIN RULE with \( f(x) = x^r \). In our alternative notation
\[
\frac{d}{dx} (g(x))^r = r(g(x))^{r-1} \frac{d}{dx} g(x).
\]

It is fun to apply this rule over and over again to see what complicated expressions one can derive. For example, take \( g(x) = x^3 + 1 \) and \( r = .20 \). Then \( \frac{d}{dx} (x^3 + 1)^2 = .2(x^3 + 1)^{- .8}3x^2 \). Let \( g(x) = ((x^3 + 1)^2 + 1) \) and \( r = .5 \). Then \( \frac{d}{dx} ((x^3 + 1)^2 + 1)^.5 = .5((x^3 + 1)^2 + 1)^{- .5}2(x^3 + 1)^{- .8}3x^2 \).

Now we can repeat this process a few more times and you will amaze your friends with the complex functions you can differentiate (i.e., find the derivative function of). It is at this point that the beginning student, amazed by the complexity of the calculations, forgets completely what the subject is about! This is the time to go back and look at FIGURES 1.2 and 1.4. Also, rethink the CHAIN RULE and what it means in terms of FIGURE 1.9. No matter how complicated the expressions, these are the ideas that underlie the formulas we have been deriving in these examples.

**Differentiating Different Expressions For The Same Function**

(9) As our final example, we shall differentiate the two expressions \( (x^2 + x^3)^2 \) and \( x^4 + 2x^5 + x^6 \). Actually, these two expressions represent the same function, as the second is what we get by computing the square indicated
by the first. Since these two expressions represent the same function, the expressions obtained by differentiating them must also represent the same function. Using the CHAIN RULE, we compute \( \frac{d}{dx}(x^2 + x^3)^2 = 2(x^2 + x^3)(2x + 3x^2) \) and, using the rule for differentiating sums, we obtain \( \frac{d}{dx}(x^4 + 2x^5 + x^6) = 4x^3 + 10x^4 + 6x^5 \). These two expressions look different at first glance but we know they are the same as functions (i.e., give the same value for each value of \( x \)) because they are just different descriptions of the derivative of the same function. In fact, multiplying the terms of the first expression gives the second.

This idea can be used to prove a very useful general identity for derivatives. To do this, replace \( x^2 \) by \( f(x) \) and \( x^3 \) by \( g(x) \) where \( f(x) \) and \( g(x) \) are any two functions that can be differentiated (“differentiable functions”). We shall try and compute \( \frac{d}{dx}(f(x) + g(x))^2 \) in two different ways as above. First, using the CHAIN RULE we obtain

\[
\frac{d}{dx}(f(x) + g(x))^2 = 2(f(x) + g(x))(f'(x) + g'(x))
\]

\[= 2f(x)f'(x) + 2(f'(x)g(x) + f(x)g'(x)) + 2g(x)g'(x).\]

Second, by first computing the square, we obtain

\[
\frac{d}{dx}(f(x))^2 + 2f(x)g(x) + (g(x))^2
\]

\[= 2f(x)f'(x) + 2(f(x)g(x))' + 2g(x)g'(x).
\]

**The Product Rule**

By setting these two expressions equal to each other and canceling common terms we obtain the important “product rule”

\[(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)\]

In our derivation of this result, we mixed our two notations for the derivative. This is fine as long as the meaning is clear. In our alternative notation, our new formula for the derivative of a product of two functions becomes

\[
\frac{d}{dx}(f(x)g(x)) = \left( \frac{d}{dx}f(x) \right)g(x) + f(x)\left( \frac{d}{dx}g(x) \right).
\]
We shall have much more to say about this important formula in the next chapter.

So Now Memorize It: $(fg)' = f'g + fg'$

We now give some exercises. Following the exercises we give the solutions. Try to work each exercise first without looking at the solution. If you do look at the solution to an exercise, immediately make up on your own a variation of that exercise and work your variation.

1.16 EXERCISES

Some Routine Work With Compositions

(1) Find the compositions indicated below. In the cases indicated, specify the values for which the composite function is defined.

(a) Find $h(x) = f(g(x))$ where $f(x) = 2x^3 + 3$ and $g(x) = (-x - 1)^3$.

(b) Find $h(x) = f(g(x))$ where $f(t) = 2t^3 + 3$ and $g(x) = (-x - 1)^3$.

(c) Find $u(v(x))$ where $u = 3v^4 + 2v^3 + 3v + 6$ and $v(x) = -x^2 + 1$.

(d) Find $p(q(x))$ where $p = \frac{1}{2y^3 + 3}$ and $q = x^{1/2}$.

(e) Find $h(z) = f(g(z))$ where $f(g) = \frac{g + 1}{g - 1}$ and $g(z) = \sqrt{z}$. For what values of $z$ is $h(z)$ defined?

(f) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^2$ and $k(y) = y^{1/2}$. For what values of $y$ are these functions defined?

(g) Find $h(k(y))$ and $k(h(y))$ where $h(y) = y^3$ and $k(y) = y^{13}$. For what values of $y$ are these functions defined?
Now Some Guesswork...

(2) Find functions $f(g)$ and $g(x)$ such that $h(x) = f(g(x))$ in each of the following cases. There are many possible correct answers in each case but generally only one “natural” choice.

(a) $h(x) = (x^2 + 1)^{1/3} + (x^2 + 1)^{-1/3}$
(b) $h(x) = (x + 1)^3 + x^2 + 2x + 1$
(c) $h(x) = \frac{x^2 + 2x}{(x + 1)^3}$
(d) $h(x) = \frac{x^2 + 2x + 5}{x^2 + 2x + 6}$. For this case, try $g(x) = x^2 + 2x + 6$, $g(x) = x^2 + 2x + 5$, and $g(x) = x + 1$.

There’s More Than One Way To Do It...

(3) In each of the following cases a function $h(x)$ is given. For each of the specified functions $g(x)$, find $f(g)$ such that $f(g(x)) = h(x)$.

(a) For $h(x) = \frac{1}{(x^{8/5} - x^2)^{1/2}}$ find $f(g)$ when $g(x) = x^{8/5} - x^2$ and $g(x) = x^{1/5}$.
(b) For $h(x) = \frac{1}{\sqrt{9 - 4x^2}}$ find $f(g)$ when $g(x) = 9 - 4x^2$ and $g(x) = 2x/3$.
(c) For $h(x) = \frac{1}{\sqrt{1 - 9x^2}}$ find $f(g)$ when $g(x) = 1 - 9x^2$ and $g(x) = 3x$.
(d) For $h(x) = (16 - 2x^2)^{-1/2}$ find $f(g)$ when $g(x) = 16 - 2x^2$ and $g(x) = x/\sqrt{8}$.
(e) For $h(x) = \frac{1}{(x^{4/3} + x^{2/3})^2}$ find $f(g)$ when $g(x) = x^{4/3} + x^{2/3}$, $g(x) = x^{2/3}$, and $g(x) = x^{2/3} + 1/2$.
(f) For $h(x) = \frac{1}{(x - x^{4/7})^5}$ find $f(g)$ when $g(x) = x - x^{4/7}$ and $g(x) = x^{4/7}$.
(g) For $h(x) = \frac{4x + x^{1/2} + 1}{4x + x^{1/2} + 5}$ find $f(g)$ when $g(x) = 4x + x^{1/2} + 5$ and $g(x) = 2x^{1/2} + 1/4$. 

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**Now Some Guesswork...**

(4) For each of the following choices of \( f(g) \) and \( g(x) \), sketch the graph of \( h(x) = f(g(x)) \). Be reasonably accurate but don’t nitpick!

- **(a)** \( f(g) = g^{1/2} \) and \( g(x) = \sin(x) \)
- **(b)** \( f(g) = g^{1/3} \) and \( g(x) = \sin(x) \)
- **(c)** \( f(g) = |g| \) and \( g(x) = \sin(x) \)

**At Last, Some Derivatives To Try**

(5) Compute the following derivatives:

- **(a)** \( \frac{d}{ds} s = \)

- **(b)** \( \frac{d}{dx} \left( x^{1/2} + \frac{1}{(2x + 1)^{1/3}} \right) = \)

- **(c)** \( \frac{d}{dx} (x^3 + 5x + 9)^{10} = \)

- **(d)** \( \frac{d}{dx} \left( \frac{6}{x^3 + 2x + 1} \right)^{10} = \)

- **(e)** If \( f(x) = \frac{1}{\sqrt{x^3 + 9x^5}} \) then \( f'(x) = \)

- **(f)** \( \frac{d}{dt} (t^{3/2} + t^{-3/2}) = \)

- **(g)** Find \( F'(2) \) if \( F(x) = 8x^3 + 3x^{-1} \).

- **(h)** If \( D(x) = (A(x) + B(x) + C(x))^2 \) find \( D'(2) \) if \( A(2) = B(2) = 1, C(2) = 2, A'(2) = B'(2) = 1/2, \) and \( C'(2) = 3 \).

- **(i)** Find the equation of the line tangent to \( y = x^3 + 3x^2 + 3 \) at \( x = 1 \).

- **(j)** Find the equation of the line normal to the curve \( y = x^{1/3} \) at \( x = 8 \).

- **(k)** For \( f(x) = \frac{1}{2x - 1} \) find \( f'(x) \) and the second derivative \( f''(x) \).

- **(l)** For \( f(x) \) as in \((k)\) above, find the “third derivative” \( \frac{d^3}{dx^3} f(x) \).

- **(m)** Find \( \left[ \frac{d}{dt} w(t^3 + t^2 + t + 1) \right]_{t=1} \) if \( w'(4) = 6 \).
1.17 SOLUTIONS TO EXERCISE 1.16

(a) For each occurrence of $x$ in $f(x) = 2x^3 + 3$ we substitute the expression $(-x - 1)^3$. Thus we write $h(x) = 2((-x - 1)^3)^3 + 3 = 2(-x - 1)^9 + 3$.

(b) For each occurrence of $t$ in $f(t) = 2t^3 + 3$ we substitute the expression $(-x - 1)^3$. Obviously, we get the same answer as in (a) above. The choice of symbol for the variable in the $f$ function does not affect the resulting function $f(g(x))$.

(c) We get $u(v(x)) = 3(-x^2 + 1)^4 + 2(-x^2 + 1)^3 + 3(-x^2 + 1) + 6$. This function is a polynomial of degree 8 in $x$ and could, if we do some tedious algebra, be written as a sum of powers of $x$ with integer coefficients. There will be situations where this sort of algebra will be important, but not here! The answer is best left in this form in the absence of any direct motivation for doing otherwise.

(d) We have $p(q(x)) = \frac{1}{2(x^{1/2})^3 + 3} = \frac{1}{2x^{3/2} + 3}$.

(e) We substitute $\sqrt{z}$ for each occurrence of $g$ in $\frac{g + 1}{g - 1}$ to obtain $f(g(z)) = \frac{\sqrt{z} + 1}{\sqrt{z} - 1}$. The function $\sqrt{z}$ is defined (as a function from real numbers to real numbers) only for $z$ a nonnegative real number. If $z = 1$ then the denominator of $f(g(z))$ is zero. Thus $f(g(z))$ is defined for all nonnegative real numbers $z$ except $z = 1$.

Compositional Inverses Will Haunt You Later

(f) First, $h(k(y)) = (y^{1/2})^2 = y$. In reverse order we find $k(h(y)) = (y^2)^{1/2} = y$. The function $i(y) = y$ is the "identity" function (it does nothing to $y$). The function $i(y)$ is the linear function with slope 1 passing through the origin. In general, functions $h(y)$ and $k(y)$ such that $h(k(y)) = k(h(y)) = y$ for all $y$ are called "compositional inverses of each other." We must be a little bit careful about the statement "for all $y" in this definition. In our example, the function $h(y) = y^2$ is defined for all $y$ but the function $k(y) = y^{1/2}$ is defined only for nonnegative real numbers (we are ignoring complex numbers at this point). Thus, the composition $k(h(y))$ is defined for all real numbers but the composition $h(k(y))$ is defined only for nonnegative real numbers. Tech-
nically, the statement $h(k(y)) = k(h(y))$ can be made only for non-negative real numbers and hence $h$ and $k$ are compositional inverses for all nonnegative real numbers. In general, when we say that two functions $h$ and $k$ are compositional inverses we mean that $h(k(y)) = k(h(y)) = y$ for all $y$ in some specified common domain of definition of $h$ and $k$. In our example this "common domain of definition" is the set of all nonnegative real numbers.

(g) As in (f) above, $h(k(z)) = k(h(z)) = z$. In this case, however, this relation is defined for all real numbers $z$ because $z^{1/3}$ is defined for all real numbers and, of course, so is $z^3$. In general, if $p$ is an odd integer then $z^p$ and $z^{-p}$ are compositional inverses for all $z$. If $p$ is even (and nonzero) they are compositional inverses for all nonnegative $z$.

(2) (a) $f(g) = g^{1/3} + g^{-1/3}$ and $g(x) = x^2 + 1$ is the most natural choice. Also $f(g) = g^{2/3} + g^{-2/3}$ and $g(x) = (x^2 + 1)^{1/2}$, $f(g) = g + g^{-1}$ and $g(x) = (x^2 + 1)^{1/3}$, etc. will work. There are infinitely many possibilities.

(b) $h(x) = (x + 1)^3 + (x + 1)^2$ so we may take $f(g) = g^3 + g^2$ and $g(x) = x + 1$ as the most natural choice.

**Remember This Trick! Completing The Square**

(c) The numerator in $h(x)$ is the expression $x^2 + 2x$. This expression is the sum of the first two terms of $(x + 1)^2 = x^2 + 2x + 1$ and thus can be written as $x^2 + 2x = (x + 1)^2 - 1$. This little trick is called "completing the square." It can be applied to any expression $ax^2 + bx$ by observing that $(\sqrt{a}x + b/2\sqrt{a})^2 = ax^2 + bx + b^2/4a$ and hence that

$$ax^2 + bx = (\sqrt{a}x + b/2\sqrt{a})^2 - b^2/4a.$$  

You should learn this trick thoroughly and try a number of examples until you feel comfortable with it! Applying it to our immediate problem gives

$$f(g) = \frac{g^2 - 1}{g^5} = g^{-3} - g^{-5}$$

and $g(x) = x + 1$.

(d) Using the "complete the square" trick described in (c) above, we replace $x^2 + 2x$ by $(x + 1)^2 - 1$ in both the numerator and denominator of $h(x)$. Doing this we see that, in the case where $g(x) = x + 1$, we have

$$f(g) = \frac{g^2 + 4}{g^2 + 5}.$$  

If $g(x) = x^2 + 2x + 6$ then $f(g) = (g - 1)/g$ or $1 - g^{-1}$. If $g(x) = x^2 + 2x + 5$ then $f(g) = g/(g + 1)$.  

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Integration By Substitution Is What We'll Call It Later...

(3) As a general remark about this problem, the CHAIN RULE is very important in both differential calculus (that's what we are studying now) and integral calculus (we'll study that a little later). The tricks for writing a given function $h(x)$ as $f(g(x))$ are a little different in these two subjects. That's the motivation for the different choices for $g(x)$ in these problems.

(a) For $g(x) = x^{8/5} + x^2$, $f(g) = g^{-1/2}$. For $g(x) = x^{1/5}$, $f(g) = g^{-4}(1 - g^2)^{-1/2}$.

(b) The notation $\frac{1}{\sqrt{9 - 4x^2}}$ is a bad one and should be replaced by $\frac{1}{(9 - 4x^2)^{1/2}}$ or simply $(9 - 4x^2)^{-1/2}$. When $g(x) = 9 - 4x^2$ then $f(g) = g^{-1/2}$. When $g(x) = (9 - 4x^2)^{1/2}$ then $f(g) = g^{-1}$. When $g(x) = 2x/3$ then $x = 3g/2$ and thus $(9 - 4x^2)^{-1/2} = (9 - 4(3g/2)^2)^{-1/2} = (9 - 9g^2)^{-1/2} = \frac{1}{3}(1 - g^2)^{-1/2} = f(g)$. This is an important trick in "integral calculus."

(c) When $g(x) = 1 - 9x^2$ then $f(g) = g^{-1/2}$. If $g(x) = 3x$ then $x = g/3$ so $(1 - 9x^2)^{-1/2} = (1 - 9(g/3)^2)^{-1/2} = (1 - g^2)^{-1/2}$. Compare this latter "change of variable" with that of (b) above.

(d) When $g(x) = 16 - 2x^2$ then $f(g) = g^{-1/2}$. When $g(x) = x/\sqrt{8}$ then $x = \sqrt{8}g$ and $(16 - 2x^2)^{-1/2} = (16 - 2(\sqrt{8}g)^2)^{-1/2} = \frac{1}{4}(1 - g^2)^{-1/2}$.

(e) When $g(x) = x^{4/3} + x^{2/3}$ then $f(g) = g^{-2}$. When $g(x) = x^{2/3}$ then $x = g^{3/2}$ and $f(g) = (g^2 + g)^{-2}$. Completing the square we find that $f(g) = ((g + 1/2)^2 - 1/4)^2$. If $g = x^{2/3} + 1/2$ then $f(g) = (g^2 - 1/4)^{-2}$.

(f) When $g(x) = x - x^{6/7}$ then $f(g) = g^{-5}$. When $g(x) = x^{1/7}$ then $f(g) = (g^7 - g^4)^{-5} = g^{-20}(g^4 - 1)^{-5}$.

(g) If $g(x) = 4x + x^{1/2} + 5$ then $f(g) = \frac{g - 4}{g} = 1 - 4g^{-1}$. For the second part, note that $4x + x^{1/2} = (2x^{1/2} + 1/4)^2 - 1/16$. Here we are again completing the square using $x^{1/2}$ as the variable. Thus for $g(x) = 2x^{1/2} + 1/4$ we get $f(g) = \frac{g^2 + 15/16}{g^2 + 79/16}$. 

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When Graphing, Do A Rough Approximation First

(4)

(5) Be sure to try the VARIATIONS for additional practice in taking derivatives.

(a) \frac{ds}{dt} = \frac{dx}{dy} = \frac{dy}{dx} = \frac{dy}{dw} = \frac{dy}{dx} = \ldots = 1.

(b) \frac{d}{dx} \left( x^{1/2} + \frac{1}{(2x + 1)^{1/3}} \right) = \frac{d}{dx} x^{1/2} + \frac{d}{dx} (2x + 1)^{-1/3} =
\frac{1}{2}x^{-1/2} + \frac{1}{3} (2x + 1)^{-4/3}(2) = \frac{1}{2}x^{-1/2} - \frac{2}{3}(2x + 1)^{-4/3}.
The second term was computed using the CHAIN RULE applied to f(g(x)) with f(g) = g^{-1/3} and g(x) = 2x + 1 (and hence g'(x) = 2).

(c) Apply the CHAIN RULE to f(g(x)) where f(g) = g^{10} and g(x) = x^3 + 5x + 9. The answer is 10(x^3 + 5x + 9)^9(3x^2 + 5).

(d) We can write this as \(6^{10}\frac{d}{dx}(x^3 + 2x + 1)^{-10}\) and use the CHAIN RULE for f(g(x)) with f(g) = g^{-10} and g(x) = x^3 + 2x + 1 to obtain 6^{10}(-10)(x^3 + 2x + 1)^{-11}(3x^2 + 2)). Another, and probably more common, way a calculus student would attempt to work this problem

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is to apply the CHAIN RULE with \( g(x) = \frac{6}{x^3 + 2x + 1} \) and \( f(g) = g^{10} \) to get
\[
10 \left( \frac{6}{x^3 + 2x + 1} \right)^9 \frac{d}{dx} \left( \frac{6}{x^3 + 2x + 1} \right).
\]
The latter derivative \( \frac{d}{dx} \left( \frac{6}{x^3 + 2x + 1} \right) \) would then be computed with a second application of the CHAIN RULE. The reader should carry this out and verify that the answer is the same as the one just given.

(e) We should first get rid of the awkward square root notation and write this function as \((x^3 + 9x^5)^{-1/2}\). Use the CHAIN RULE with \( g(x) = x^3 + 9x^5 \) and \( f(g) = g^{-1/2} \) to get \((-1/2)(x^3 + 9x^5)^{-3/2}(3x^2 + 45x^4)\).

(f) The answer is \((3/2)t^{1/2} + (-3/2)t^{-5/2}\).

(g) \( F'(x) = 24x^2 + 3(-1)x^{-2} = 24x^2 - \frac{3}{x^2} \) and thus \( F'(2) = 96 - 3/4 = 95.25 \).

(h) We again use the CHAIN RULE with \( f(g) = g^2 \) and \( g(x) = A(x) + B(x) + C(x) \) so that \( D(x) = f(g(x)) \). Thus, \( D'(x) = 2(A(x) + B(x) + C(x))(A'(x) + B'(x) + C'(x)) \) and \( D'(2) = 2(1 + 1 + 2)(1/2 + 1/2 + 3) = 32 \).

(i) If you look at Figure 1.9(a), the line \( \hat{g} \) is tangent to the function \( g \) at the point (3, 2) on the graph of \( g \). In general, if \( g(x) \) is a function that has a derivative at the point \( P = (x, g(x)) \) then the tangent to \( g \) at \( P \) is the unique straight line passing through \( P \) and having slope \( g'(x) \). In this problem, we are asked to find the tangent to \( y(x) = x^3 + 3x^2 + 3 \) at the point \( x = 1 \). This is a typical way of stating this sort of problem in which the statement ‘‘at \( x = 1 \)’’ really means ‘‘at the point \( (1, y(1)) = (1, 7) \).’’ We must find the equation of the straight line passing through the point (1, 7) and having slope \( y'(1) = 9 \). This is the line \( y = 9x - 2 \).

(j) The normal line to a curve at a point \( P \) is the line passing through \( P \) and perpendicular to the tangent line to the curve at \( P \). In this problem, \( P = (8, 2) \). The tangent line to \( y = x^{1/3} \) at \( P \) has slope \( y'(8) = (1/3)8^{-2/3} = 1/12 \) and its equation is given by \( y = (1/12)x + 4/3 \). You should know that the slope of the line normal to a line of slope \( m \) is \( -1/m \). Thus the normal line we are looking for has slope \(-12\) and
passes through the point (8, 2). Its equation is \( y = -12x + 98 \), which is the answer to this problem.

**k** Write \( f(x) = (2x - 1)^{-1} \). We find that \( f'(x) = (-1)(2x - 1)^{-2}(2x - 1)' = -2(2x - 1)^{-2} \). The function \( f'(x) \) can again be differentiated. The first derivative of \( f'(x) \) is called the second derivative of \( f(x) \) and is denoted by \( f''(x) \) or by \( \frac{d^2}{dx^2} \). In this problem, we compute \( f''(x) = (-2(2x - 1)^{-2})' = (-2)(-2)(2x - 1)^{-3}(2x - 1)' = 8(2x - 1)^{-3} \).

**l** The third derivative of \( f(x) \) is the derivative of \( f''(x) \). The third derivative is denoted by \( f'''(x) \) or by \( \frac{d^3}{dx^3} \). In this problem we compute \( (8(2x - 1)^{-3})' = (8)(-3)(2x - 1)^{-4}(2x - 1)' = -48(2x - 1)^{-4} \).

In general, the \( n \)th derivative of \( f(x) \) is obtained by differentiating \( f(x) \) \( n \) times. The \( n \)th derivative is denoted by \( f^{(n)}(x) \) or by \( \frac{d^n}{dx^n}f(x) \). Can you give a formula for \( \frac{d^n}{dx^n}(2x + 1)^{-1} \)? In general, it will happen that a function with a first derivative at a point may not have an \( n \)th derivative at that point for some \( n \). For example, \( f(x) = x^{-3} \) has first derivative function \( f'(x) = (4/3)x^{1/3} \) and second derivative \( f''(x) = (4/9)x^{-2/3} \). At \( x = 0 \) we have \( f'(0) = 0 \) but \( f''(0) \) is not defined.

**m** This problem illustrates the "bracket notation" for the two-step process of first computing a derivative function and second evaluating that function at a certain value. The notation \( \left[ \frac{d}{dx}f(x) \right]_{x=a} \) means first compute \( f'(x) \) and then evaluate \( f'(a) \). Some students like to do it the other way around by first finding \( f(a) \) and then computing the derivative with respect to \( x \) of the constant \( f(a) \). The answer is of course always zero! This is not the way to go. In our particular problem, we compute

\[
\frac{d}{dt}w(t^3 + t^2 + t + 1) = w'(t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)'
= w'(t^3 + t^2 + t + 1)(3t^2 + 2t + 1).
\]

Substituting \( t = 1 \) this becomes \( w'(4)(6) = 6w'(4) \). We are using the CHAIN RULE without knowing explicitly what \( w \) is (sometimes this is a very useful trick). We could go no further except for the fact that we have (conveniently!) been given that \( w'(4) = 6 \) so the final answer is 36.
Chapter 2

COMPUTING DERIVATIVES

We Must Enlarge The Class Of Functions We Can Differentiate

In Chapter 1 we developed the basic conceptual ideas of calculus. Our approach is mostly intuitive, but EXERCISE 1.16 should give a hint of the power of these ideas. In this chapter we shall concentrate on techniques of differentiation. Our first task is to enlarge the class of basic functions that we know how to differentiate. We shall also learn a few more rules for differentiation.

To review, the keystone of our approach to derivatives is an understanding of RULES FOR DERIVATIVES 1.12, particularly the all-important CHAIN RULE. In EXAMPLE 1.15(5) we stated without proof the rule \((x^r)' = rx^{r-1}\). This important rule should be memorized and is valid for any real number \(r\).

In EXAMPLE 1.15(8) we used this rule together with the CHAIN RULE to state the rule \(((g(x))^r)' = r(g(x))^{r-1}g'(x)\), valid for any differentiable function \(g\). This rule is a very common special case of the CHAIN RULE. Another very useful rule for computing derivatives was derived in EXAMPLE 1.15(9). This rule is called the PRODUCT RULE and states that \((f(x)g(x))' = f'(x)g(x) + f(x)g'(x)\). If you have not carefully studied EXERCISE 1.16 you should do so now.

As our first task in this chapter, we shall prove the validity of the rule \((x^r)' = rx^{r-1}\). Our proof consists of a series of very short “lemmas” dealing with special cases of this formula. By piecing together these lemmas, we obtain a proof of the general result. A student interested only in the techniques
of differentiation could memorize the result \((x^r)' = rx^{r-1}\) and skip the proof of this formula. In this particular case, however, we have chosen our lemmas such that they themselves illustrate important ways of applying our rules of differentiation. For this reason it is probably worthwhile that all students study this series of lemmas.

**Now We Prove That \((x^r)' = rx^{r-1}\) For Any \(r\)**

### 2.1 LEMMA
For any nonnegative integer \(n\), \((x^n)' = nx^{n-1}\).

**Proof:** For \(n = 0\), we have \((x^0)' = (1)' = 0 = 0x^0 = 0x^{-1}\). Technically, \(0x^{-1}\) is not defined for \(x = 0\), but we interpret \(0x^{-1}\) to be the function which is zero for all real numbers (the zero function). For \(n = 1\) we have \((x^1)' = (x)' = 1 = 1x^0 = 1x^0\). We explained in FIGURE 1.13 why \((x^2)' = 2x\). Thus we know that the formula \((x^n)' = nx^{n-1}\) is valid for \(n = 0, 1, \text{ and } 2\). The proof for general \(n\) is by induction. Assume that \((x^{n-1})' = (n - 1)x^{n-2}\) is known to be true. Write \(x^n = x^{n-1}x\) and use the product rule with \(f(x) = x^{n-1}\) and \(g(x) = x\). We obtain \((x^n)' = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) = (n - 1)x^{n-2}x + x^{n-1}(1) = nx^{n-1}\). This proves the formula \((x^n)' = nx^{n-1}\) for all nonnegative integers \(n\).

**If \(h(x) = f(g(x))\), Knowing \(h'\) And \(f'\) Gives \(g'\)**

### 2.2 LEMMA
Let \(m\) be a positive integer and let \(r = 1/m\). Then, \((x^r)' = rx^{r-1}\).

**Proof:** Here is a trick worth remembering! We apply the CHAIN RULE to \(h(x) = f(g(x))\) where \(g(x) = x^{1/m}\) and \(f(x) = x^m\). Thus the formula \(h(x) = f(g(x))\) becomes \(x = (x^{1/m})^m\). The CHAIN RULE \(h'(x) = f'(g(x))g'(x)\) becomes \(1 = m(x^{1/m})^{m-1}(x^{1/m})'\). Solving for \((x^{1/m})'\) we obtain \((x^{1/m})' = (1/m)x^{(1/m)-1}\), which is the identity to be proved.

The trick we used in LEMMA 2.2 is an important one. We applied the CHAIN RULE to \(h(x) = f(g(x))\) knowing \(f'(x)\) and \(h'(x)\) but not knowing \(g'(x)\). The identity \(h'(x) = f'(g(x))g'(x)\) is then used to solve for \(g'(x)\).

### 2.3 LEMMA
Let \(m\) and \(n\) be positive integers and let \(r = n/m\). Then, \((x^r)' = rx^{r-1}\).

**Proof:** We apply the CHAIN RULE to \(h(x) = f(g(x))\) where \(h(x) = x^{n/m}\), \(f(x) = x^m\), and \(g(x) = x^{1/m}\). The identity \(h(x) = f(g(x))\) becomes \(x^{n/m} =
(x^{1/m})^n. Applying the CHAIN RULE, h'(x) = f'(g(x))g'(x) becomes (x^{n/m})' = n(x^{1/m})^{n-1}(x^{1/m})'. Using the identity (x^{1/m})' = (1/m)x^{(1/m)-1} from LEMMA 2.2, we obtain (x^{n/m})' = (n/m)x^{(n/m)-1}, which was to be proved.

Irrational Numbers Are Approximated By Rationals

At this point we have shown that the formula (x^r)' = rx^{r-1} is valid for any “nonnegative rational number r.” A rational number is a number that can be expressed as a ratio of two integers n/m. For example, 2/3, 0/1, 257/1011, −23/45, and 34/343 are rational numbers. The number −23/45 is a negative rational number, the others are nonnegative. Every integer is a rational number. The number π is known to be an irrational number (to prove this takes some work). The number √2 is irrational (this is easy to show). An irrational number such as π can be approximated to within any degree of accuracy by rational numbers. For example, π can be approximated by 3.14, 3.14159, 3.1415926, 3.141592653589793238, . . . . Numbers such as 3.14159 (terminating decimal expansions) are rational numbers because they can be written as ratios of integers (3.14159 = 314159/100000, for example). Thus, by what we have shown, if r = 3.14159 then (x^r)' = rx^{r-1}. If you think about the geometric meaning of the derivative as the slope of a curve at a point, you can see that (x^π)' = πx^{π-1} since this formula must hold for all rational approximations to π. The same argument works for any nonnegative irrational number s, not just π, so we must have (x^s)' = sx^{s-1} for all nonnegative real numbers. To treat the case of negative exponents we need the next lemma.

2.4 LEMMA Let r = −1. Then (x^r)' = rx^{r-1}.

Proof: Apply the product rule to h(x) = f(x)g(x) with h(x) = 1, f(x) = x^{-1}, and g(x) = x. The identity h'(x) = f'(x)g(x) + f(x)g'(x) becomes 0 = (x^{-1})' x + (x^{-1})(1). Solving for (x^{-1})' gives (x^{-1})' = (-1)x^{-2}, which was to be shown.

By putting LEMMA 2.1 to 2.4 together we obtain the result we were seeking, THEOREM 2.5.

2.5 THEOREM For any real number r, (x^r)' = rx^{r-1}.

Proof: We know the result is true for any nonnegative real number s. Suppose r = −s is a negative real number. Then x^r = (x^s)^{-1} and by LEMMA 2.4 and the CHAIN RULE.
\[(x^r)' = (-1)(x^{r-2})x^{r-1} = (-1)x^{-2}sx^{s-1} = (-s)x^{-s-1} = rx^{-r-1}\]
as was to be shown.

**The Quotient Rule**

We remind the reader again of the principal application of THEOREM 2.5 as described in EXAMPLE 1.15(8), namely, \((g(x))' = r(g(x))^{r-1}g'(x)\) for any differentiable function \(g(x)\). Using this latter formula together with the product rule, we can easily obtain our final general rule of differentiation, the "QUOTIENT RULE." Consider the quotient \(\frac{f(x)}{g(x)}\). We shall derive a formula for computing the derivative \(\frac{d}{dx} \frac{f(x)}{g(x)}\). First, we write \(\frac{d}{dx} \frac{f(x)}{g(x)} = (f(x)(g(x))^{-1})'\). This is just a change of notation. Now, apply the product rule to this latter expression to obtain \(f'(x)(g(x))^{-1} + f(x)(-1)(g(x))^{-2}g'(x)\). Putting this latter expression over a common denominator \((g(x))^2\), we obtain

\[
\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.
\]

The above formula is called the "QUOTIENT RULE" for obvious reasons. The QUOTIENT RULE should be memorized. Here are some other ways the QUOTIENT RULE is commonly written:

\[
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}
\]

\[
\frac{d}{dx} \left(\frac{f}{g}\right) = \frac{df}{dx}g - f\frac{dg}{dx} g^2
\]

\[
\frac{d}{dx} \left(\frac{f}{g}\right) = \frac{(df)g - f(dg)}{g^2}
\]

As already stated, you should memorize one of these forms of the QUOTIENT RULE. The last one above is the most abbreviated form. Each occurrence of "d" needs to be "divided by" \(dx\) to get the previous form. This is called "differential notation." At this stage we can think of this simply as a memory aid for remembering the QUOTIENT RULE. Said in words, the differential form of the QUOTIENT RULE sounds like "dee f over g is dee f times g minus f times dee g divided by g squared." This may or may not help you remember the QUOTIENT RULE!
2.6 EXAMPLES OF THE QUOTIENT RULE

(1) Let’s try to differentiate \( \frac{(x^2 + 1)^{1/2}}{(x^3 + 2)^{2/3}} \). We use the QUOTIENT RULE

\[
\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\]

where \( f(x) = (x^2 + 1)^{1/2} \) and \( g(x) = (x^3 + 2)^{2/3} \). Using the CHAIN RULE, we find

\[
f'(x) = (1/2)(x^2 + 1)^{-1/2}(2x) = x(x^2 + 1)^{-1/2}
\]

and

\[
g'(x) = (2/3)(x^3 + 2)^{-1/3}(3x^2) = 2x^2(x^3 + 2)^{-1/3}.
\]

Thus,

\[
\left( \frac{f(x)}{g(x)} \right)' = \frac{x(x^2 + 1)^{-1/2}(x^3 + 2)^{2/3} - (x^2 + 1)^{1/2}2x^2(x^3 + 2)^{-1/3}}{(x^3 + 2)^{4/3}}.
\]

(2) You may or may not know that \( \frac{d}{dx} \sin(x) = \cos(x) \). It’s true and we shall see why below. It is also true that \( \frac{d}{dx} \cos(x) = -\sin(x) \). We can use these facts together with the QUOTIENT RULE to compute \( \frac{d}{dx} \tan(x) = \frac{\frac{d}{dx} \sin(x)}{\frac{d}{dx} \cos(x)} \) . We use the QUOTIENT RULE in the form

\[
\frac{d}{dx} \left( \frac{f}{g} \right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}
\]

where \( f(x) = \sin(x) \), \( g(x) = \cos(x) \), \( \frac{df}{dx} = \cos(x) \), and \( \frac{dg}{dx} = -\sin(x) \). Substituting these expressions into the QUOTIENT RULE above, we obtain

\[
\frac{d}{dx} \tan(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2}.
\]
Recalling that \((\cos(x))^2 + (\sin(x))^2 = 1\) for all \(x\), we have that
\[
\frac{d}{dx}\tan(x) = (\cos(x))^{-2}.
\]

**Memorize The Basic Rules**

At this point we have learned all of the general rules for differentiation that are required of the beginning calculus student. We now summarize these rules. You should memorize them all!

**2.7 DIFFERENTIATION RULES TO MEMORIZE**

1. **CHAIN RULE**
   \[
   (f(g(x)))' = f'(g(x))g'(x)
   \]
   \[
   \frac{d}{dx}f(g) = \frac{df}{dg} \frac{dg}{dx}
   \]

2. **PRODUCT RULE**
   \[
   (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)
   \]
   \[
   \frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f
   \]

3. **QUOTIENT RULE**
   \[
   \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
   \]
   \[
   \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}
   \]

We shall work a number of problems below in order to practice RULES 2.7. Before doing so, however, it is important that we enlarge the class of basic functions that we know how to differentiate. We shall now learn how to differentiate two important classes: the “trigonometric” functions and the “exponential” functions.

**Derivative Of \(\sin(x)\) Is \(\cos(x)\)**

Look at FIGURE 2.8. There we see a circle of radius one. The angle POQ is defined by specifying the point Q on the circle. As indicated in FIGURE
2.8, the angle POQ intercepts an arc of length $x$ on the circle of radius 1. We say, therefore, that the "measure of the angle POQ is $x$ radians." The reader who feels uneasy about measuring angles in radians rather than degrees should consult a precalculus or trigonometry book for some review. Unless otherwise stated, we shall always measure angles in radians (this is common practice in calculus). Recall that by definition the coordinates of the point $Q$ are $(\cos(x), \sin(x))$. The point $\sin(x)$ corresponding to the second coordinate of $Q$ is shown on the vertical axis of FIGURE 2.8. We want to compute the derivative $(\sin(x))'$ using ERUDITE OBSERVATION 1.14. Thus we shall first compute the "difference quotient" \[
\frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}
\] and try to understand its value for small values of $\Delta x$.

Now is where FIGURE 2.8 really comes in handy! Note in FIGURE 2.8 that we have gone a little further along the circle from the point $Q$ to a point $O'$, distance $\Delta x$ further to be exact. If $\Delta x$ is small then the arc of the circle from $Q$ to $O'$ looks like a straight line. Pretend it is a straight line in FIGURE 2.8. Note that the "segment" $O'Q$ is the hypotenuse of the little right triangle $O'P'Q$. This little right triangle is similar to the big right triangle $OPQ$. 

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(one way to see this is to observe that the segment O'P' is perpendicular to OP and O'Q is perpendicular to QQ). Thus, the angle P'O'Q also has measure x radians. Referring to FIGURE 2.8, again, we see that

$$\frac{\sin(x) - \sin(x)}{\Delta x} = \frac{\text{length(O'P')}}{\text{length(O'Q)}} \approx \cos(x).$$

Actually, the above formula is not quite correct because the "segment" O'Q is not quite a straight line segment. The smaller $\Delta x$ is, however, the closer O'Q is to being a straight line segment. Mathematicians express this fact by saying

$$\lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} = (\sin(x))' = \cos(x).$$

We formally state this important (and somewhat amazing!) result in THEOREM 2.9.

2.9 THEOREM Let $f(x) = \sin(x)$ where x is measured in radians. Then, $f'(x) = \cos(x)$.

By inspecting FIGURE 2.8 you will easily see that $\sin(x + \pi/2) = \cos(x)$ for all x. We can think of this fact as $\cos(x) = f(g(x))$ where $f(x) = \sin(x)$ and $g(x) = x + \pi/2$. Applying the CHAIN RULE gives $(\cos(x))' = f'(g(x))g'(x) = \cos(x + \pi/2) (x + \pi/2)' = \cos(x + \pi/2).$ Again, by looking at FIGURE 2.8 you can easily see that $\cos(x + \pi/2) = -\sin(x)$. This proves that

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

By combining the above rules for differentiating $\sin(x)$ and $\cos(x)$ with the rule for differentiating $\tan(x)$ given in EXAMPLE 2.6(2), we obtain the BASIC TRIGONOMETRIC DERIVATIVES 2.10. These formulas should be memorized.

**Memorize These Trigonometric Derivatives**

2.10 BASIC TRIGONOMETRIC DERIVATIVES

$$\frac{d}{dx} \sin(x) = \cos(x)$$
\[
\frac{d}{dx} \cos(x) = -\sin(x)
\]
\[
\frac{d}{dx} \tan(x) = (\cos(x))^{-2}
\]

**Exponential And Logarithmic Derivatives**

The second class of functions that we must learn how to differentiate are the exponential functions. In particular, we shall look at functions \( f_a(x) = a^x \) where \( a \) is a real number, \( a > 1 \). The first thing to notice about exponential functions is that they get large very fast. Consider \( f_2(x) = 2^x \) compared with our old friend \( x^2 \). At \( x = 2 \) these two functions have the same value. At \( x = 16, 2^{16} = 65,536 \) but \((16)^2 \) is only 256. By the time \( x = 50, 2^{50} \) is greater than \( 10^{15} \) while \((50)^2 \) is just 2500. For negative values of \( x \) we note that \( a^x \) tends to zero as \( x \) tends to minus infinity. Check the values of \( 2^{-2}, 2^{-10}, \) and \( 2^{-50} \). The constant \( a \) is called the “base” of the exponential function \( a^x \). We have assumed that \( a > 1 \). For all such \( a, a^x \) gets large (“goes to infinity”) as \( x \) gets large. The bigger the value of \( a, \) the more rapidly \( a^x \) gets large. Take a look at FIGURE 2.11, which shows the graphs of the three functions \( f_{10}(x) = 10^x, f_{2.7}(x) = (2.7)^x, \) and \( f_{1.2}(x) = (1.2)^x \). A careful understanding of FIGURE 2.11 will be very important for our discussion of exponential functions.

**They All Pass Through (0,1)**

In FIGURE 2.11, we are looking at the graphs of our three functions in the interval \( -0.7 < x < +0.7 \). All three of these graphs pass through the point \((0,1)\) as shown. This is true for any exponential function: \( f_a(0) = a^0 = 1 \). One question that will occur to the reader is “How were the graphs of FIGURE 2.11 computed?” These graphs were computed on a personal computer using BASIC. The command PRINT X ^ Y will print \( X^Y \) for numbers \( X \) and \( Y \). Of course the numbers \( X \) and \( Y \) must be rational numbers of a size acceptable to the given computer or hand calculator you are using. How does one program a computer to evaluate strange expressions such as \((5.987241)^{3.98713}\)? This is a technical question about numerical methods that is beyond the scope of this book. To get some feeling for this sort of thing, note that a number such as \( a^{3.987123} \) can be written as \( a^3 \times a^{0.987123} \). You already know how to compute \( a^3 \) for any rational number.
FIGURE 2.11 Three Exponential Functions

\[ f_{10}(x) = 10^x \]
\[ f_{2.7}(x) = 2.7^x \]
\[ f_{1.2}(x) = 1.2^x \]
\[ f'_{10}(0) = 2.30 \]
\[ f'_{2.7}(0) = 0.99 \]
\[ f'_{1.2}(0) = 0.18 \]
At $(0,1)$, $y = (2.7)^x$ Has Slope Almost 1

Look again at FIGURE 2.11. Although $f_a(0) = 1$ for all $a$, the slope $f'_a(0)$ increases as $a$ increases. For example, $f'_{1.2}(0) = 0.18$ while $f'_{10}(0) = 2.30$ (both values are approximate). A little thought reveals that for $1.2 < a < 10$, $f'_{1.2}(0) < f'_a(0) < f'_{10}(0)$. Thus as the base $a$ varies between 1.2 and 10, $f'_a(0)$ varies between 0.18 and 2.3. To the mathematician, this means that there is some particular number $a$ between 1.2 and 10 such that $f'_a(0) = 1$. As a “guess” at this number, we tried $a = 2.7$ in FIGURE 2.11. This turns out to be a pretty good guess as $f'_2(0) = 0.99$. Here are some attempts to improve on this guess:

<table>
<thead>
<tr>
<th>Value of $a$</th>
<th>Value of $f'_a(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.70000000</td>
<td>.993251773</td>
</tr>
<tr>
<td>2.71000000</td>
<td>.996948635</td>
</tr>
<tr>
<td>2.71800000</td>
<td>.999896315</td>
</tr>
<tr>
<td>2.71820000</td>
<td>.999969897</td>
</tr>
<tr>
<td>2.71828000</td>
<td>.9999999328</td>
</tr>
<tr>
<td>2.71828182</td>
<td>.999999997</td>
</tr>
</tbody>
</table>

So what is this mysterious number $a$ such that $f'_a(0) = 1$? From the above, $a = 2.71828182$ is extremely close to this number. Mathematicians give this mysterious number the name “e” as specified in DEFINITION 2.12.

The Irrational Number $e = 2.71828182...$

2.12 DEFINITION Let $f_a(x) = a^x$ where $a > 1$. The unique number $a$ such that $f'_a(0) = 1$ is denoted by $e$. Thus, $f'_e(0) = 1$. The value of $e$ is about 2.7. For most practical purposes, $e = 2.71828182$.

In fact, mathematicians have shown that the number $e$ is irrational and hence can never be represented by a terminating or repeating decimal expansion. This means that the exact value of $e$ can never be stored in a computer (only rational numbers can be stored in a computer).

The Compositional Inverse of $a^x$ Is $\log_a(x)$

In CHAPTER 1, in connection with 1.17(f) and (g) (SOLUTIONS TO EXERCISE 1.16), we discussed the idea of “compositional inverses.” The compositional inverse of the function $f_a(x) = a^x$ is denoted by $\log_a(x)$ and is called the “‘logarithm base a” function. The idea is very simple and is illustrated in FIGURE 2.13. First, compare FIGURE 2.13 with FIGURE 2.11.
FIGURE 2.13 Logarithmic Functions

Note that the graphs shown in FIGURE 2.13 and FIGURE 2.11 are the same ($10^x$, $(2.7)^x$, and $(1.2)^x$). Remember, that the function $a^x$ is positive for all
values of $x$. Look at FIGURE 2.13 and pick some point on the vertical axis, say 2.2 as shown. The horizontal distance from that point to the curve $a^y$ is by definition $\log_a(2.2)$. In FIGURE 2.13 the horizontal distance from the point 2.2 to the curve $10^y$ is shown and seems to be about 0.34. Thus, by definition, $\log_{10}(2.2) = 0.34$. At the point 1.5 on the vertical axis the horizontal distance to the curve $(2.7)^y$ is shown and is about 0.41. This means that $\log_{2.7}(1.5) = 0.41$. As 2.7 is approximately $e$, we have written $\log_e(1.5)$ for $\log_{2.7}(1.5)$ in FIGURE 2.13. In general, for any positive real number $y$, to compute $\log_a(y)$ graphically, find $y$ on the vertical axis and go horizontally to the curve $a^y$. This horizontal directed distance is by definition $\log_a(y)$. By "directed" distance we mean distances to the left are negative. Thus, referring to FIGURE 2.13, $\log_{10}(0.5) = -0.3$. For any nonnegative real number $y$, the point in the plane with coordinates $(\log_a(y), y)$ is by definition on the curve $f_a(x) = a^x$. We summarize these ideas in DEFINITION 2.14.

**Definition Of $\log_a(x)$**

2.14 DEFINITION

Let $f_a(x) = a^x$ where $a > 1$. For each $y > 0$ define $\log_a(y)$ to be the unique real number such that $(\log_a(y), y)$ is on the graph of $a^x$.

Look once again at FIGURE 2.13. Note that $10^{\log_{10}(2.2)} = 2.2$. Taking $\log_{10}(2.2) = 0.34$, note also that $\log_{10}(10^{-0.3}) = 0.34$. In general, we have the following basic properties of $\log_a(x)$ and $a^x$.

2.15 COMPOSITIONAL PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS Let $f_a(x) = a^x$, $a > 1$, and let $g_a(x) = \log_a(x)$ where $f_a$ is defined for all $x$ and $g_a$ is defined for all positive $x$. Then

1. For all $x$, $g_a(f_a(x)) = \log_a(a^x) = x$
2. For all positive $x$, $f_a(g_a(x)) = a^{\log_a(x)} = x$.

$log_a(a^x) = x$, $a^{\log_a(x)} = x$

The reader should, using FIGURE 2.13, check the validity of COMPOSITIONAL PROPERTIES 2.15 for various values of $x$. When checking $\log_a(a^x) = x$ start with $x$ on the horizontal axis and when checking $a^{\log_a(x)} = x$ start with $x$ on the vertical axis. The graphs shown in FIGURE 2.13 were initially drawn for the three functions $10^x$, $(2.7)^x$, and $(1.2)^x$ but they can equally well be regarded as the graphs for $\log_{10}(x)$, $\log_{2.7}(x)$, and $\log_{1.2}(x)$ if we understand that the dependent variable $x$ now ranges along the
positive vertical axis. Of course, tradition has it that when graphing a function the horizontal axis represents the dependent variable. Look at the stick figure SALLY in FIGURE 2.13. Imagine that she is slightly behind the page and is viewing the three graphs as logarithmic functions of the dependent variable represented on the vertical axis. For her, the dependent variable axis is oriented “properly” with values increasing to her right and decreasing to her left. Her view of the three logarithmic functions is shown in the upper left-hand corner. You should note carefully the shape of these functions and fix their general structure and relationship with each other in your mind.

**Derivatives Of a^x And log_a(x)**

We now must learn how to compute the derivatives of the two functions $a^x$ and $\log_a(x)$. Using the trick we learned in LEMMA 2.2, it will be very easy to find formulas for these derivatives. Before we give proofs, however, we should take an advance look at the answers to make sure we understand the concepts involved. Recall the number $e$ of DEFINITION 2.12 ($e$ is about 2.7). Here are the derivatives of $a^x$ and $\log_a(x)$:

$$\frac{d}{dx} a^x = \log_e(a)a^x$$

$$\frac{d}{dx} \log_a(x) = \frac{1}{(\log_e(a))x}.$$

It is obvious from the above two expressions for the derivatives of $a^x$ and $\log_a(x)$ that we need to be able to compute the numbers of the form $\log_e(a)$ for the various possible bases $a$. For values of $a$ between 1 and 2 we could use the graph for $(2.7)^x$ of FIGURE 2.13. As shown in FIGURE 2.13, $\log_e(1.5) = 0.41$ (we are approximating $e$ by 2.7 here). This means that

$$\frac{d}{dx} (1.5)^x = 0.41(1.5)^x$$

$$\frac{d}{dx} \log_{1.5}(x) = \frac{1}{(0.41)x}.$$

There are, of course, much better ways to compute numbers $\log_e(a)$. 
\( (e^x)' = e^x, (\ln(x))' = 1/x \)

Before proving the above formula for the derivatives of \(a^x\) and \(\log_a(x)\) we should remember that \(\log_e(e) = 1\) and thus take note of the important special cases when \(a = e\):

\[
\frac{d}{dx} e^x = e^x
\]

\[
\frac{d}{dx} \log_e(x) = \frac{1}{x}.
\]

We now must derive the above formulas. First, consider \(f_a(x) = a^x\). The derivative \(f'_a(x)\) is, by ERUDITE OBSERVATION 1.14, approximately equal (for very small values of \(\Delta x\)) to the difference quotient

\[
\frac{f_a(x + \Delta x) - f_a(x)}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x}.
\]

**The Proof**

By the elementary properties of exponents

\[
\frac{a^{x+\Delta x} - a^x}{\Delta x} = \frac{a^{\Delta x} - 1}{\Delta x} a^x.
\]

But the expression \(\frac{a^{\Delta x} - 1}{\Delta x}\) is approximately equal to \(f'_a(0)\) for small values of \(\Delta x\). More precisely,

\[
\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = f'_a(0).
\]

We summarize this result in THEOREM 2.16.

2.16 **THEOREM** Let \(f_a(x) = a^x\) where \(a > 1\). Then \(f'_a(x) = f'_a(0)a^x\).

If you think about THEOREM 2.16 a little bit, you will realize that it is a remarkable result. It says that the derivative function of \(a^x\) is again \(a^x\) except for a constant multiple which is the slope of the graph of \(a^x\) at the point \((0,1)\). Of course, to compute \(f'_a(0)\) we don’t have to graph \(a^x\) and measure the slope at \((0,1)\) because it turns out that \(f'_a(0) = \log_e(a)\). We now explain why this is so.
The Slope Of $a^x$ At (0,1) ls $\log_a(a)$

We know that $f'(0) = 1$ because that is the way we defined the number $e$. Thus applying THEOREM 2.16 with $a = e$ we get $\frac{d}{dx} e^x = e^x$.

By PROPERTIES 2.15, $a = e^{(\log_e(a))}$ and hence $a^x = e^{(\log_e(a))x}$. We write $h(x) = f(g(x))$ where $h(x) = a^x$, $f(g) = e^g$, and $g(x) = (\log_e(a))x$. The CHAIN RULE says that $\frac{dh}{dx} = \frac{dh}{dg} \frac{dg}{dx}$ so we have

$$\frac{d}{dx} a^x = e^g(\log_e(a)) = (\log_e(a))e^{(\log_e(a))x} = (\log_e(a))a^x.$$

By comparing the above formula with THEOREM 2.16, which states that $\frac{d}{dx} a^x = f'(0)a^x$, we see that $f'(0) = \log_e(a)$. These observations are summarized in THEOREM 2.17.

2.17 THEOREM Let $f_a(x) = a^x$ where $a > 1$. Then $f'_a(x) = (\log_e(a))a^x$.

Compositional Inverses And The Chain Rule Gives $(\log_a(x))$.

We are now in a position to easily obtain the derivative function of $\log_a(x)$. The method we shall use is very similar to the method used in proving LEMMA 2.2. We shall apply the CHAIN RULE to $h(x) = f(g(x))$ where $f(g) = a^g$, $g(x) = \log_a(x)$, and $h(x) = f(g(x)) = a^{(\log_a(x))} = x$ (by PROPERTIES 2.15). For these functions $\frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dx}$ becomes

$$1 = (\log_e(a))a^x \frac{d}{dx}(\log_a(x)).$$

Solving for $\frac{d}{dx}(\log_a(x))$ we obtain THEOREM 2.18.

2.18 THEOREM For $a > 1$, $\frac{d}{dx}(\log_a(x)) = \frac{1}{(\log_e(a))x}$.

Note that if $a = e$ in THEOREM 2.18 then, using the fact that $\log_e(e) = 1$, we have $\frac{d}{dx}(\log_e(x)) = 1/x$. We summarize these results in 2.19.
2.19 EXPONENTIAL AND LOGARITHMIC DERIVATIVES

(1) \[ \frac{d}{dx} a^x = (\log_a(a))a^x \]

(2) \[ \frac{d}{dx} e^x = e^x \]

(3) \[ \frac{d}{dx} \log_a(x) = \frac{1}{(\log_a(a))x} \]

(4) \[ \frac{d}{dx} \log_a(x) = \frac{1}{x^2} \]

2.20 EXAMPLES OF COMPUTING DERIVATIVES

(1) We compute \[ \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{1/2} \] as a first example. We use an online derivative calculator. Search for "math differentiation integration solver." Here is what we get as an answer from one:

\[ DS \quad \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{1/2} = \frac{\sqrt{x}}{2(x + \sqrt{x})^2 \sqrt{\frac{x - \sqrt{x}}{x + \sqrt{x}}}}. \]

We can work DS using first the chain rule and then the quotient rule:

\[ CR \quad \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{1/2} = 1/2 \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{-1/2} \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right) \]

where

\[ QR \quad \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right) = \frac{(x - x^{1/2})(x + x^{1/2})' - (x - x^{1/2})(x + x^{1/2})'}{(x + x^{1/2})^2} = \frac{(1 - (1/2)x^{-1/2})(x + x^{1/2}) - (x - x^{1/2})(1 + (1/2)x^{-1/2})}{(x + x^{1/2})^2} = \frac{x^{1/2}}{(x + x^{1/2})^2}. \]

Substituting into CR gives

\[ CRR \quad \frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{1/2} = 1/2 \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{-1/2} \frac{x^{1/2}}{(x + x^{1/2})^2}. \]

This is the same as equation DS (from the online derivative calculator).
We can also write \( \text{CRR} \)

\[
\frac{1}{2} \left( \frac{x + x^{1/2}}{x - x^{1/2}} \right)^{1/2} \frac{x^{1/2}}{2(x - x^{1/2})^{1/2} (x + x^{1/2})^{3/2}} = \frac{x^{1/2}}{2(x^2 - x)^{1/2} (x + x^{1/2})} = \frac{1}{2x^{1/2} (x^{1/2} - 1)^{1/2} (x^{1/2} + 1)}.
\]

As you see, using online software can result in very different analytic expressions for the same answer. To check if these expressions represent the same function you can use algebra (as here) or use a grapher to see if the two functions have the same graph.

Removing common factors in the beginning can simplify calculations:

\[
\frac{d}{dx} \left( \frac{x - x^{1/2}}{x + x^{1/2}} \right)^{1/2} = \frac{d}{dx} \left( \frac{x^{1/2} - 1}{x^{1/2} + 1} \right)^{1/2}.
\]

Let \( f(v) = ((v - 1)/(v + 1))^{1/2} \) where \( v = x^{1/2} \) so \( \frac{dv}{dx} = 1/2v \). Thus,

\[
\frac{df}{dx} = \frac{df}{dv} \frac{dv}{dx} = \frac{d}{dv} \left( \frac{v - 1}{v + 1} \right)^{1/2} \frac{dv}{dx} = \frac{1}{2} \left( \frac{v - 1}{v + 1} \right)^{-1/2} \frac{2}{(v + 1)^2} \frac{1}{2v}.
\]

Setting \( v = x^{1/2} \) plus a bit of algebra gives \( \text{CRR} \) again.

\[\text{(sec(x)) = tan(x) sec(x)}\quad \text{Memorize!}\]

(2) Compute \( \frac{d}{dx} \text{sec}(x) \). The secant function \( \text{sec}(x) \) is \( (\cos(x))^{-1} \). The cosecant function \( \text{csc}(x) \) is \( (\sin(x))^{-1} \). One would have thought that \( \text{sec}(x) \) should have been \( (\sin(x))^{-1} \) and \( \text{csc}(x) \) should have been \( (\cos(x))^{-1} \) as that would have been easier to remember! We have

\[
\frac{d}{dx} \text{sec}(x) = \frac{d}{dx} (\cos(x))^{-1} = (-1)(\cos(x))^{-2} \frac{d}{dx} \cos(x)
\]

\[
= -(\cos(x))^{-2}(-\sin(x)) = \tan(x)\text{sec}(x).
\]

Thus,

\[
\frac{d}{dx} \text{sec}(x) = \tan(x)\text{sec}(x).
\]

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(\csc(x))' = -\cot(x)\csc(x) \hspace{1em} \text{Memorize!}

(3) To compute \(\frac{d}{dx} \csc(x)\) we write

\[
\frac{d}{dx} \csc(x) = \frac{d}{dx} (\sin(x))^{-1} = (-1)(\sin(x))^{-2} \frac{d}{dx} \sin(x)
\]

\[
= (-1)(\sin(x))^{-2}\cos(x) = -\cot(x)\csc(x).
\]

We have shown that

\[
\frac{d}{dx} \csc(x) = -\cot(x)\csc(x).
\]

(tan(x))' = sec^2(x). \hspace{1em} (\cot(x))' = -\csc^2(x) \hspace{1em} \text{Memorize!}

(4) Another important trigonometric function is the cotangent, \(\cot(x)\). For this function we find

\[
\frac{d}{dx} \cot(x) = \frac{d}{dx} \frac{\cos(x)}{\sin(x)} = \frac{(\cos(x))'\sin(x) - \cos(x)(\sin(x))'}{(\sin(x))^2} = -(\sin(x))^{-2}.
\]

We have used the trigonometric identity \((\sin(x))^2 + (\cos(x))^2 = 1\) in deriving the above identity. The usual way this result is written is

\[
\frac{d}{dx} \cot(x) = -(\csc(x))^2.
\]

From BASIC TRIGONOMETRIC DERIVATIVES 2.10 we also have

\[
\frac{d}{dx} \tan(x) = (\sec(x))^2.
\]

(5) Find \(\frac{d}{dx} \sin^2(\sqrt{x})\). The notation \(\sin^2(\sqrt{x})\) means \((\sin(\sqrt{x}))^2\) and is better written \((\sin(x^{1/2}))^2\). Let \(h(x) = f(g(x))\) with \(f(g) = g^2\) and \(g(x) = \sin(x^{1/2})\). We have \(h'(x) = 2\sin(x^{1/2})(\sin(x^{1/2}))'\). To compute \((\sin(x^{1/2}))'\), we again use the CHAIN RULE applied to \(w(x) = u(v(x))\) where \(w(x) = \sin(x^{1/2})\),
\( u(v) = \sin(v), \) and \( v(x) = x^{1/2}. \) We find \( w'(x) = u'(v(x))v'(x) = \cos(x^{1/2}) \)
\( (1/2)x^{-1/2}. \) Our final conclusion is
\[
\frac{d}{dx} \sin^2(\sqrt{x}) = x^{-1/2}\sin(x^{1/2})\cos(x^{1/2}).
\]

**The "Loglog" Function**

(6) Compute \( \frac{d}{dx} \ln(\ln(x)) \) and \( \frac{d}{dx} \log_2(\log_2(x)) \). Once again we use the
CHAIN RULE applied to \( f(g(x)) \) where \( f(g) = \ln(g) \) and \( g(x) = \ln(x) \). The
CHAIN RULE in the form \( \frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx} \) becomes \( \frac{d}{dx} \ln(\ln(x)) = 
(1/g)(1/x) = 1/(x \ln(x)) \). If we replace \( \ln(x) \) by \( \log_2(x) \) and set \( f(g) = \log_2(g), \)
\( g(x) = \log_2(x) \) then the CHAIN RULE \( \frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx} \) becomes
\[
\frac{d}{dx} \log_2(\log_2(x)) = \frac{1}{(\ln(2))\ln(2)x} = (\ln(2))^{-1}(x \log_2(x))^{-1}.
\]

To summarize,
\[
\frac{d}{dx} \ln(\ln(x)) = \frac{1}{x \ln(x)}
\]
\[
\frac{d}{dx} \log_2(\log_2(x)) = \frac{1}{(\ln(2))x \log_2(x)}.
\]

In general, we would have
\[
\frac{d}{dx} \log_a(\log_a(x)) = \frac{1}{(\ln(a))x \log_a(x)}.
\]

**Differentiating \( x \) And \( (\ln x)' = 1 \) Memorize!**

(7) What is \( \frac{d}{dx} \log_a(|x|) \)? Most students dislike the function \( |x| \). The best
way to think of this function is as two functions pieced together
\[
|x| = \begin{cases} 
x & \text{if } x \geq 0 \\
-x & \text{if } x < 0
\end{cases}
\]
The function $|x|$ has derivative function

$$\frac{d}{dx} |x| = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$ 

**SUMMARY:**

Think of (a), to the left, as the graph of $\ln(x)$ for $x > 0$. Then (b) is the graph of $\ln(|x|)$. Note that:

$$[\ln(-x)]' = -\frac{1}{x} = 1/(-x).$$

Thus, we have for all $x \neq 0$,

$$[\ln(|x|)]' = 1/x.$$
We use the PRODUCT RULE to obtain
\[
\frac{d}{dx}(x\ln(x)) = (1)\ln(x) + x(1/x) = \ln(x) + 1.
\]
the final answer is thus
\[
\frac{d}{dx}x^x = (\ln(x) + 1)x^x.
\]

**The Second Derivative**

\( f''(x), f''(x), \frac{d^2}{dx^2}f \)

(9) We should also practice taking higher order derivatives of logarithmic and trigonometric functions. For example, what is \( \frac{d^2}{dx^2}(\ln(x))^8 \)? We use the CHAIN RULE

\[
\frac{d}{dx}f(g(x)) = \frac{df}{dg} \frac{dg}{dx}
\]

with \( f(g) = (g)^8 \) and \( g(x) = \ln(x) \) to obtain
\[
\frac{d^2}{dx^2}(\ln(x))^8 = \frac{d}{dx} \left( 8(\ln(x))^7 \frac{d}{dx}\ln(x) \right) = \frac{d}{dx} \frac{8(\ln(x))^7}{x}.
\]

Now apply the QUOTIENT RULE to \( 8(\ln(x))^7/x \) to obtain the final answer
\[
\frac{d^2}{dx^2}(\ln(x))^8 = 8 \left( \frac{7(\ln(x))^6}{x^2} - \frac{(\ln(x))^7(1)}{x^2} \right).
\]
This can also be written
\[
\frac{d^2}{dx^2}(\ln(x))^8 = 8 \frac{(\ln(x))^6}{x^2} (7 - \ln(x)).
\]

**Higher Order Derivatives**

(10) We now compute \( \frac{d^p}{dx^p}\sin(bx) \) where \( p \) is a nonnegative integer and \( b \) is a real number. There are infinitely many such \( p \) so we shall give a formula or rule for these derivatives in terms of \( p \). To get a feeling for how this
computation goes, we compute the first few terms of the sequence of derivatives
\[
\frac{d^0}{dx^0}\sin(bx), \frac{d^1}{dx^1}\sin(bx), \frac{d^2}{dx^2}\sin(bx), \ldots, \frac{d^p}{dx^p}\sin(bx), \ldots.
\]

Here we mean that \(\frac{d^0}{dx^0}\sin(bx) = \sin(bx)\) and \(\frac{d^1}{dx^1}\sin(bx) = \frac{d}{dx}\sin(bx)\). Computing these terms we obtain the following sequence:

\[
\sin(bx), \cos(bx), -b^2\sin(bx), -b^3\cos(bx), b^4\sin(bx), b^5\cos(bx),
\]

\[
- b^6\sin(bx), - b^7\cos(bx), b^8\sin(bx), b^9\cos(bx), \text{ etc.}
\]

If you stare at this sequence for awhile, the pattern will become clear, but how do we describe it concisely? Here is the standard trick for giving a "formula" for the terms of this sequence. We say that for \(j = 0, 1, 2, 3, \ldots\)

\[
\frac{d^j}{dx^j}\sin(bx) = (-1)^j b^j\sin(bx)
\]

\[
\frac{d^{j+1}}{dx^{j+1}}\sin(bx) = (-1)^{j+1} b^{j+1}\cos(bx).
\]

Although this terminology (that's all it is) looks impressive at first glance it is really a trivial idea. To see that it works try the various values of \(j\) to see that the sequence of derivatives that we computed above is obtained. A formal proof based on induction could be given of these formulas but we won't bother with that.

**Know Your Trig Identities And Simplify First**

(11) In our first example, (1) above, we ran into problems because we failed to simplify an expression before differentiating it. This sort of oversight can become particularly embarrassing in the case of trigonometric functions. Consider the following difficult-looking problem:

\[
\frac{d}{dx} \left( \frac{\sin^2(x) + \cos^2(x)}{\sin^2(x) + \cos^2(x)} \right) = ?
\]

If you remember your trigonometric identities

\[
\sin^2(x) + \cos^2(x) = 1
\]

\[
- \sin^2(x) + \cos^2(x) = \cos(2x)
\]
then the problem reduces to
\[
\frac{d}{dx} \cos(2x) = -2\sin(2x).
\]

Much easier if you know your basic trigonometric identities!

**Extended Product Rule—Logarithmic Differentiation**

(12) You may have noticed by now that the PRODUCT RULE \((fg)’ = f’g + fg’\) extends to three functions \((fgh)’ = f’gh + fg’h + fgh’\) or more than three functions in the obvious way (each function in the product gets its turn to be differentiated). These more general rules follow easily from our basic PRODUCT RULE. For example, in the case of three functions, \((fgh)’ = ((f)(gh))’ = f’(gh) + f(gh)’\) by the PRODUCT RULE applied to the two functions \(f\) and \(gh\). Now apply the PRODUCT RULE for two functions to \((gh)’\) to get \((fgh)’ = f’(gh) + f(g’h + gh’) = f’gh + fg’h + fgh’\). The general case of a product of \(n\) functions can be derived in the same way (formally, one would use induction).

There are certain types of problems where it is best to avoid using the extended PRODUCT RULE and use instead a method called “logarithmic differentiation.” In spite of the fancy name, the idea is quite simple. If we have a function \(h(x) = \ln(|g(x)|)\) then by the CHAIN RULE \(\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}\) with \(f(g) = \ln(|g|)\) we obtain \(h’(x) = \frac{g’(x)}{g(x)}\). In other words, for any \(g(x)\) we have

\[
\frac{d}{dx} \ln(|g(x)|) = \frac{g’(x)}{g(x)}.
\]

As an example, consider the product
\[g(x) = (4x + 5)^{9/4}(2x + 1)^{1/2}(3x^2 + 2)^{1/3}(3x^2 + 1)^{1/6}.
\]
We could find \(g’(x)\) by the PRODUCT RULE, but another way is to first take the natural logarithm of the absolute value of both sides to get \(\ln(|g(x)|)\) equal to
\[
(9/4)\ln(|4x + 5|) + (1/2)\ln(|2x + 1|) + (1/3)\ln(|3x^2 + 2|) + (1/6)\ln(|3x^2 + 1|).
\]

Now differentiate both sides to obtain
\[
\frac{g’(x)}{g(x)} = \frac{9}{4x + 5} + \frac{1}{2x + 1} + \frac{1}{3x + 2} + \frac{x}{3x^2 + 1}.
\]
The expression on the right side of the above equation is much simpler than what we would have obtained if we had differentiated g(x) by the PRODUCT RULE. Of course what we have obtained is not g'(x) but g'(x)/g(x). This is usually not much of a problem as we know g(x) and can multiply both sides of the equation by g(x) to obtain g'(x). If we do that and then cancel common factors, we will obtain what we would have obtained by applying the PRODUCT RULE in the first place (and there would have been no net computational advantage in using logarithmic differentiation). Logarithmic differentiation is more useful in finding the value of g'(x) for a particular value of x. For example,

\[ g'(1) = (9^{3/4})(3^{1/2})(5^{1/3})(4^{1/6})(1 + (1/3) + (1/5) + (1/4)). \]

There is one minor problem with logarithmic differentiation that we have not mentioned. Notice in our example that g(−1/2) = 0 and thus g'(−1/2)/g(−1/2) is not defined. Generally, this happens only at a few points and is best dealt with in particular cases rather than by formulating some awkward general rule. How would you compute g'(−1/2) in our particular case?

\[ g(x) = (4x + 5)^{3/4}(2x + 1)^{1/2}(3x + 2)^{1/3}(3x^2 + 1)^{1/6} \]

**Chain Rule With Limited Information**

(13) Here is another type of problem that one sees in calculus textbooks. Suppose \( A(x) = B((y(x))^{-2}) \), find \( A'(1) \). As it stands, we don’t have enough information to work this problem. If \( A, B, \) and \( y \) were known functions we could compute the function \( A'(x) \) explicitly:

\[ A'(x) = B'((y(x))^{-2})(-2)(y(x))^{-3}y'(x). \]

There may be some question in this notation about what \( B' \) means it is not \( \frac{dB}{dx} \). In the differential notation we can be more specific. Let \( g(x) = (y(x))^{-2} \). Then \( B'(g) = \frac{dB}{dg} \) and

\[ \frac{dA}{dx} = \frac{dB}{dg} \frac{dg}{dx} = B'(g(x))(-2)(y(x))^{-3}y'(x). \]

To compute \( A'(1) \) we at least need to know \( y(1), y'(1), \) and \( B'(g(1)) \). If you were writing a calculus book, you might ask the students the following question:

If \( A(x) = B((y(x))^{-1}) \), find \( A'(1) \) where \( y(1) = 2, y'(1) = 4, \) and \( B'(1/4) = 3 \).
From the above formula, we have that $A'(1) = B'(g(1)) (-2) (y(1))^{-3}y'(1) = B'(1/4)(-2)(2^{-3})(4) = (3)(-2)(1/8)(4) = -3$. The $1/4$ is there because $g(1) = (y(1))^{-2} = 2^{-2} = 1/4$. This type of problem may seem a bit artificial to you. How would one happen to know that $y(1) = 2$, $y'(1) = 4$, and $B'(1/4) = 3$ but not know enough about $y$ and $B$ to know $A$ or at least a graph of $A$ from which $A'$ could be computed graphically? It is an unlikely situation that could only arise with graphical data in some very special situations. These problems are mostly just ways to test your understanding of the rules of differentiation.

**Implicit Differentiation**

(14) If $L(x)$ is such that $L'(x) = x^{-2}$ then what is $\frac{d}{dx}L(x^3 + 2)$? This problem might seem a little strange at first glance because $L(x)$ is not known explicitly. If we set $g(x) = x^3 + 2$ then $L(g(x)) = L(x^3 + 2)$ and by the CHAIN RULE

$$\frac{d}{dx}L(x^3 + 2) = L'(g(x))g'(x) = \frac{1}{(x^3 + 2)^2} (3x^2).$$

Here we have used the fact that $L'(x) = 1/x^2$ so $L'(g) = 1/g^2$. This problem belongs to a general class called "implicit differentiation." We are given a relationship such as $yx^2 - y^3 + x^2 - 2 = 0$ where $y$ is thought of as a function of $x$, but $y$ is not explicitly known as a function of $x$. By differentiating, we obtain a relationship between $y$, $y'$ and $x$. In this example, we get $y'x^2 + y(2x) - 3y^2y' + 2x = 0$. Solving gives $y' = 2x(1 + y)/(3y^2 - x^2)$.

If for a particular value of $x$, we know $y(x)$, then we can compute $y'(x)$. The original relationship between $x$ and $y$ can often be used to solve for $y$ for that particular value of $x$.

**Checking The Solution To A Differential Equation**

(15) Prove that $y = 3\sin(2t) + \cos(2t)$ satisfies the differential equation $y'' = -4y$. This problem, although it might sound hard if you are not familiar with the terminology, is very simple. It asks you to compute the second derivative $y''$ of the given expression and verify that it is the same as multiplying $-4$ times the expression. Thus we compute
\begin{align*}
y' &= 3(\cos(2t))(2) - \sin(2t)(2) \\
y'' &= (y')' = (6\cos(2t) - 2\sin(2t))' = -12\sin(2t) - 4\cos(2t)
\end{align*}

The expression above is clearly the same as 
\(-4y = -4(3\sin(2t) + \cos(2t))\).

We shall conclude this chapter with some exercises. In working these exercises you will find it very helpful to memorize certain basic facts about logarithmic and trigonometric functions. This will free your mind to think about the calculus rather than the precalculus aspects of these problems. For logarithmic functions we have BASIC FACTS 2.21

\begin{center}
Memorize!
\end{center}

\section*{2.21 BASIC LOGARITHMIC FACTS}

\begin{align*}
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a(x/y) &= \log_a(x) - \log_a(y) \\
\log_a(x') &= r\log_a(x) \\
\log_a(b)\log_b(x) &= \log_a(x) \\
\log_a(b) &= \frac{1}{\log_b(a)}.
\end{align*}

Everyone who has had a course in precalculus mathematics has seen the first three of the BASIC FACTS 2.21. The fourth identity, \(\log_a(b)\log_b(x) = \log_a(x)\), is important for relating logarithms for different bases \(a\) and \(b\). As we discussed in connection with FIGURE 2.13, for a fixed number \(x\) there is only one number \(z\) such that \(a^z = x\). This number \(z\) is, by definition, \(\log_a(x)\). Thus, to prove the fourth identity of BASIC FACTS 2.21, we need only show that for \(z = \log_a(b)\log_b(x)\) we have \(a^z = x\). We compute

\[a^{(\log_a(b)\log_{b}(x))} = (a^{(\log_{a}(b))})^{\log_{b}(x)} = b^{(\log_{b}(x))} = x.\]

That’s all there is to it! The fifth identity of BASIC FACTS 2.21 follows immediately from the fourth by setting \(x = a\) and using the fact that \(\log_a(a) = 1\). In memorizing such formulas it helps to talk to yourself a little bit about them! Regarding the fourth identity of BASIC FACTS 2.21, you might say “The two b’s cancel each other.” In regard to the fifth identity, you might say “The a and the b change roles.” The idea is to aid your long-term memory with some verbal associations (no matter if they have precise mathematical meaning).
Finally, we list without proof some of the more frequently occurring trigonometric identities.

\begin{center}
Memorize!
\end{center}

### 2.22 BASIC TRIGONOMETRIC FACTS

\[
\begin{align*}
\tan(x) &= \frac{\sin(x)}{\cos(x)} & \cot(x) &= \frac{\cos(x)}{\sin(x)} & \sec(x) &= (\cos(x))^{-1} \\
\csc(x) &= (\sin(x))^{-1} & (\sin(x))^2 + (\cos(x))^2 &= 1 & (\tan(x))^2 + 1 &= (\sec(x))^2 \\
1 + (\cot(x))^2 &= (\csc(x))^2 \\
\tan(x + y) &= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \\
\sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= (\cos(x))^2 - (\sin(x))^2 = 1 - 2(\sin(x))^2 = 2(\cos(x))^2 - 1
\end{align*}
\]

In the exercises that follow, be sure to pay attention to the values for which the various functions are defined. Some helpful formulas are given in Section 2.24 and the solutions are given in Section 2.25.

### 2.23 EXERCISES

1. Differentiate the following:
   \begin{enumerate}
   \item\textbf{(a)} \quad \frac{x^2 - 1}{x + 1} \\
   \item\textbf{(b)} \quad \frac{1}{x^2 + 2x + 1} \\
   \item\textbf{(c)} \quad \left(\frac{x - 1}{x^{1/2} + 1}\right)^2 \\
   \item\textbf{(d)} \quad \sin\left(\frac{3 - x^2}{x + 1}\right) \\
   \item\textbf{(e)} \quad \cos\left(\frac{x^2 + 2x + 3}{x^3 + 6}\right) \\
   \item\textbf{(f)} \quad \left(\frac{x - 1}{x + 1}\right)^3 (3x^2 + 1)^{1/2}
   \end{enumerate}

2. Differentiate the following:
   \begin{enumerate}
   \item\textbf{(a)} \quad \sin^2(t) + \cos^2(t) \\
   \item\textbf{(b)} \quad (\sin(x)\cos(z))^{10} \\
   \item\textbf{(c)} \quad xe^{\sin(x)} \\
   \item\textbf{(d)} \quad \ln(|\sin(x)|)
   \end{enumerate}
(e) \((\log_a(x))^{1/3}\)

(f) \(\frac{\log_b(x)}{\log_a(x)}\) if \(a \neq b, \ a > 1, \ b > 1\)

(3) (a) Find \(\frac{d^2}{dx^2} \csc(x)\) at \(x = \pi/4\).

(b) Find \(\frac{d^2}{dx^2} \sec(x)\).

(c) Find \(\frac{d}{dx} (\cos^2(x) - \sin^2(x))^5\).

(d) Find \(\frac{d}{dx} \left[\frac{d}{dx} \ln(|\tan(x)|)\right]\) at \(x = \pi/6\).

(e) If \(f(x) = e^{\ln(x)}\) then \(f'(x) = ?\)

(f) If \(f(x) = \ln(e^{2x})\) then \(f'(x) = ?\)

(g) Find \(\frac{d}{dx} (\ln(x^9) - \ln(x^5))^2\).

(h) Find \(\frac{d}{dx} 2^{\cot(x)}\).

(4) (a) \(\frac{d}{dx} \log_2(\log_2(\log_2(x))) = ?\)

(b) For what values of \(x\) is \((\ln(x))^{1/2}\) defined and what is \(\frac{d}{dx} (\ln(x))^{1/2}\)?

(c) What is \(\frac{d}{dx} x^{\ln(x)}\)?

(assume \(x > 0\))

(d) What is \(\frac{d}{dx} (x + 1)^x\)?

(assume \(x > -1\))

Before giving the solutions to EXERCISES 2.23, we will summarize our logarithmic and trigonometric differentiation rules. These rules should be memorized. Recall that \(\log_a(x)\) and \(\ln(x)\) mean the same thing. In RULES 2.24, the second list of rules follows from the first by the CHAIN RULE: \(f(g(x))' = f'(g(x))g'(x)\). Strictly speaking, we should not have to bother giving you the second set of rules. Some students have trouble remembering the factor \(g'(x)\) when applying the CHAIN RULE. Perhaps the second set of rules will help impress on these students the need for this factor!
2.24 TRIG AND LOG DIFFERENTIATION RULES

\[
\frac{d}{dx} \sin(x) = \cos(x)
\]

\[
\frac{d}{dx} \cos(x) = -\sin(x)
\]

\[
\frac{d}{dx} \sec(x) = \tan(x)\sec(x)
\]

\[
\frac{d}{dx} \csc(x) = -\cot(x)\csc(x)
\]

\[
\frac{d}{dx} \cot(x) = -\frac{1}{\sin^2(x)}
\]

\[
\frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)}
\]

\[
\frac{d}{dx} a^x = (\log_a(e))a^x
\]

\[
\frac{d}{dx} e^x = e^x
\]

\[
\frac{d}{dx} \log_a(x) = \frac{1}{\log_a(e)x}
\]

\[
\frac{d}{dx} \log_e(x) = \frac{1}{x}
\]

\[
\frac{d}{dx} \sin(g(x)) = \cos(g(x))g'(x)
\]

\[
\frac{d}{dx} \cos(g(x)) = -\sin(g(x))g'(x)
\]

\[
\frac{d}{dx} \sec(g(x)) = \tan(g(x))\sec(g(x))g'(x)
\]

\[
\frac{d}{dx} \csc(g(x)) = -\cot(g(x))\csc(g(x))g'(x)
\]
\[
\frac{d}{dx} \cot(g(x)) = -(\csc(g(x)))^2 g'(x)
\]
\[
\frac{d}{dx} \tan(g(x)) = (\sec(g(x)))^2 g'(x)
\]
\[
\frac{d}{dx} a^{g(x)} = (\log_e(a))a^{g(x)} g'(x)
\]
\[
\frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x)
\]
\[
\frac{d}{dx} \log_a(g(x)) = \frac{1}{(\log_e(a))g(x)} g'(x)
\]
\[
\frac{d}{dx} \log_e(g(x)) = \frac{g'(x)}{g(x)}
\]

Next we give the solutions to EXERCISE 2.23.

**Study The Solution—Change The Problem—Rework It**

### 2.25 SOLUTIONS TO EXERCISE 2.23

**1.**

(a) This looks like a problem that requires the QUOTIENT RULE. Notice, however, that the numerator factors into \((x - 1)(x + 1)\). The answer is 1. If you didn’t notice this simplification don’t feel too bad as you probably got some good practice with the QUOTIENT RULE.

(b) \[
\frac{d}{dx}(x^2 + 2x + 1)^{-1} = (-1)(x^2 + 2x + 1)^{-2}(2x + 2) = -2(x + 1)^{-3}.
\]

Alternatively, you might factor before differentiating:

\[
\frac{d}{dx}(x^2 + 2x + 1)^{-1} = \frac{d}{dx}(x + 1)^{-2} = -2(x + 1)^{-3}.
\]

(c) Once again, we can apply the QUOTIENT RULE but notice that \[x - 1 = (x^{1/2} - 1)(x^{1/2} + 1)\] so we need only compute

\[
\frac{d}{dx}(x^{1/2} - 1)^2 = 2(x^{1/2} - 1)(1/2)x^{-1/2} = 1 - x^{-1/2}.
\]

(d) We use the CHAIN RULE \[
\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx} \text{ with } f(g) = \sin(g)
\]
and \( g(x) = \frac{3 - x^2}{x + 1} \). We obtain
\[
\cos \left( \frac{3 - x^2}{x + 1} \right) \frac{d}{dx} \left( \frac{3 - x^2}{x + 1} \right).
\]
Using the QUOTIENT RULE we obtain
\[
\frac{d}{dx} \left( \frac{3 - x^2}{x + 1} \right) = \frac{(-2x)(x+1) - (3-x^2)(1)}{(x+1)^2}.
\]
Doing a little algebra, the final answer may be written
\[
-\cos \left( \frac{3 - x^2}{x + 1} \right) \left( \frac{x^2 + 2x + 3}{(x + 1)^2} \right).
\]
(e) Using the CHAIN RULE we obtain first
\[
-\sin \left( \frac{x^2 + 2x + 3}{x^3 + 6} \right) \frac{d}{dx} \left( \frac{x^2 + 2x + 3}{x^3 + 6} \right).
\]
Now use the QUOTIENT RULE
\[
\frac{d}{dx} \left( \frac{x^2 + 2x + 3}{x^3 + 6} \right) = \frac{(2x+2)(x^3+6) - (x^2+2x+3)3x^2}{(x^3 + 6)^2}.
\]
The answer can be left the way it is or simplified slightly and be written
\[
\left( \frac{x^4 + 4x^3 + 9x^2 - 12x - 12}{(x^3 + 6)^2} \right) \sin \left( \frac{x^2 + 2x + 3}{x^3 + 6} \right).
\]
(f) Apply the PRODUCT RULE, \((f(x)g(x))' = f'(x)g(x) + f(x)g'(x), with f(x) = \left( \frac{x - 1}{x + 1} \right)^3 and g(x) = (3x^2 + 1)^{1/2}. For f'(x) we compute
\[
f'(x) = 3 \left( \frac{x - 1}{x + 1} \right)^2 \left( \frac{(1)(x + 1) - (x - 1)(1)}{(x + 1)^2} \right).
\]
For g'(x) we compute
\[
g'(x) = (1/2)(3x^2 + 1)^{-1/2} 6x.
\]
After a little algebra, the final expression can be written
\[
6 \left( \frac{x - 1}{x + 1} \right)^2 \left( \frac{3x^2 + 1}{(x + 1)^2} \right)^{1/2} + \left( \frac{x - 1}{x + 1} \right)^3 \frac{3x}{(3x^2 + 1)^{1/2}}.
\]
(2) (a) The notation \( \sin^2(t) \) means \( (\sin(t))^2 \). From BASIC TRIGONOMETRIC FACTS 2.22 we have that \( \sin^2(t) + \cos^2(t) = 1 \). Thus
\[
\frac{d}{dt}(\sin^2(t) + \cos^2(t)) = \frac{d}{dt}(1) = 0.
\]

(b) Again, from BASIC TRIGONOMETRIC FACTS 2.22 we find that \(2\sin(z)\cos(z) = \sin(2z)\). Thus \(\frac{d}{dz}((\sin(z)\cos(z))^{10} = \frac{d}{dz}\left(\frac{\sin(2z)}{2}\right)^{10} = 2^{-9}\ 10\sin^9(2z)\cos(2z)\). If you didn’t remember the trigonometric identity you would compute \(10(\sin(z)\cos(z))^9\frac{d}{dz}(\sin(z)\cos(z)) = 10(\sin(z)\cos(z))^9(\cos^2(z) - \sin^2(z))\). This answer is correct also, but not quite as nice! Can you show they’re the same?

(c) \(\frac{d}{dx}xe^{\sin(x)} = (1)e^{\sin(x)} + x\frac{d}{dx}e^{\sin(x)} = e^{\sin(x)} + xe^{\sin(x)}\cos(x) = e^{\sin(x)}(1 + x\cos(x))\).

(d) Use the CHAIN RULE \(\frac{d}{dx}f(g(x)) = \frac{df}{dg}\frac{dg}{dx}\) with \(f(g) = \ln(|g|)\) and \(g(x) = \sin(x)\). We compute \(\frac{df}{dg} = \frac{1}{g}\) and \(\frac{dg}{dx} = \cos(x)\). Hence

\[
\frac{d}{dx}\ln(\sin(x)) = \frac{\cos(x)}{\sin(x)} = \cot(x).
\]

(e) Using the CHAIN RULE, we compute that

\[
\frac{d}{dx}(\log_a(|x|)^{1/3} = (1/3)(\log_a(|x|))^{-2/3}\frac{d}{dx}\log_a(|x|)
\]

The final answer can be written

\[
\frac{1}{3\ln(a)x(\log_a(|x|))^{2/3}}.
\]

(f) The answer is zero because \(\frac{\log_a(|x|)}{\log_a(|x|)}\) is a constant function. To see why, recall BASIC LOGARITHMIC FACTS 2.21, write \(\frac{\log_a(|x|)}{\log_a(|x|)} = \log_a(b)\log_a(|x|)\), and substitute this into the expression \(\frac{\log_a(|x|)}{\log_a(|x|)}\). The answer is \(\log_a(b)\), which is a constant. That the ratio of the logarithm functions for different bases is a constant is worth remembering.

(g) Use the CHAIN RULE to obtain
\[ 3 \left( \frac{\ln(x) + 1}{\ln(x) + 2} \right)^2 \frac{d}{dx} \left( \frac{\ln(x) + 1}{\ln(x) + 2} \right). \]

By the QUOTIENT RULE,
\[ \frac{d}{dx} \left( \frac{\ln(x) + 1}{\ln(x) + 2} \right) = \frac{(1/x)(\ln(x) + 2) - (\ln(x) + 1)(1/x)}{(\ln(x) + 2)^2}. \]

After a little algebra, the final answer can be written
\[ \frac{3(\ln(x) + 1)^2}{x(\ln(x) + 2)^4}. \]

In working this problem, one could notice that \( \ln(x) + 1 = \ln(e^x) \) and \( \ln(x) + 2 = \ln(e^2x) \) and make these substitutions at the beginning. That does not simplify the calculations very much in this case.

(b) We use the CHAIN RULE to compute
\[ \frac{d}{dv} \left( \frac{\tan(v) + 1}{\cot(v) + 2} \right)^5 = 5 \left( \frac{\tan(v) + 1}{\cot(v) + 2} \right)^4 \frac{d}{dv} \left( \frac{\tan(v) + 1}{\cot(v) + 2} \right). \]

Now use the quotient rule to compute
\[ \frac{d}{dv} \left( \frac{\tan(v) + 1}{\cot(v) + 2} \right) = \frac{\sec^2(v)(\cot(v) + 2) - (\tan(v) + 1)\csc^2(v)}{(\cot(v) + 2)^2}. \]

These expressions can be simplified somewhat by various substitutions but it is not worth the trouble at this point.

(3) (a) The notation in this problem means first compute the second derivative of \( \csc(x) \), and then evaluate the resulting function at \( \pi/4 \). To compute \( \frac{d^2}{dx^2} \csc(x) \), we first compute \( \frac{d}{dx} \csc(x) = -\csc(x)\cot(x) \). Next compute
\[ \frac{d}{dx} (-\csc(x)\cot(x)) = (\csc(x)\cot(x))\cot(x) + (-\csc(x))(-\csc^2(x)) \]
\[ = \csc(x)\cot^2(x) + \csc^3(x). \]

Evaluated at \( \pi/4 \) this expression becomes \( 2^{1/2}(1)^2 + 2^{3/2} \), which is approximately 4.24.

(b) We have
\[ \frac{d}{dx} \sec(x) = \sec(x)\tan(x). \]
\[
\frac{d}{dx}(\sec(x)\tan(x)) = \sec(x)\tan^2(x) + \sec^3(x).
\]

(c) Computing without thinking we have
\[
\frac{d}{dx}(\cos^2(x) - \sin^2(x))^5 = 5(\cos^2(x) - \sin^2(x))^4 \frac{d}{dx}(\cos^2(x) - \sin^2(x)).
\]
Doing the indicated differentiation gives a messy but correct expression for the derivative. If we know our B propagometric FACTS 2.22, we recognize that \(\cos^2(x) - \sin^2(x) = \cos(2x)\). The above expression is considerably simplified by this identity and becomes
\[
\frac{d}{dx}(\cos(2x))^5 = 5(\cos(2x))^4 \frac{d}{dx}\cos(2x) = -10\cos^4(2x)\sin(2x).
\]

(d) \(\frac{d}{dx}\ln(|\tan(x)|) = (\tan(x))^{-1} \sec^2(x) = \cot(x)\sec^2(x) = (\sin(x)\cos(x))^{-1} = \sin(2x)^{-1} = \csc(2x)\). Evaluating this expression at \(\pi/8\) gives \(2^{1/2}\).

(e) If you have forgotten what the graph of \(\ln(x)\) looks like, refresh your memory by looking once again at FIGURE 2.13. From this graph it is clear that
\[
|\ln(x)| = \begin{cases} 
\ln(x) & \text{if } x > 1 \\
-\ln(x) & \text{if } 0 < x < 1
\end{cases}
\]
From this observation it is clear that the graph of \(|\ln(x)|\) has a sharp point or "cusp" at the point of the graph corresponding to \(x = 1\) (like the point \(C\) of FIGURE 1.2). This means that \(|\ln(x)|\) has no derivative at \(x = 1\) (i.e., at the point \((1,0)\) on the graph of \(|\ln(x)|\)). For points \(0 < x < 1\), \(\frac{d}{dx}|\ln(x)| = \frac{d}{dx}(\ln(x)) = -1/x\). For points \(x > 1\), \(\frac{d}{dx}|\ln(x)| = \frac{d}{dx}\ln(x) = 1/x\). To summarize,
\[
\frac{d}{dx}|\ln(x)| = \begin{cases} 
1/x & \text{if } x > 1 \\
-1/x & \text{if } 0 < x < 1
\end{cases}
\]
We thus have that (simplify this further!) \(\frac{d}{dx}e^{\ln(x)} = \begin{cases} 
e^{\ln(x)}(1/x) & \text{if } x > 1 \\
\ne^{-\ln(x)}(-1/x) & \text{if } 0 < x < 1
\end{cases}\)

(f) By BASIC LOGARITHMIC FACTS 2.21, \(\ln(e^{2x}) = 2x\). Thus, \(f'(x) = 2\).
(g) By BASIC LOGARITHMIC FACTS 2.21, $\ln(x^9) - \ln(x^5) = \ln(x^9/x^5) = \ln(x^4) = 4\ln(x)$. Thus we have
\[
\frac{d}{dx}(\ln(x^9) - \ln(x^5))^2 = \frac{d}{dx}(4\ln(x))^2 = 32 \ln(x)/x.
\]

(h) We use the CHAIN RULE
\[
\frac{d}{dx}f(g(x)) = \frac{df}{dg} \frac{dg}{dx}
\]
with $f(g) = 2^g$ and $g(x) = \cot(x)$. This gives
\[
\frac{d}{dx} 2^{\cot(x)} = \ln(2) 2^{\cot(x)}(-\csc^2(x)).
\]

(4) (a) In EXAMPLES 2.20(6), we have already computed the derivative of the function $\log_2(\log_2(x))$. To use this result we shall apply the CHAIN RULE
\[
\frac{d}{dx}f(g(x)) = \frac{df}{dg} \frac{dg}{dx}
\]
with $f(g) = \log_2(\log_2(g))$ and $g(x) = \log_2(x)$. From EXAMPLES 2.20(6) we have that
\[
\frac{df}{dg} = \frac{1}{(\ln(2))^2 \log_2(g)}.
\]
We know that $\frac{dg}{dx} = \frac{1}{\ln(2)x}$. The final result is
\[
\frac{d}{dx} \log_2(\log_2(x)) = \frac{1}{(\ln(2))^2 x \log_2(x) \log_2(\log_2(x))}.
\]
This result is valid if base 2 is replaced by base a.

(b) The function $(\ln(x))^{1/2}$ is defined (as a function from real numbers to real numbers) only when $\ln(x)$ is nonnegative. This means that $(\ln(x))^{1/2}$ is defined for $x$ greater than or equal to 1. For $x > 1$ we have
\[
\frac{d}{dx}(\ln(x))^{1/2} = (1/2)(\ln(x))^{-1/2} \frac{d}{dx}(\ln(x)) = \frac{1}{2x(\ln(x))^{1/2}}.
\]
For $x = 1$, the function $(\ln(x))^{1/2}$ has no derivative. If you look at the expression above for the derivative when $x > 1$ you will see that it gets very large ("goes to infinity") as $x$ gets close to 1.

(c) In EXAMPLES 2.20(8) we computed $\frac{d}{dx}x^x = (\ln(x) + 1)x^x$. The trick we used there was to replace $x$ with $e^{\ln(x)}$ and differentiate $e^{x\ln(x)}$. We can use that same trick in this problem, writing $x^{\ln(x)} = (e^{\ln(x)})^{\ln(x)}$. Thus,
\[ \frac{d}{dx} x^{\ln(x)} = \frac{d}{dx} e^{(\ln(x))^2} = e^{(\ln(x))^2} \frac{d}{dx} (\ln(x))^2 = \frac{2\ln(x)x^{\ln(x)}}{x}. \]

(d) We use essentially the same trick as in (c). Write \((x + 1)^x = e^{x\ln(x + 1)}\). Using the CHAIN RULE \(\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}\) with \(f(g) = e^g\) and \(g(x) = x\ln(x + 1)\) gives
\[
\frac{d}{dx} (x + 1)^x = e^{x\ln(x + 1)} \frac{d}{dx} x\ln(x + 1) = (x + 1)^x \left( \ln(x + 1) + \frac{x}{x + 1} \right).
\]

(e) Write \(x^{(1/x)} = e^{\ln(x)/x}\). Then
\[
\frac{d}{dx} x^{(1/x)} = e^{\ln(x)/x} \frac{d}{dx} (\ln(x)/x) = x^{(1/x)} \left( \frac{1 - \ln(x)}{x^2} \right).
\]

(f) From BASIC LOGARITHMIC FACTS 2.21, we have that
\[ \log_{10}(e^x) = \log_{10}(e) \log_{e}(e^x) = \log_{10}(e)x. \]
Thus we compute easily
\[ \frac{d}{dx} \log_{10}(e^x) = \log_{10}(e) = \frac{1}{\log_{e}(10)} = .4343 \text{ (approximately)}. \]

(g) Use \(\log_{10}(\ln(x)) = \log_{10}(e)\log_{e}(\ln(x))\) to get \(\frac{d}{dx} \log_{10}(\ln(x)) = \frac{1}{x\ln(x)}\).

Some Students Memorize \((f(x)^{g(x)})' = f(x)^{g(x)}(g(x)\ln(f(x)))'\)

NOTE: Items 2.26 through 2.34 of the book *Top-down Calculus* (Computer Science Press, 1987) contain numerous variations on 2.23 EXERCISES. We omit these exercises here but retain the book’s original numbering for convenient referencing.
2.35 INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

The inverse trigonometric functions (called arcsin, arccos, or \( \sin^{-1} \), \( \cos^{-1} \), etc.) and hyperbolic functions (sinh, cosh, etc.) will be the last class of functions that we shall add to our basic list of functions whose derivatives we shall memorize. For the beginning calculus student, the former class, the inverse trig functions, are the most important. The hyperbolic functions are little more than a notational contrivance from the point of view of the beginning student. The hyperbolic functions have many properties that are remarkably like the trig functions. The reason for this will be clear to any student who takes more mathematical analysis beyond the first course in calculus.

A More Careful Look At Compositional Inverses

We now take a more careful look at the idea of compositional inverses of functions. We saw this idea applied in 1.17(1)f,g in connection with the solutions to EXERCISE 1.16, in the proof of LEMMA 2.2, and in the proof of THEOREM 2.18. We are now going to make a more systematic use of this idea. It will be helpful to be a little more precise about the notion of a function. If a mathematician says “I have a function in this envelope!” he or she means that the following information is contained in the envelope:

1. A set \( D \) is specified. \( D \) is called the domain of the function.
2. A set \( R \) is specified. \( R \) is called the range of the function.
3. A rule is specified which assigns to each element of the domain \( D \) an element of the range \( R \). (Exactly one element is assigned to each element of \( D \). When an element of \( R \) is assigned to an element of \( D \) it is not removed from \( R \). It may be assigned to other elements of \( D \) as well.)

Here are some examples of functions:

Notice The Notation For Intervals: \([a,b], (a,b], [a,b), (a,b)\)

FUNCTION 1

1. \( D \) is the set \((0,1] = \{x:0 < x \leq 1\}\).
2. \( R \) is the set \((0,1]\) also.
3. To each element \( x \) in \( D \) assign \( \sqrt{x} \).

FUNCTION 2

1. \( D \) is \((0,1]\) just as in FUNCTION 1.
(2) \( R \) is \((0,2]\).

(3) To each element \( x \) in \( D \) assign \( \sqrt{x} \). This is the same rule as in \text{FUNCTION 1}.

\textbf{When Are Two Functions Equal?}

Notice that the only difference between \text{FUNCTION 1} and \text{FUNCTION 2} is that in \text{FUNCTION 1} the range is \((0,1]\) and in \text{FUNCTION 2} the range is \((0,2]\). The graphs of these two functions are shown in FIGURE 2.36. Technically, two functions are equal if they have the same domain, the same range, and the same rule of assignment. Thus, \text{FUNCTION 1} and \text{FUNCTION 2} are different in the sense of this definition as they have different ranges. This may seem a bit like nit-picking to you, in which case you are right. In calculus, we usually just take the range of a function to be the set of real numbers (which we also denote by \( R \)). If we had done this for \text{FUNCTION 1} and \text{FUNCTION 2}, they would be the same function.

\textbf{Image Of f Is The Set Of All f(x), x In The Domain Of f}

If a function has domain \( D \) and range \( R \), then a mathematician would say that the function "maps \( D \) to \( R \)." The \text{IMAGE} of a function is the set of all values in \( R \) that are assigned to some value in \( D \). \text{FUNCTION 1} and \text{FUNCTION 2} have the same \text{IMAGE}. In both cases, the \text{IMAGE} is the set \((0,1]\). A function is "surjective" or "onto" if its \text{IMAGE} is the same as its range \( R \). In this sense, \text{FUNCTION 1} above is onto, but \text{FUNCTION 2} is not onto. Let's look at a couple of additional functions:

\text{FUNCTION 3}

(1) \( D \) is the set \((0,1]\).

(2) \( R \) is the real numbers.

(3) To each element \( x \) in \( D \) assign \( \sqrt{x} \).

\text{FUNCTION 4}

(1) \( D \) is the set \((0,2]\).

(2) \( R \) is the set of real numbers.

(3) To each element \( x \) in \( D \) assign \( \sqrt{x} \).

\textbf{Graph Of f Is The Set Of All (x,f(x)) x In Domain Of f}

The graphs of \text{FUNCTION 3} and \text{FUNCTION 4} are shown in FIGURE 2.36. The graph of a function is, by definition, the set of all pairs \((x,y)\) where \( x \) is in the
domain $D$ of the function and $y$ is the value of $R$ assigned to $x$ by the function. The function $\sqrt{x}$ is defined (as a real valued function) for all nonnegative real numbers $x$. Sometimes one hears, "The domain of definition of $\sqrt{x}$ is the set of all nonnegative real numbers." It is not just nit-picking to be careful about specifying the domains of the functions we talk about in calculus. As in the case of FUNCTION 3 and 4, we may not always take the domain of a function to be all values for which the expression defining the function is valid (i.e., the "domain of definition").

"One-To-One" Or "Injective"—It's The Same

The functions graphed in FIGURE 2.36 also have another important property. Notice that, for these functions, no value in the range is assigned to two or more domain values. To see a situation where this property is not valid, look at FIGURE 2.37(a). For this function, the value 1 in the range is assigned to $1/4$, $1$, and $7/4$. A function which has the property that no range value is assigned to more than one domain value is called "one-to-one" or "injective." Another way to say the same thing is "$f$ is one-to-one if $s \neq t$ implies $f(s) \neq f(t)$" or, equivalently, "$f$ is one-to-one if $f(s) = f(t)$ implies $s = t$."

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FIGURE 2.37 A Function That Is Not One-To-One

You should think carefully about why these various definitions define the same thing.

"Functional" Or "Compositional" Inverses—Same Thing

Look again at FIGURE 2.36. Note that FUNCTIONS 1, 2, 3, and 4 are one-to-one. Any one-to-one function \( f \) has another function associated with it called its "functional inverse" or, simply, "inverse" \( f^{-1} \). To each number in the IMAGE of \( f \), we associate the number \( s \) which has \( f(s) = t \). We call this unique number \( s \), \( f^{-1}(t) \). If, for example, we take \( f \) to be FUNCTION 4 of FIGURE 2.36, then \( t = 1.3 \) is associated with \( s = 1.7 \) because \( f(1.7) = (1.7)^{1/2} = 1.3 \). Thus \( f^{-1}(1.3) = 1.7 \). In the case of a one-to-one function \( f \), we may take the domain of \( f^{-1} \) to be the IMAGE of \( f \) and the range of \( f^{-1} \) to be the real numbers. Thus for \( f = \) FUNCTION 4, the domain of \( f^{-1} \)
is the interval \([0,2^{1/2}] = \{x:0 \leq x \leq 2^{1/2}\}\). For values of \(x\) in this interval, \(f^{-1}(x) = x^2\). Thus, \(f^{-1}(1.3) = (1.3)^2 = 1.7\). Be careful with the notation for functional inverses! The notation \(f^{-1}(1.3)\) is not the same as \((f(1.3))^{-1}\). The latter means "compute \(f(1.3)\) and take its reciprocal." Thus we compute \(f(1.3) = 1.14\) and \(1/1.14\) is 0.88, so \((f(1.3))^{-1} = 0.88\), which is not the same as \(f^{-1}(1.3) = (1.3)^2 = 1.7\). Functional inverses are also called "compositional inverses."

The general situation is illustrated in FIGURE 2.38. In this figure, a one-to-one function \(f\) is defined by giving its graph. The domain of \(f\) is \([-1/4,1]\) and the range is the real numbers. FIGURE 2.38(b) is an exact copy of FIGURE 2.38(a) except that the vertices of the rectangle are interpreted in terms of \(f\) in (a) and in terms of \(f^{-1}\) in (b). In (a), \(P = (0,f(s))\) and in (b), \(P = (0,t)\). \(P\) is a fixed point in the plane so this means that \((0,f(s)) = (0,t)\) or, in particular, \(f(s) = t\). Similarly, \(Q = (s,0) = (f^{-1}(t),0)\) so \(f^{-1}(t) = s\).

If in the identity \(f(s) = t\) we substitute \(s = f^{-1}(t)\), we obtain
\[
f(f^{-1}(t)) = t \text{ for any } t \text{ in the domain of } f^{-1}.
\]
Likewise, if in the identity \(f^{-1}(t) = s\) we substitute \(t = f(s)\) we obtain
\[
f^{-1}(f(s)) = s \text{ for any } s \text{ in the domain of } f.
\]
These observations are summarized in FIGURE 2.38.

**FIGURE 2.38** Basic Properties of Compositional Inverses
**If It’s Not One-To-One, Restrict The Domain!**

Unfortunately, many of the most important functions in calculus are not one-to-one. A typical situation is shown in FIGURE 2.37(a) and (b). The function \( h(x) \) shown in FIGURE 2.37(a) is defined graphically and has domain \([0, 2]\). Clearly, \( h \) is not one-to-one. Note, however, that if we restrict the function \( h \) to the interval \([0, 3/4]\) then this function is one-to-one. The graph of this new function is shown as the darker portion of the graph of \( h \) in FIGURE 2.37(a). Shown also is our old friend SALLY of FIGURE 2.13. Again, SALLY is slightly behind the page. FIGURE 2.37(b) shows the darkened portion of the graph as viewed by SALLY. This new graph is called \( h^{-1} \) even though it is the inverse of the restriction of \( h \) to the interval \([0, 3/4]\) and not the inverse of the original \( h \). The original \( h \) has no inverse as it is not one-to-one. There is nothing magical about the choice of the interval \([0, 3/4]\). The interval \([3/4, 5/4]\) could have been chosen as well. The reader should sketch the graph of \( h^{-1} \) based on the restriction of \( h \) to \([3/4, 5/4]\).

**Compositional Inverses Of Trig Functions**

We now use the idea of FIGURE 2.37 to define the compositional inverses of the trigonometric functions. These compositional inverses are shown in FIGURE 2.40. The first function shown is the compositional inverse of the cosine. This function is defined by restricting the cosine to the interval \([0, \pi]\). The inverse function is denoted by \( \cos^{-1}(x) \) or \( \arccos(x) \). The first notation is common but very poor. It is standard practice to write \( \cos^\theta(x) \) for \((\cos(x))^\theta\). Thus \( \cos^{-1}(x) = \frac{1}{\cos(x)} \) in this notation. We shall use \( \arccos(x) \) in this book, but you should be forewarned of the alternative notation \( \cos^{-1} \) for \( \arccos \). Usually, it will be clear from the context whether \( \cos^{-1}(x) \) is intended to mean \( \arccos(x) \) or \((\cos(x))^{-1}\).

**The Chain Rule And Compositional Inverses**

There is a standard technique for finding the derivative of the compositional inverse \( h^{-1} \) given the derivative \( h' \). We have that \( h(h^{-1}(x)) = x \) for all \( x \) in the domain of \( h^{-1} \). By the CHAIN RULE, \( 1 = (x)' = (h(h^{-1}(x)))' = h'(h^{-1}(x))(h^{-1}(x))' \). Thus, we have

**2.39 BASIC RULE**

\[
(h^{-1}(x))' = \frac{1}{h'(h^{-1}(x))}.
\]
Applying this rule to $\arccos(x)$ gives

$$(\arccos(x))' = \frac{1}{-\sin(\arccos(x))}.$$

Look now at the small triangle near the graph of $\arccos(x)$ in FIGURE 2.40. From this triangle, it is evident that $y = \arccos(x) = \arcsin((1 - x^2)^{1/2})$. Thus, $\sin(\arccos(x)) = \sin(\arcsin((1 - x^2)^{1/2})) = (1 - x^2)^{1/2}$. This gives

$$(\arccos(x))' = \frac{-1}{(1 - x^2)^{1/2}}.$$

Using the small triangles in FIGURE 2.40 together with BASIC RULE 2.39 we get the following rules for differentiating the compositional inverses of the trigonometric functions.

**FIGURE 2.40** Basic Inverse Trigonometric Functions
2.41 DIFFERENTIATION RULES FOR INVERSE TRIG FUNCTIONS

\[
\begin{align*}
\text{(arcsin}(x))' &= \frac{1}{(1 - x^2)^{1/2}} \\
\text{(arccos}(x))' &= -\frac{1}{(1 - x^2)^{1/2}} \\
\text{(arctan}(x))' &= \frac{1}{1 + x^2} \\
\text{(arccot}(x))' &= -\frac{1}{1 + x^2}
\end{align*}
\]
\[
\begin{align*}
\text{(arcsec}(x))' &= \frac{1}{x(x^2 - 1)^{1/2}} \\
\text{(arccsc}(x))' &= \frac{-1}{x(x^2 - 1)^{1/2}}
\end{align*}
\]

The latter two rules of DIFFERENTIATION RULES 2.41 are the rules of differentiation for the compositional inverses of the secant and cosecant. The graphs of these functions are shown in FIGURE 2.43.

2.42 EXERCISES

(1) Derive all of the DIFFERENTIATION RULES 2.41 that we did not derive in the text. (Use tricks of Figure 2.40)

\[
\begin{align*}
\text{arcsin}(x) &= \frac{\pi}{2} - \text{arccos}(x) \\
(\text{arcsin}(x))' &= -\frac{1}{(1-x^2)^{1/2}} \\
(\text{arccsc}(x))' &= \frac{1}{x(x^2 - 1)^{1/2}} \\
\text{arctan}(x) &= \frac{\pi}{2} - \text{arccot}(x) \\
(\text{arctan}(x))' &= \frac{\sec^2(\text{arcsec}(x))}{1 + x^2} \\
(\text{arcsec}(x))' &= \frac{1}{\sec(\text{arcsec}(x))} \\
(\text{arcsec}(x)) &= \frac{1}{\sec(\text{arcsec}(x))} \\
\end{align*}
\]

FIGURE 2.43 Inverse secant and cosecant
The Hyperbolic Functions

Our final class of functions is the class of "hyperbolic" functions. These functions will not be of any great importance to us in the sequel. They do occur naturally in some applied problems involving calculus (hanging cables and ocean waves are two classical examples). We give a brief description of these functions and some of their properties at this point to guard the reader against an anxiety attack in the event they crop up in some later course.

2.44 DEFINITION (hyperbolic functions)

\[
\begin{align*}
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2} \\
\tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh(x)}{\cosh(x)}
\end{align*}
\]

In addition to these basic functions we have

\[
\begin{align*}
\coth(x) &= \frac{1}{\tanh(x)} \\
\sech(x) &= \frac{1}{\cosh(x)} \\
\csch(x) &= \frac{1}{\sinh(x)}.
\end{align*}
\]

The function \(\sinh(x)\) is pronounced "sinch of \(x\)" and \(\cosh(x)\) is pronounced "kohsch of \(x\)." Alternatively, one may just say "hyperbolic sin of \(x\)" for \(\sinh(x)\) or "hyperbolic cosine of \(x\)" for \(\cosh(x)\). This seems to be the form for \(\tanh(x)\), \(\coth(x)\), \(\sech(x)\), and \(\csch(x)\) where one says, "hyperbolic tangent of \(x\)," hyperbolic cotangent of \(x\)," etc. The differentiation rules for these functions are trivial to derive:

2.45 DIFFERENTIATION RULES FOR HYPERBOLIC FUNCTIONS

\[
\begin{align*}
\frac{d}{dx} \sinh(x) &= \cosh(x) \\
\frac{d}{dx} \cosh(x) &= \sinh(x) \\
\frac{d}{dx} \tanh(x) &= \text{sech}^2(x) \\
\frac{d}{dx} \coth(x) &= -\text{csch}^2(x)
\end{align*}
\]
\[ \frac{d}{dx} \text{sech}(x) = -\text{sech}(x) \tanh(x) \quad \frac{d}{dx} \text{csch}(x) = -\text{csch}(x) \coth(x) \]

Graphs of the hyperbolic functions are shown in FIGURE 2.46.

Several questions usually occur to the student at this point. One is “Why are these functions called ‘hyperbolic’?” If you were to graph all points in the plane of the form \((x,y) = (\cosh(t), \sinh(t))\) as \(t\) ranges over all real numbers, you would obtain the graph of a curve called a hyperbola. The points \((x,y)\) on this curve satisfy \(x^2 - y^2 = 1\) because \(\cosh^2(t) - \sinh^2(t) = 1\).

A second, more practical, question is usually “How much of this stuff should we memorize?” If you have a good memory for this sort of stuff then memorize it all, as the more you know the better. There is a danger for the beginning student of getting confused between the differentiation rules for the trigonometric functions and the much less important hyperbolic functions. If you feel this might be a problem for you, then memorize just the definitions of the hyperbolic functions (DEFINITION 2.44) and the rules for differentiating the \(\sinh\), \(\cosh\), and \(\tanh\). Note that these rules are the same as for the \(\sin\), \(\cos\), and \(\tan\), except that \((\cosh(x))' = -\sin(x)\) but \((\cosh(x))' = +\sinh(x)\).

Just as with the trigonometric functions, one can define the compositional inverses of the hyperbolic functions. These functions can be expressed directly

FIGURE 2.46  Graphs of the Hyperbolic Functions
in terms of the function \( \ln \) (log base \( e \)). Because of this, the derivatives of these functions can be computed directly by differentiating \( \ln \) and applying the CHAIN RULE. We conclude by stating these results.

2.47 INVERSE HYPERBOLIC FUNCTIONS

\[
\begin{align*}
\text{arcsinh}(x) &= \ln(x + (1 + x^2)^{1/2}) \\
\text{arccosh}(x) &= \ln(x + (x^2 - 1)^{1/2}) \quad (x \geq 1) \\
\text{arctanh}(x) &= (1/2)\ln \frac{1 + x}{1 - x} \quad (x^2 < 1) \\
\text{arccoth}(x) &= (1/2)\ln \frac{x + 1}{x - 1} \quad (x^2 > 1) \\
\text{arcsech}(x) &= \ln \left( \frac{1 + (1 - x^2)^{1/2}}{x} \right) \quad (0 < x \leq 1) \\
\text{arccsch}(x) &= \ln \left( \frac{1 + (1 + x^2)^{1/2}}{|x|} \right) \quad (x \neq 0)
\end{align*}
\]

2.48 DIFFERENTIATION RULES FOR INVERSE HYPERBOLIC FUNCTIONS

\[
\begin{align*}
\frac{d}{dx} \text{arcsinh}(x) &= \frac{1}{(1 + x^2)^{1/2}} \\
\frac{d}{dx} \text{arccosh}(x) &= \frac{1}{(x^2 - 1)^{1/2}} \quad (x > 1) \\
\frac{d}{dx} \text{arctanh}(x) &= \frac{1}{1 - x^2} \quad (x^2 < 1) \\
\frac{d}{dx} \text{arccoth}(x) &= \frac{1}{1 - x^2} \quad (x^2 > 1) \\
\frac{d}{dx} \text{arcsech}(x) &= -\frac{1}{x(1 - x^2)^{1/2}} \quad (0 < x < 1) \\
\frac{d}{dx} \text{arccsch}(x) &= -\frac{1}{|x|(1 + x^2)^{1/2}} \quad (x \neq 0)
\end{align*}
\]

Are you tempted to say that \( \text{arctanh}(x) \) and \( \text{arccoth}(x) \) have the same derivative? If so, reread the discussion of functions prior to FIGURE 2.36, noting that the domain for the derivative of \( \text{arctanh}(x) \) is all \( x \) with \( x^2 < 1 \) and the domain of the derivative of \( \text{arccoth}(x) \) is all \( x \) with \( x^2 > 1 \).
Chapter 3

APPLICATIONS OF DERIVATIVES

In this chapter we shall take a look at various applications of derivatives. We are, however, going to be a little more critical of these "applications" than the standard calculus textbook. To begin with, we look at the problem of graphing functions. Let's take as our first example the function \( y(x) = x - (1/3)x^3 \). We want to sketch the graph of this function. In what sense is this an application of derivatives? It seems to be a simple problem for a hand calculator or a computer and not really a problem for calculus as advertised in the calculus textbooks. We shall see that calculus really plays a very minor role in these graphing problems. The best strategy is to avoid calculus as much as possible in these problems and only use derivatives to "fine tune" your results. This fine tuning can be important in some problems and there we will see calculus concepts coming into play in interesting ways.

The Inventors Of Calculus Didn't Have Computers—you Do!

In PROGRAMS 3.2, we see two programs in BASIC to compute values of the function \( y = x - (1/3)x^3 \). PROGRAM 3.2(a) gives a sequence of points on the graph of \( y \). Statement 10 in this program records what the function is. This is a good idea in case you have been doing several functions and have a number of these tables of values lying around on your desk. Statement 20 generates the values of the dependent variable between \(-2\) and \(+2\) in steps of \(.2\). In statement 30, "(" prints a left parenthesis, \( X \) prints the value of the dependent variable, "," prints comma, \( X - (X^3)/3\) prints the value of the function, and ")" prints the closing parenthesis. The
BASIC program of PROGRAM 3.2(a) was executed on a good but very inexpensive computer and does not involve any formatting. Any computer that implements BASIC should be able to do at least this well. PROGRAM 3.2(b) is done in a little bit more sophisticated version of BASIC which allows the PRINT USING statement for formatted output. Computers are so common in our society at this point that any student can gain access to one or buy one for very little expense. The language BASIC can be learned to the point required by PROGRAMS 3.2 in less than 30 minutes. We assume in our discussion of graphing of functions that the reader has taken the trouble to adventure at least this far into the world of computers. The function \( y = x - (1/3)x^3 \) and its derivative \( y' = 1 - x^2 \) have been graphed in FIGURE 3.1.

**FIGURE 3.1** Graphs of \( y = x - (1/3)x^3 \) and \( y' = 1 - x^2 \)
3.2 Programs for Graphing \( x - (1/3)x^3 \)

(a)

```
10 PRINT "(X,X-(X^3)/3)"
20 FOR X = -2 TO 2 STEP .2
30 PRINT "(";"X","X-(X^3)/3")"
40 NEXT X

(X,X-(X^3)/3)

(-2 , .666666667 )
(-1.8 , .144 )
(-1.6 , -.234666667 )
(-1.4 , -.485333333 )
(-1.2 , -.624 )
  (- .999999999 , -.666666666 )
  (- .799999999 , -.629333333 )
  (- .599999999 , -.528 )
  (- .399999999 , -.378666666 )
  (- .199999999 , -.197333333 )
(  5.82076609E-10 ,  5.82076609E-10 )
(   .200000001 , .197333334 )
(   .400000001 , .378666667 )
(   .600000001 , .528 )
(   .800000001 , .629333333 )
(    1 , .666666666 )
(   1.2 , .623999999 )
(   1.4 , .485333331 )
(   1.6 , .234666663 )
(   1.8 , -.144000005 )
```

(b)

```
10 PRINT "  X:";"  X-(X^3)/3"
20 FOR X = -2 TO 2 STEP .2
30 PRINT USING " +##.## ";X,X-(X^3)/3
40 NEXT X

<table>
<thead>
<tr>
<th>X</th>
<th>X-(X^3)/3</th>
</tr>
</thead>
<tbody>
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<td>+0.14</td>
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<tr>
<td>-1.60</td>
<td>-0.23</td>
</tr>
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<td>-1.40</td>
<td>-0.49</td>
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<tr>
<td>-1.20</td>
<td>-0.62</td>
</tr>
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<td>-1.00</td>
<td>-0.67</td>
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<tr>
<td>-0.80</td>
<td>-0.63</td>
</tr>
<tr>
<td>X</td>
<td>X - (X^3)/3</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>−0.60</td>
<td>−0.53</td>
</tr>
<tr>
<td>−0.40</td>
<td>−0.38</td>
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<tr>
<td>−0.20</td>
<td>−0.20</td>
</tr>
<tr>
<td>+0.00</td>
<td>+0.00</td>
</tr>
<tr>
<td>+0.20</td>
<td>+0.20</td>
</tr>
<tr>
<td>+0.40</td>
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<tr>
<td>+0.60</td>
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<td>+0.49</td>
</tr>
<tr>
<td>+1.60</td>
<td>+0.23</td>
</tr>
<tr>
<td>+1.80</td>
<td>−0.14</td>
</tr>
</tbody>
</table>

**Local Minima And Maxima**

Take a look now at FIGURE 3.1. Three points, A = (−1, −2/3), B = (0, 0), and C = (1, 2/3) are shown on the graph of y(x) = x − (1/3)x^3. The point A is called a “local minimum” of y, the point B is called a “point of inflection” of y, and the point C is called a “local maximum” of y. We shall now discuss these terms and what they mean in terms of derivatives. Notice that the point A = (−1, −2/3), where the value of y is −2/3, is not the smallest value that the function y(x) = x − (1/3)x^3 ever assumes. For all values of x > 2, y(x) < −2/3. We know that because y(2) = −2/3 and it is apparent from the graph of y that y decreases as x increases for values of x > 1. How can we be absolutely sure that y decreases for all values of x > 1? We might worry that for x large, say x > 100, the function starts to increase again! First of all, a little common sense will convince us that such a situation will not occur. The term x^3 gets large much faster than the term x so that once the term (1/3)x^3 begins to get larger than x for positive x, the whole expression x − (1/3)x^3 continues to decrease. At this point, if you still need reassuring you can use a little calculus. The derivative function y'(x) = 1 − x^2 is shown in FIGURE 3.1. Note that for x > 1, 1 − x^2 < 0. A function with a negative derivative at a point is decreasing. Thus, y(x) = x − (1/3)x^3 is decreasing for all values of x > 1.

Even though there are infinitely many values of x for which y(x) < y(−1) = −2/3, we say that y has a “local minimum at x = −1” or “local minimum at A = (−1, −2/3)” because the value y(−1) = −2/3 is minimal in the vicinity of A. In other words, for some interval of values of x, centered at x = −1, the value y(−1) = −2/3 is the minimal value of f. Similarly, the function y has a “local maximum” at x = +1 (at the point C).
Critical Points

How did we know that the local minimum of \( y \) was exactly at \( x = -1 \)? Just looking at the graph of \( y \) in FIGURE 3.1, we really can’t be sure that the local minimum of \( y \) near \( x = -1 \) doesn’t actually occur at \( x = -1.001 \), say. Here calculus can help. We compute \( y'(x) = 1 - x^2 \). Setting \( y'(x) = 0 \) and solving for \( x \) we get \( x = +1 \) or \( x = -1 \). The values of \( x \) where \( y'(x) = 0 \) are called the “critical points” of \( y(x) \). Since \( y'(-1.001) \) is not zero, \( -1.001 \) is not a critical point of \( y(x) \). It should be obvious to you at this point that if \( x \) is not a critical point of \( y \) then \( x \) is not a local maximum or minimum. Thus, \( x = -1 \) is the only critical point in its immediate vicinity and hence \( x = -1 \) must be exactly the local minimum. In general, for differentiable functions \( y(x) \), the local maxima and minima will be found among the critical points.

Weren’t we lucky that we were easily able to solve for the values of \( x \) such that \( y'(x) = 0 \)? That is, we were easily able to find the critical points of \( y(x) \) by solving the simple quadratic equation \( 1 - x^2 = 0 \). Since every student learns to solve a quadratic \( ax^2 + bx + c = 0 \) in high school algebra, calculus problems are frequently concocted so that \( y'(x) \) is a quadratic equation. In the real world, things are not usually this nice. If \( y'(x) \) is a polynomial of degree 3 or higher or some horrible trigonometric or logarithmic function, then it may not be so easy to locate the critical points other than by some graphical or numerical method involving a computer. As we have already remarked, the other critical point, \( x = +1 \), is a local maximum. What about the mysterious point B of FIGURE 3.1?

Points Of Inflection

To understand what is special about B, the so-called “point of inflection,” it is necessary to look at the graph of \( y' \) shown also in FIGURE 3.1. Note that for values of \( x < 0 \), \( y' \) has been increasing. At \( x = 0 \), \( y' \) starts to decrease. A value of \( x \) where \( y' \) changes from increasing to decreasing or from decreasing to increasing is called a “point of inflection” of \( y \). One also says “\( (x, y(x)) \) is a point of inflection of \( y \)” Thus, we might say “\( B = (0, 0) \) is a point of inflection of \( y = x - (1/3)x^3 \)” or “\( 0 \) is a point of inflection of \( y = x - (1/3)x^3 \).” By definition, a point of inflection \( x \) of \( y(x) \) will be a local maximum or minimum of \( y' \). If \( y' \) itself has a derivative, \( y'' \) (the second derivative of \( y \), then \( y''(x) = 0 \) at a point of inflection. This can sometimes be a help in pinning down the exact location of a point of inflection. As with critical points, the usefulness of the “second derivative tests” has been exaggerated by most calculus texts.
We now consider some additional examples of graphing functions.

3.3 **EXAMPLE** Graph the function \( f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \) and find the critical points.

**Use Calculus To Fine-Tune Graphical Information**

Following the precept to avoid calculus if at all possible, we wrote the two BASIC programs of PROGRAM SEQUENCE 3.4(a) and (b). PROGRAM 3.4(a) gives a general feeling for the graph of the function \( f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \). Note that this function gets very large for large negative and positive values of \( x \); in fact already at \( x = -3 \) the function value is 319 and for \( x = +3 \) the function value is 547. PROGRAM 3.4(a) seems to indicate that \( f(x) \) has a single minimum in the vicinity of \( x = -3.3 \). In PROGRAM 3.4(b) we take a more careful look at \( f(x) \) in the vicinity of \( x = -3.3 \). It seems that \( -44 \) is quite close to the value of \( x \) where \( f(x) \) assumes its minimum value. We could continue in this manner without using any calculus to try and locate the value of \( x \) where \( f(x) \) assumes its minimum. The reader should attempt this on his or her computer. You will discover in attempting this that it becomes quite difficult to gain additional accuracy in locating the minimum. A more sensitive way to locate the minimum is to pass to the derivative \( f'(x) = 2 + 6x + 12x^2 + 20x^3 \) and try to find where \( f''(x) = 0 \) in the vicinity of \( -44 \). There are fairly sophisticated polynomial root finding subroutines available on many computers but a straightforward approach works fine for us. PROGRAMS 3.4(c), (d), and (e) search directly for the root of \( f'(x) \) near \( -44 \) and end up with \( -43708 \) where \( f(x) \) takes on the value of 0.547439. This is, to within the accuracy of interest to us, the minimum value of \( f(x) \). It is both an "absolute minimum" and a local minimum. By writing a program to evaluate \( f'(x) \) from \(-3 \) to \(+3\), such as PROGRAM 3.4(a) does for \( f(x) \), you can easily convince yourself that there are no other roots of \( f'(x) \) than the one we have located near \(-43708\).

What about points of inflection of \( f(x) \)? The second derivative is \( f''(x) = 6 + 24x + 60x^2 \). By graphing this function, you can easily see that \( f''(x) \) has no real roots. Thus there are no points of inflection for \( f(x) \). A common calculus problem is to specify a function and interval on which the function is to be considered. The student is asked to find the local maxima and minima of the function on that interval and to also find the absolute maxima and minima. Consider, for example, our function \( f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \) and the interval \(-3 \leq x \leq +3\). This interval is also written \([-3, +3]\). From our above discussion, we have that there are no local maxima of \( f(x) \) in \([-3, +3]\). The maximum value is unique (i.e., there is only one
x in \([-3, +3]\) such that f(x) is maximal) and this occurs at \(x = +3\) where \(f(+3) = 547\). There is one local minima at \(x = -0.43708\) where \(f(x) = 0.547439\). This is also the absolute minima.

3.4 PROGRAM SEQUENCE FOR \(f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4\)

(a)

10 FOR \(X = -3\) TO 3 STEP .3
20 PRINT \(X, 1 + 2 \cdot X + 3 \cdot X^2 + 4 \cdot X^3 + 5 \cdot X^4\)
30 NEXT \(X\)

READY.

<table>
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<th>(1 + 2x + 3x^2 + 4x^3 + 5x^4)</th>
</tr>
</thead>
<tbody>
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<td>-2.7</td>
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<tr>
<td>-2.4</td>
<td>124.072</td>
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<td>70.2265003</td>
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<td>36.2800001</td>
</tr>
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</tr>
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<td>-1.2</td>
<td>6.37600003</td>
</tr>
<tr>
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<td>1.99450001</td>
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<td>6.64000002</td>
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<td>372.7225</td>
</tr>
<tr>
<td>3</td>
<td>547</td>
</tr>
</tbody>
</table>

(b)

10 FOR \(X = -.45\) TO -.35 STEP .01
20 PRINT \(X, 1 + 2 \cdot X + 3 \cdot X^2 + 4 \cdot X^3 + 5 \cdot X^4\)
30 NEXT \(X\)

READY.

<table>
<thead>
<tr>
<th>(X)</th>
<th>(1 + 2x + 3x^2 + 4x^3 + 5x^4)</th>
</tr>
</thead>
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<td>.54803125</td>
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<tr>
<td>-.44</td>
<td>.5474688</td>
</tr>
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<td>$x$</td>
<td>$1 + 2x + 3x^2 + 4x^3 + 5x^4$</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------</td>
</tr>
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<td>$-0.43$</td>
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<td>$-0.36$</td>
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</table>

(c)

10 FOR $X = -0.44$ TO $-0.43$ STEP .001  
20 PRINT $X, 2 + 6X + 12X^2 + 20X^3$  
30 NEXT $X$

READY.

<table>
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<tr>
<th>$x$</th>
<th>$2 + 6x + 12x^2 + 20x^3$</th>
</tr>
</thead>
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<td>$0.021341921$</td>
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<tr>
<td>$-0.431$</td>
<td>$0.0418721824$</td>
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</tbody>
</table>

(d)

10 FOR $X = -0.438$ TO $-0.437$ STEP .0001  
20 PRINT $X, 2 + 6X + 12X^2 + 20X^3$  
30 NEXT $X$

READY.

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<th>$x$</th>
<th>$2 + 6x + 12x^2 + 20x^3$</th>
</tr>
</thead>
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</table>
10 FOR X = -.4371 TO -.437 STEP .00001
20 PRINT X, 2 + 6*X + 12*X^2 + 20*X^3
30 NEXT X

READY.

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</tr>
<tr>
<td>-.437000001</td>
<td>5.58936088E-04</td>
</tr>
</tbody>
</table>

**Parameterized Families**

In our next example, we see a situation where calculus is a major help beyond straight computational techniques. Instead of considering one function, we consider a "family of functions." Let \( f_a(x) = x^2(a - x)^2 \) where \( a \) is a real number. For each actual value of \( a \) we can write down a specific function of this type. For \( a = 1 \) we obtain \( f_1(x) = x(x - 1)^2 \) and for \( a = 2 \) we obtain \( f_2(x) = 2x(x - 2)^2 \). For \( a = -3 \) we obtain \( f_{-3}(x) = -3x(x + 3)^2 \). Graphs of \( f_1(x) \) and \( f_2(x) \) are shown in FIGURE 3.6. The "variable" \( a \) that is required to specify a function \( f_a(x) \) is called a "parameter" of the family of functions. The parameter "ranges over all real numbers."

### 3.5 EXAMPLE

Consider the parameterized family of functions \( f_a(x) = ax(x - a)^2 \) where the parameter \( a \) ranges over all real numbers. Find the local maxima, local minima, and points of inflection of the functions \( f_a(x) \) in terms of the parameter \( a \) when \( a \neq 0 \).

Using the product rule, we compute that \( f_a'(x) = a(x - a)^2 + (ax)2(x - a) = a(x - a)(3x - a) \). The second derivative is \( a(6x - 4a) \). The critical points are, by definition, all points where \( f_a'(x) = 0 \). There are two such points, \( x = a \) and \( x = a/3 \). We are lucky that it is so easy to solve the equation \( f_a'(x) = 0 \). It is easy to construct other examples where it is very difficult or impossible to solve the equation explicitly in terms of the param-
FIGURE 3.6 The Parameterized Family $f_a$

We should graph a few of the functions $f_a(x)$ from our family of functions. We have graphed $f_1(x)$ and $f_2(x)$ in FIGURE 3.6. The reader should graph $f_a(x)$ for some negative value of $a$ such as $a = -1$ or $a = -2$. Looking at the graphs of FIGURE 3.6, it is apparent that $x = a/3$ corresponds to a local maximum and $x = a$ corresponds to a local minima. Neither points are absolute maxima or minima. The second derivative, $a(6x - 4a)$ is zero when $x = 2a/3$. Thus, $f_1(x)$ has a point of inflection at $x = 2/3$ and $f_2(x)$ has a
point of inflection at \( x = 4/3 \). It is clear from FIGURE 3.6 that the points of inflection are about where calculus techniques tell us they are. It would be awkward to locate these points of inflection without calculus and VERY awkward to locate them as functions of the parameter \( a \) without calculus.

**Second Derivative Test**

There is another important fact that we can learn from FIGURE 3.6. The function \( f_2(x) \) has a local maximum at 2/3. Look at values of \( f_2'(x) \) for values of \( x \) starting a little bit less than 2/3 and ranging to values a little bit more than 2/3. Note that \( f_2'(x) \) is decreasing as \( x \) ranges through these values. This means that the second derivative, \( f_2''(2/3) \), is negative. The general rules are as follows:

If \( t \) is a local maximum of a function \( f(x) \) then \( f''(t) \leq 0 \).

If \( t \) is a local minimum of a function \( f(x) \) then \( f''(t) \geq 0 \).

An equivalent and more useful statement of these rules is given in SECOND DERIVATIVE TEST 3.7.

3.7 SECOND DERIVATIVE TEST Let \( f(x) \) be a function and suppose that at \( x = t \), \( f'(t) = 0 \). If \( f''(t) < 0 \) then \( f \) has a local maximum at \( t \). If \( f''(t) > 0 \) then \( f \) has a local minimum at \( t \).

At \( a/3 \), \( f_2''(a/3) = -2a^2 \) is negative for all values of \( a \) and hence \( a/3 \) is a local maximum. At \( a \), \( f_2''(a) = 2a^2 \) is positive for all \( a \) and hence \( a \) is a local minimum. These facts are obvious from the graphs of FIGURE 3.6.

For any point \( x \), if \( f''(x) < 0 \) then \( f \) is said to be "concave downwards" at \( x \) and if \( f''(x) > 0 \) then \( f \) is said to be "concave upwards" at \( x \). Referring to FIGURE 3.6, \( f_2 \) is concave downward at \( x = .5 \) and concave upwards at \( x = 1.8 \).

We now review the basic principles of graphing functions.

3.8 REVIEW OF GRAPHING FUNCTIONS

**Compute Enough Values To Know The General Shape**

(1) Given a function \( f(x) \) to be graphed, write a simple BASIC program (or program in some other computer language) such as those of PROGRAMS 3.1 and use it to generate points on the graph of \( f(x) \). Sketch a graph of the function to get a good feeling for its shape. In the case where you are dealing with a parameterized family of functions, draw several graphs for represen-
tative choices of the parameter (as was done in FIGURE 3.6). When you
have had some experience with this sort of problem you may not have to
draw the graph but simply look at the list of points on the graph (i.e., the
output of your program) to get all of the information you need at this stage.

**Study The Critical Points**

(2) Next try to locate the local and absolute maxima and minima. The
interval over which the function is defined will be important here as some
local maxima and minima may not be included in the interval. Also, an
absolute maximum and/or minimum may occur at the endpoints of the interval.
If a local maximum or minimum of a function \( f(x) \) occurs at a point \( x \) where
\( f'(x) \) is defined, then \( f'(x) = 0 \) at this point \( x \). This fact can be a help in
locating these local maxima and minima. The points \( x \) where \( f'(x) = 0 \) are
called the critical points of \( f \). Just because \( f'(x) = 0 \) it does not necessarily
follow that \( x \) is a local maximum or minimum of \( f(x) \). For example, take
\( f(x) = x^3 \) and \( x = 0 \). If you are dealing with a parameterized family of
functions, try to express the critical points, local maxima, and local minima
in terms of the parameter or parameters as was done in EXAMPLE 3.3. This
may not always be possible as it may be very difficult to solve the equation
\( f'(x) = 0 \) in terms of the parameter. Even when no explicit solution can be
given, there are sometimes more sophisticated techniques that can give useful
information about the roots of \( f'(x) = 0 \) in terms of the parameters.

**Find The Points Of Inflection**

(3) After steps (1) and (2) you should have a very good idea of what the
graph of \( f(x) \) looks like. You should now notice where the function is concave
upwards and concave downwards. A function is concave upwards at \( x \) if the
second derivative \( f''(x) > 0 \) and concave downwards at \( x \) if \( f''(x) < 0 \). Points
\( x \) where \( f(x) \) changes from being concave upwards to concave downwards or
from concave downwards to concave upwards are called points of inflection
of \( f \). If \( x \) is a point of inflection of \( f \) and \( f''(x) \) is defined, then \( f''(x) = 0 \).
This fact can help in locating points of inflection precisely. Just because \( f''(x) = 0 \)
doesn’t mean that you have a point of inflection. For example, look at
\( f(x) = x^4 \) and \( x = 0 \). If \( x \) is such that \( f'(x) = 0 \) and \( f \) is concave downwards
at \( x \) (i.e., \( f''(x) < 0 \)) then \( x \) is a local maximum of \( f \). If \( f'(x) = 0 \) and \( f \) is
concave upwards at \( x \) then \( x \) is a local minimum of \( f \). These facts are the
basis for SECOND DERIVATIVE TEST 3.7.
**Taylor Polynomials, L'Hopital's Rule**

We shall conclude our discussion of graphing functions by giving a number of exercises. Before doing so, however, we introduce some additional ideas that will be helpful in solving these problems. These ideas, which are standard topics in calculus courses, are called "limits," "Taylor polynomials," and "L'Hopital's Rule." Here we emphasize intuitive ideas, returning for a more careful treatment in Chapter 5. If you take a more advanced course in calculus or in mathematical analysis you will see a "rigorous" treatment of these topics.

To illustrate these topics we shall study the following two examples:

\[
s(x) = \frac{e^x - e^{-x}}{\sin(x)} \quad \text{and} \quad a(x) = \frac{\ln(x)}{|x - 1|}.
\]

Values of \(s(x)\) are computed in PROGRAM SEQUENCE 3.10(a) and (b). Values of \(a(x)\) are computed in PROGRAM SEQUENCE 3.10(c) and (d). The first thing to notice about these functions is that, at \(x = 0\) for \(s(x)\) and at \(x = 1\) for \(a(x)\), the numerators and denominators are zero. This is why statement 10 in each of the four programs of PROGRAM SEQUENCE 3.10 is constructed so that the values \(x = 0\), for \(s(x)\), and \(x = 1\), for \(a(x)\), are not generated. The function \(s(x)\) is a very crazy function which undergoes wild oscillations as \(x\) gets larger and larger ("goes to infinity") or smaller and smaller ("goes to minus infinity"). These oscillations are caused by the oscillations of \(\sin(x)\) in the denominator.

At the moment, we are concerned only about the behavior of \(s(x)\) near \(x = 0\) and \(a(x)\) near \(x = 1\). In PROGRAM SEQUENCE 3.10(a) we have generated some values of \(s(x)\) in the interval \([-3, +3]\). Notice that as \(x\) approaches 0 through negative values, the values of \(s(x)\) approach 2. Likewise, as \(x\) approaches 0 through positive values, the values of \(s(x)\) approach 2. This should be compared with the behavior of \(a(x)\) near \(x = 1\) as shown in PROGRAM SEQUENCE 3.10(c). As \(x\) approaches 1 through values less than 1, the values of \(a(x)\) approach \(-1\). As \(x\) approaches 1 through values greater than 1, the values of \(a(x)\) approach \(+1\). Alternatively, we say "the limit of \(s(x)\) as \(x\) approaches 0 from below is 2" or "the limit of \(a(x)\) as \(x\) approaches 1 from above is \(+1\)." Symbolically, these statements are made as in LIMIT EXPRESSIONS 3.9.

### 3.9 LIMIT EXPRESSIONS

\[
\lim_{x \to 0^-} s(x) = +2 \quad \text{and} \quad \lim_{x \to 0^+} s(x) = +2
\]
$\lim_{x \to 1^-} a(x) = -1$ and $\lim_{x \to 1^+} a(x) = +1$

where $s(x) = \frac{e^x - e^{-x}}{\sin(x)}$ and $a(x) = \frac{\ln(x)}{|x - 1|}$

### 3.10 PROGRAM SEQUENCE

(a)

10 FOR X = -3 TO +3 STEP .21
20 PRINT X,(EXP(X) - EXP(-X))/SIN(X)
30 NEXT X

<table>
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<th>x</th>
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</tr>
</thead>
<tbody>
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94
### (b)

```
10 FOR X = -.0055 TO .0055 STEP .001
20 PRINT X, (EXP(X) - EXP(-X))/SIN(X)
30 NEXT X
```

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### (c)

```
10 FOR X = +.15 TO 2 STEP 0.1
20 PRINT X, LOG(X)/ABS(X - 1)
30 NEXT X
```

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</thead>
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</table>
(d)

10 FOR X = +0.99 TO +1.01 STEP 0.0011
20 PRINT X;LOG(X)/ABS(X-1)
30 NEXT X

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In our discussion of LIMIT EXPRESSIONS 3.9 we did not define the concept of a limit precisely. A precise definition of a limit is given in CHAPTER 5 (DEFINITION 5.66). The intuitive idea of a limit as expressed in our previous discussion is important, however. The validity of the assertions of LIMIT EXPRESSIONS 3.9 is, for the moment, based solely on the computer data of PROGRAM SEQUENCE 3.10. From PROGRAM SEQUENCE 3.10(a) we see that \( s(-.06) = 2.002 \) and \( s(.15) = 2.02 \). In PROGRAM SEQUENCE 3.10(b) we take a closer look at what is happening to \( s(x) \) near \( x = 0 \). There we look at the values of \( s(x) \) for \( x \) ranging from \(-.0055 \) (or \(-5.5E-03 \) in scientific notation) to \( .0055 \). We find that \( s(-.0005) = 2.00000027 \) and \( s(.0005) = 1.99999983 \). If you try these computations on your computer, you might get slightly different answers due to different methods of computing the functions \( \sin(x) \) and \( e^x \). The message will be the same, however, that

\[
\lim_{x \to 0^-} s(x) = +2 \quad \text{and} \quad \lim_{x \to 0^+} s(x) = +2.
\]
The numerator of \( s(x) \) is \( e^x - e^{-x} \) and the denominator is \( \sin(x) \). Both of these functions are zero when \( x = 0 \). For this reason we say that \( s(x) \) is "not defined" for \( x = 0 \). If we want to define \( s(x) \) for \( x = 0 \) the natural choice is \( s(0) = +2 \) since the limits from above and below are both equal to +2 at \( x = 0 \). Formally, we could write

\[
\begin{align*}
  s(x) &= \begin{cases} 
    \frac{e^x - e^{-x}}{\sin(x)} & \text{if } x \neq 0 \\
    +2 & \text{if } x = 0
  \end{cases}
\end{align*}
\]

The following definition will be useful in discussing this expanded definition of \( s(x) \):

3.11 DEFINITION  Let \( f(x) \) be a function. Suppose that for \( x = t \) we have

\[
\lim_{x \to t^-} f(x) = r \quad \text{and} \quad \lim_{x \to t^+} f(x) = r.
\]

Then we say that "the limit of \( f(x) \) as \( x \) approaches \( t \) is \( r \)" and write

\[
\lim_{x \to t} f(x) = r.
\]

The idea of DEFINITION 3.11 is that if both the limit from above and the limit from below exist at a point \( t \) and are equal to the same number \( r \) then that value \( r \) is simply called the limit at \( t \). Thus for our function \( s(x) \),

\[
\lim_{x \to 0} s(x) = +2.
\]

This leads to the definition of continuous functions.

**Continuous Functions**

3.12 DEFINITION  Let \( f(x) \) be a function. Suppose that at \( x = t \), \( f(t) \) is defined and

\[
\lim_{x \to t} f(x) = f(t).
\]

Then we say that \( f(x) \) is *continuous* at \( x = t \).

The intuitive idea of DEFINITION 3.12 is that the graph of a continuous function has no jumps or tears in it. In other words, you can draw the graph without taking your pencil off the paper. To better understand this concept,
we now take a closer look at the function \( a(x) \) which, as we have already noted informally in connection with LIMIT EXPRESSIONS 3.9, is not continuous (or "discontinuous") at \( x = 1 \).

PROGRAM SEQUENCE 3.10(c) gives values of \( a(x) = (\ln(x))/|x - 1| \) for various values of \( x \) between \( x = 0.15 \) and \( x = 2 \). The value \( x = 1 \) is not included here as both the numerator and denominator of \( a(x) \) vanish at \( x = 0 \). By inspecting these values, one becomes strongly suspicious that

\[
\lim_{x \to 1^-} a(x) = -1 \quad \text{and} \quad \lim_{x \to 1^+} a(x) = +1.
\]

This suspicion is confirmed by PROGRAM SEQUENCE 3.10(d), where we take a closer look at values of \( a(x) \) near \( x = 1 \). This means that the graph of \( a(x) \) takes a sudden jump as \( x \) goes past the point \( x = 1 \). We could define \( a(1) \) to be something, just as we assigned a value to \( s(0) \) above. If we define \( a(1) = -1 \) then \( a(x) \) would be "continuous from below" or "left continuous" at \( x = 0 \). If we define \( a(1) = +1 \) then \( a(x) \) would be "continuous from above" or "right continuous" at \( x = 1 \). No assignment of a value to \( a(1) \) can make the function \( a(x) \) continuous at \( x = 1 \) in the sense of DEFINITION 3.12, however.

**Limit As x Approaches Infinity**

In discussing limits, limit \( f(x) \), we have assumed that \( t \) is a real number. We can also discuss limit \( f(x) \) where the symbol \( +\infty \) stands for "plus infinity." We can also look at limit \( f(x) \) where \( -\infty \) stands for "minus infinity." To say that

\[
\lim_{x \to +\infty} f(x) = L
\]

means that as \( x \) gets larger and larger the values of \( f(x) \) get closer and closer to \( L \). As an example, consider \( f(x) = (2x + x^{-1})/(4x + x^{-1}) \). A BASIC program to evaluate the limit of \( f(x) \) as \( x \) gets larger and larger is given in PROGRAM 3.13. This program contains an infinite loop and execution must be aborted by the user. It is evident that

\[
\lim_{x \to +\infty} f(x) = .5.
\]

It is obvious from the definition of \( f(x) \) that \( f(x) \) tends to .5 as \( x \) tends to plus infinity. The term \( x^{-1} \) goes to zero as \( x \) gets large. Thus for large values of \( x \), \( f(x) \) is essentially equal to \( 2x/4x \), which is .5.
3.13 PROGRAM FOR AN INFINITE LIMIT

10 X = 1000
20 PRINT (2·X + X⁻¹ - 1)/(4·X + X⁻¹ - 1)
30 X = X + 1000
40 GOTO 20

READY.

.500000125
.500000016
.500000007
.500000004
.500000002
.500000002
.500000001
.500000001
.500000001
.5
.5
.5
.5
.5
.5
.5
.5
.5
.5
.5
.5
.5
.5

If the reader will think about the type of calculations used to evaluate limits, namely those of PROGRAM SEQUENCE 3.10 and PROGRAM 3.13, it will be obvious that the following rules for computing limits are valid:

3.14 RULES FOR LIMITS For any real number \( r \) and any functions \( f(x) \) and \( g(x) \)

1. \( \lim_{x \to t} rf(x) = r \lim_{x \to t} f(x) \)

2. \( \lim_{x \to t} (f(x) + g(x)) = \lim_{x \to t} f(x) + \lim_{x \to t} g(x) \)

3. \( \lim_{x \to t} (f(x)g(x)) = (\lim_{x \to t} f(x))(\lim_{x \to t} g(x)) \)
\[
(4) \quad \lim_{x \to t} \frac{f(x)}{g(x)} = \lim_{x \to t} \frac{f(x)}{g(x)}.
\]

In rule (4) we must have \( g(x) \) nonzero. These rules apply if limit is replaced by limit or limit and if \( t \) is plus or minus infinity.

You will find that you will apply the RULES FOR LIMITS 3.14 without even thinking about it. Notice how much simpler RULES 3.14(3) and (4) are than the corresponding rules for derivatives!

If we try to apply RULE 3.14(4) to either of the functions \( s(x) \) or \( a(x) \) of LIMIT EXPRESSIONS 3.9 we end up with no useful information. For example, apply RULE 3.14(4) to \( a(x) = f(x)/g(x) \) where \( f(x) = e^x - e^{-x} \) and \( g(x) = |x - 1| \) (take \( t = 1 \)):

\[
\lim_{x \to 1^-} \frac{f(x)}{g(x)} = \lim_{x \to 1^-} \frac{f(x)}{g(x)} = 0.
\]

The expression 0/0 is not defined. In words, we say that the limit from below of \( f(x)/g(x) \) is "of the form 0/0." For this example, the limit from above of \( f(x) \) is also of the form 0/0. Another example of this type of difficulty is obtained by taking \( f(x) \) to be \( x^2 \) and \( g(x) \) to be \( e^x \). Again trying to apply RULE 3.14(4), now with \( t = \infty \), we obtain

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.
\]

In the above case, we say that the limit as \( x \) approaches infinity of \( f(x)/g(x) \) is of the form \( \infty/\infty \). We need not have \( t = \infty \) to obtain a limit of the form \( \infty/\infty \). For example, let \( f(x) = 1 + 2(x - 2)^{-3} \) and let \( g(x) = 1 + 3(x - 2)^{-3} \). Then the limit as \( x \) approaches 2 of \( f(x)/g(x) \) is of the form \( \infty/\infty \).

We now discuss some methods to deal with limits of the form 0/0 or \( \infty/\infty \). More examples will be given in the exercises. One simple way that works with ratios of polynomials (so called "rational functions") is to use some algebraic tricks to make the limiting value obvious when using rules such as 3.14(4). As a trivial example, consider \( f(x)/g(x) \) where \( f(x) = x^3 \) and \( g(x) = x^2 \). Using 3.14(4), this is of the form 0/0 as \( x \) approaches 0 and of the form \( \infty/\infty \) as \( x \) approaches infinity. But, of course, we would just write
\[ \frac{f(x)}{g(x)} = x. \] Then it is obvious that \( f(x)/g(x) \) approaches 0 as \( x \) approaches 0 and approaches infinity (i.e., gets arbitrarily large) as \( x \) approaches infinity. A slightly more interesting example is the case \( f(x) = 1 + 2(x - 2)^{-3} \) and \( g(x) = 1 + 3(x - 2)^{-3} \). In this case we multiply numerator and denominator by \((x - 2)^{3}\) to obtain

\[ \frac{f(x)}{g(x)} = \frac{(x - 2)^{3} + 2}{(x - 2)^{3} + 3}. \]

It is obvious from this expression for \( f(x)/g(x) \) that the limit as \( x \) approaches 2 is \( 2/3 \). This type of trick works for ratios of polynomials but what about ratios such as those for \( s(x) \) and \( a(x) \) of LIMIT EXPRESSIONS 3.9? We have \( s(x) \) as a ratio of \( e^{x} - e^{-x} \) and \( \sin(x) \) and neither of these functions are polynomials. A mathematician named Taylor (he died in 1731) discovered a way to replace such functions as \( e^{x} - e^{-x} \) and \( \sin(x) \) by polynomials. When this is done we can then use algebraic tricks to find limits. Before discussing Taylor's method, we discuss a useful related method known as L'Hopital's rule.

### L'Hopital's Rule

**3.15 L'HOPITAL'S RULE** For reasonable functions \( f(x) \) and \( g(x) \), if

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{or} \\ \infty \end{cases} \]

then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \]

Instead of limit we could use \( \lim_{x \to a} \) or \( \lim_{x \to a+} \).

You should, of course, wonder what is meant by "reasonable" in L'HOPITAL'S RULE 3.15. To learn more, read the article by A. E. Taylor referenced in Appendix 2. We won't bother with a precise definition of "reasonable" at this point. Our point of view is that we can use L'HOPITAL'S RULE to help us guess the limit, but we shall be a bit wary of our answer and, if concerned, check it against other evidence (such as a computer generation of points on the graph of \( f(x)/g(x) \)). We shall give an example of L'HOPITAL'S RULE failing, but first let's apply the rule to our functions \( s(x) \) and \( a(x) \) of LIMIT EXPRESSIONS 3.9.
Consider our function $s(x) = f(x)/g(x)$ where $f(x) = e^x - e^{-x}$ and $g(x) = \sin(x)$. Viewed in this way, $s(x)$ has the form $0/0$ as $x$ approaches $0$. Thus we can apply L’HOPITAL’S RULE with $f'(x) = e^x + e^{-x}$ and $g'(x) = \cos(x)$. By L’HOPITAL’S RULE

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{\sin(x)} = \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos(x)} = \frac{2}{1} = 2.$$

This result certainly checks with PROGRAM SEQUENCE 3.10(a) and (b), so $f(x) = e^x - e^{-x}$ and $g(x) = \sin(x)$ must be reasonable functions (it’s true, they are!).

Now let’s take a look at $a(x) = f(x)/g(x)$ where $f(x) = \ln(x)$ and $g(x) = |x - 1|$. With this choice of $f(x)$ and $g(x)$, $a(x)$ has the form $0/0$ as $x$ approaches $1$. We apply L’HOPITAL’S RULE with $f'(x) = 1/x$ and

$$g'(x) = \begin{cases} 
-1 & \text{if } x < 1 \\
+1 & \text{if } x > 1 
\end{cases}.$$

By L’HOPITAL’S RULE, we find that for the limit from below

$$\lim_{x \to 1^-} \frac{\ln(x)}{|x - 1|} = \lim_{x \to 1^-} \frac{1/x}{-1} = \frac{-1}{1} = -1$$

and for the limit from above

$$\lim_{x \to 1^+} \frac{\ln(x)}{|x - 1|} = \lim_{x \to 1^+} \frac{1/x}{+1} = \frac{+1}{1} = +1.$$

These results check with what we found in PROGRAM SEQUENCE 3.10(c) and (d).

**You May Have To Apply L’Hopital’s Rule Many Times**

As yet another example of L’HOPITAL’S RULE, consider $f(x)/g(x)$ where $f(x) = x^2$ and $g(x) = e^x$. As $x$ approaches plus infinity, the limit is of the form $\infty/\infty$. By L’HOPITAL’S RULE, we have

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{2x}{e^x}.$$

But, the latter limit is also of the form $\infty/\infty$. We can apply L’HOPITAL’S RULE to $2x/e^x$ to get

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.$$
We thus conclude that

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = 0.
\]

To reach this conclusion we had to apply L’HOPITAL’S RULE twice. If we had tried to find the limit as x approaches infinity of \(x^3/e^x\) using L’HOPITAL’S RULE we would end up applying the rule 3 times to conclude

\[
\lim_{x \to \infty} \frac{x^3}{e^x} = 0.
\]

In fact, by applying L’HOPITAL’S RULE \(n\) times to \(x^n/e^x\) (for \(n\) a fixed positive integer) we would find

\[
\lim_{x \to \infty} \frac{x^n}{e^x} = 0.
\]

Results like these will be considered in the exercises. We conclude our brief discussion of L’HOPITAL’S RULE with an example of the failure of L’HOPITAL’S RULE. This example can be skipped without affecting your ability to work the exercises that follow. Mathematically inclined students will rightfully insist on such an example.

---

**There Are “Unreasonable” Functions**

Consider \(f(x)/g(x)\) where \(f(x) = x^2 \sin(1/x)\) and \(g(x) = \sin(x)\). As \(x\) approaches zero, this expression becomes of the form 0/0. That \(g(x) = \sin(x)\) approaches 0 as \(x\) approaches 0 is obvious. The function \(f(x)\) is a bit crazy, however! Some values of \(\sin(1/x)\) near \(x = 0\) are shown in PROGRAM SEQUENCE 3.16(a). Of course, all values of \(\sin(1/x)\) will be between \(-1\) and \(+1\), but notice how wildly the values of \(\sin(1/x)\) oscillate in PROGRAM SEQUENCE 3.16(a). This is because as \(x\) approaches 0, \(1/x\) goes to infinity, forcing \(\sin(1/x)\) to oscillate infinitely often between \(-1\) and \(+1\). When we take the product \(x^2 \sin(1/x)\) these oscillations are forced to be smaller and smaller by the fact that \(x^2\) goes to zero. Thus \(f(x) = x^2 \sin(1/x)\) approaches 0 as \(x\) approaches 0.
### 3.16 PROGRAM SEQUENCE FOR L'HOPITAL'S FAILURE

**(a)**

```plaintext
10 FOR X = -.01 TO +.01 STEP .0015
20 PRINT X, SIN(1/X)
30 NEXT X
READY.
```

<table>
<thead>
<tr>
<th>x</th>
<th>sin(1/x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.01</td>
<td>.506365668</td>
</tr>
<tr>
<td>-8.5E-03</td>
<td>.986799078</td>
</tr>
<tr>
<td>-7E-03</td>
<td>.996362208</td>
</tr>
<tr>
<td>-5.5E-03</td>
<td>.384062462</td>
</tr>
<tr>
<td>-4E-03</td>
<td>.970528058</td>
</tr>
<tr>
<td>-2.5E-03</td>
<td>.850919195</td>
</tr>
<tr>
<td>-1E-03</td>
<td>-.826878546</td>
</tr>
<tr>
<td>4.99999998E-04</td>
<td>.93003695</td>
</tr>
<tr>
<td>2E-03</td>
<td>-.467772183</td>
</tr>
<tr>
<td>3.5E-03</td>
<td>.169818632</td>
</tr>
<tr>
<td>5E-03</td>
<td>-.87329726</td>
</tr>
<tr>
<td>6.5E-03</td>
<td>.0917569965</td>
</tr>
<tr>
<td>8E-03</td>
<td>-.616040486</td>
</tr>
<tr>
<td>9.5E-03</td>
<td>-.999803908</td>
</tr>
</tbody>
</table>

**(b)**

```plaintext
10 FOR X = -.01 TO +.01 STEP .0015
20 PRINT X, ((X^2) * SIN(1/X)) / SIN(X)
30 NEXT X
READY.
```

<table>
<thead>
<tr>
<th>x</th>
<th>(x^2sin(1/x))/sin(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.01</td>
<td>-5.06374108E-03</td>
</tr>
<tr>
<td>-8.5E-03</td>
<td>-8.38789317E-03</td>
</tr>
<tr>
<td>-7E-03</td>
<td>-6.97459243E-03</td>
</tr>
<tr>
<td>-5.5E-03</td>
<td>-2.11235419E-03</td>
</tr>
<tr>
<td>-4E-03</td>
<td>-3.8821226E-03</td>
</tr>
<tr>
<td>-2.5E-03</td>
<td>-2.12730021E-03</td>
</tr>
<tr>
<td>-1E-03</td>
<td>8.26878689E-04</td>
</tr>
<tr>
<td>4.99999998E-04</td>
<td>4.650E-04</td>
</tr>
<tr>
<td>2E-03</td>
<td>-9.35544989E-04</td>
</tr>
<tr>
<td>3.5E-03</td>
<td>5.94366426E-04</td>
</tr>
</tbody>
</table>

104
\[
\begin{array}{cc}
\hline
x & (x^2 \sin(1/x)) / \sin(x) \\
\hline
5E-03 & -4.3665045E-03 \\
6.5E-03 & 5.96424027E-04 \\
8E-03 & -4.92837645E-03 \\
9.5E-03 & -9.49828001E-03 \\
\hline
\end{array}
\]

(c)

\[
10 \text{ FOR } X = -.01 \text{ TO } +.01 \text{ STEP } .0015 \\
20 \text{ PRINT } X, \text{COS}(1/X) \\
30 \text{ NEXT } X
\]

READY.

\[
\begin{array}{cc}
\hline
x & \cos(1/x) \\
\hline
- .01 & .862318856 \\
- 8.5E-03 & -.161949312 \\
- 7E-03 & -.0652194656 \\
- 5.5E-03 & .923307113 \\
- 5E-03 & .240988149 \\
- 2.5E-03 & -.525296685 \\
- 1E-03 & .562380848 \\
4.99999998E-04 & -.367466 \\
2E-03 & -.883849161 \\
3.5E-03 & -.985475317 \\
5E-03 & .487187742 \\
6.5E-03 & -.995781433 \\
8E-03 & .787714491 \\
9.5E-03 & .019802684 \\
\hline
\end{array}
\]

(d)

\[
10 \text{ FOR } X = -.01 \text{ TO } +.01 \text{ STEP } .0015 \\
20 \text{ PRINT } X,(2x \cdot \sin(1/X) - \cos(1/X))/\cos(x) \\
30 \text{ NEXT } X
\]

READY.

\[
\begin{array}{cc}
\hline
x & (2x \sin(1/x) - \cos(1/x))/\cos(x) \\
\hline
- .01 & -.872489794 \\
- 8.5E-03 & .145178972 \\
- 7E-03 & .0712721408 \\
- 5.5E-03 & -.92754583 \\
- 5E-03 & -.248754363 \\
- 2.5E-03 & .521043718 \\
- 1E-03 & -.560727371 \\
\hline
\end{array}
\]
\[
\begin{array}{c|c}
 x & (2x\sin(1/x) - \cos(1/x))/\cos(x) \\
\hline
4.99999998E-04 & .368396096 \\
2E-03 & .861979837 \\
3.5E-03 & .966670091 \\
5E-03 & -.495926914 \\
6.5E-03 & .996995335 \\
8E-03 & -.797596662 \\
9.5E-03 & -.0388007091 \\
\end{array}
\]

Now look at PROGRAM SEQUENCE 3.16(b) where we have generated values of \( f(x)/g(x) \). These values seem to indicate that

\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = 0.
\]

It is in fact true that the above limit is 0. But, if we try to apply L'HOPITAL'S RULE we compute \( g'(x) = \cos(x) \) and \( f'(x) = 2x\sin(1/x) - \cos(1/x) \). The function \( \cos(1/x) \) that occurs in \( f'(x) \) is another "bad actor" and oscillates wildly near zero as is shown in PROGRAM SEQUENCE 3.16(c). These oscillations cause \( f'(x)/g'(x) \) to oscillate wildly also near zero (see PROGRAM SEQUENCE 3.16(d)) so that limit \( \frac{f'(x)}{g'(x)} \) does not exist. Thus, even though \( f(x)/g(x) \) behaves well near \( x = 0 \) and has a limit there, L'HOPITAL'S RULE is no help in finding it. Basically, this is how all failures of L'HOPITAL'S RULE occur. If you are interested, you should look up the precise statement of L'HOPITAL'S RULE (e.g. Wikipedia). You will see that the existence of the limit \( f'(x)/g'(x) \) is the key assumption necessary for the rule to succeed. The correctly stated L'HOPITAL'S RULE never fails.

We have already remarked above that functions such as \( \sin(x) \), \( \cos(x) \), \( e^x \), and \( \ln(x) \) can be approximated by polynomials (called "Taylor Polynomials"). In TAYLOR POLYNOMIALS 3.17, we give such a list.

### Taylor Polynomials

#### 3.17 TAYLOR POLYNOMIALS

For \( x \) near zero we have

1. \( e^x = 1 + x + \frac{x^2}{2} \)
2. \( \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \)
3. \( \sin(x) = x - \frac{x^3}{6} \)
\[ \cos(x) = 1 - \frac{x^2}{2} \]
\[ \tan(x) = x + \frac{x^3}{3} \]

We are being a bit sloppy in TAYLOR POLYNOMIALS 3.17 because none of these statements are true equalities. They are only approximations. The reader should now take a close look at PROGRAM SEQUENCE 3.18 which shows how good these approximations are for various values of \( x \) near \( x = 0 \). Of course, when a computer evaluates a function like \( \sin(x) \), it is programmed to make a certain approximation selected by an expert in the subject of numerical analysis (hopefully!). Thus, what we are doing is comparing our simple Taylor polynomials with these approximations. As you can see, the Taylor polynomials of PROGRAM SEQUENCE 3.18 are pretty good over the ranges chosen. You should try out different sequences of \( x \) near 0 on your own computer. For each of the functions \( f(x) \) of PROGRAM SEQUENCE 3.18, the general rule is to form the polynomial

\[ p(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3. \]

In the above formula for \( p(x) \), the notation \( f^{(n)}(0) \) means the \( n \)th derivative of \( f(x) \) evaluated at 0. In the case of \( e^x \) in TAYLOR POLYNOMIALS 3.17, we have left off the term involving \( x^3 \). One can compute Taylor polynomials of any degree for the functions \( f(x) \) above. In general, the coefficient of \( x^n \) is \( f^{(n)}(0)/n! \) where \( n! = n(n - 1)(n - 2) \ldots (3)(2)(1) \) is the product of the first \( n \) integers. The higher the degree of the polynomial the better it approximates \( f(x) \) near \( x = 0 \). Of course, anybody can write down a polynomial and claim it "approximates \( f(x) \)." The question, of course, is \textit{how well} does it approximate \( f(x) \)? We have answered this question by a simple computer test. A more theoretical study of the errors involved in approximating with Taylor polynomials can be found in most calculus books. Sometimes the Taylor polynomials such as we have been considering (approximating \( f(x) \) near \( x = 0 \)) are called Maclaurin polynomials or "series."

3.18 PROGRAM SEQUENCE FOR TAYLOR POLYNOMIALS

\begin{verbatim}
10 FOR X = -.18 TO .18 STEP .04
20 PRINT X,EXP(X),1+X+(X^2)/2
30 NEXT X
\end{verbatim}
<table>
<thead>
<tr>
<th>x</th>
<th>e^x</th>
<th>1 + x + x^2 / 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.18</td>
<td>.835270212</td>
<td>.8362</td>
</tr>
<tr>
<td>-.14</td>
<td>.869358235</td>
<td>.8698</td>
</tr>
<tr>
<td>-.1</td>
<td>.904837418</td>
<td>.905</td>
</tr>
<tr>
<td>-.06</td>
<td>.941764534</td>
<td>.9418</td>
</tr>
<tr>
<td>-.02</td>
<td>.980198673</td>
<td>.9802</td>
</tr>
<tr>
<td>0.02</td>
<td>1.02020134</td>
<td>1.0202</td>
</tr>
<tr>
<td>0.06</td>
<td>1.06183655</td>
<td>1.0618</td>
</tr>
<tr>
<td>.1</td>
<td>1.10517092</td>
<td>1.105</td>
</tr>
<tr>
<td>.14</td>
<td>1.1502738</td>
<td>1.1498</td>
</tr>
</tbody>
</table>

(b)

10 FOR X = -.18 TO .18 STEP .04
20 PRINT X,LOG(1 + X)^2,X - (X^2)/2 + (X^3)/3
30 NEXT X

<table>
<thead>
<tr>
<th>x</th>
<th>ln(1 + x)</th>
<th>x - x^2/2 + x^3/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.18</td>
<td>-.198450939</td>
<td>-.198144</td>
</tr>
<tr>
<td>-.14</td>
<td>-.15082289</td>
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</tr>
<tr>
<td>.1</td>
<td>-.105360516</td>
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</tr>
<tr>
<td>-.06</td>
<td>-.0618754037</td>
<td>-.061872</td>
</tr>
<tr>
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<td>-.0202026667</td>
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<tr>
<td>.1</td>
<td>.0953101797</td>
<td>.095333334</td>
</tr>
<tr>
<td>.14</td>
<td>.131028262</td>
<td>.131114667</td>
</tr>
</tbody>
</table>

(c)

10 FOR X = -.18 TO .18 STEP .04
20 PRINT X,SIN(X),X - (X^3)/6
30 NEXT X

<table>
<thead>
<tr>
<th>x</th>
<th>sin(x)</th>
<th>x - x^3/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.18</td>
<td>-.179029573</td>
<td>-.179028</td>
</tr>
<tr>
<td>-.14</td>
<td>-.139543115</td>
<td>-.139542667</td>
</tr>
<tr>
<td>-.1</td>
<td>-.0998334167</td>
<td>-.099833333</td>
</tr>
<tr>
<td>-.06</td>
<td>-.0599640065</td>
<td>-.059964</td>
</tr>
<tr>
<td>-.02</td>
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<tr>
<td>.02</td>
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</tr>
<tr>
<td>.14</td>
<td>.139543115</td>
<td>.139542667</td>
</tr>
</tbody>
</table>
We now give some exercises on computing limits and graphing functions. The reader should memorize TAYLOR POLYNOMIALS 3.17. For additional hints and examples look at SOLUTIONS 3.20.

### 3.19 EXERCISES

1. In this exercise we ask the student to "play calculus instructor" and make up some exercises on limits along the guidelines suggested.
(a) One way to make hard-looking limit problems is to factor a polynomial and form the quotient of the polynomial with one or more of its factors. For example, divide \((x - 2)(x + 1)\) by \(x - 2\) and take the limit as \(x\) approaches 2. In symbols,

\[
\lim_{x \to 2} \frac{(x - 2)(x + 1)}{(x - 2)}.
\]

This limit is of the form \(0/0\) as both the numerator and denominator evaluated at 2 become 0. The obvious thing to do in this case is to cancel \(x - 2\) from the numerator and denominator (i.e., divide numerator and denominator by \(x - 2\)) to obtain

\[
\lim_{x \to 2} (x + 1) = 3.
\]

So far this seems simple enough. Suppose you give this problem to a friend but, before doing so, multiply \((x - 2)\) times \((x + 1)\) to get \(x^2 - x - 2\). The problem becomes

\[
\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = ?
\]

If your friend hasn’t seen the first version of the problem, he or she will have to divide the numerator by the denominator to evaluate the limit.

(b) If you worked part (a) above, you may have found that it is easy to construct hard problems that require some awkward polynomial arithmetic to solve. You might, for example, start with

\[
\lim_{x \to 1} \frac{(x^3 - x^2 - x - 1)(x^2 - 1)}{(x^2 - 1)}.
\]

Now take the product \((x^3 - x^2 - x - 1)(x^2 - 1) = x^5 - x^4 - 2x^3 + x + 1\) and give your friend the problem

\[
\lim_{x \to 1} \frac{x^5 - x^4 - 2x^3 + x + 1}{x^2 - 1} = ?
\]

One could use polynomial division to work this problem (divide numerator into denominator and evaluate the resulting polynomial at \(x = \)}
1). An easier way is to use L'HOPITAL'S RULE 3.15. In this case, \( f(x) = x^3 - x^4 - 2x^3 + x + 1 \) and \( g(x) = x^2 - 1 \). Thus, \( f'(x) = 5x^4 - 4x^3 - 6x^2 + 1 \) and \( g'(x) = 2x \). We then see that

\[
\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{5x^4 - 4x^3 - 6x^2 + 1}{2x} = -2.
\]

Note in this last limit the denominator evaluated at \( x = 1 \) is not zero so the limit is obtained by substituting \( x = 1 \) and evaluating the resulting expression. Notice that, instead of dividing by \( x^2 - 1 \) in this problem, we could have divided by some polynomial times \( x^2 - 1 \). For example, we could have divided by \((x + 1)(x^2 - 1) = x^3 + x^2 - x - 1\). Using the method of polynomial division of (a) above would not work on this problem, but L'HOPITAL'S RULE still works fine.

(2) This exercise is concerned with limits that can be evaluated using the TAYLOR POLYNOMIALS 3.17. First of all, you should recall that these polynomials are only valid approximations for small values of \( x \) (\( x \) near zero). For example, \( \cos(x) = 1 - \frac{x^2}{2} \) cannot be a good approximation for large \( x \) as the right-hand side becomes very large and negative for all large values of \( x \) while the \( \cos(x) \) function varies between \(-1\) and \( +1 \) for all values of \( x \). Another important observation for these problems is that in evaluating a limit \( p(x)/q(x) \) as \( x \) goes to zero where \( p(x) \) and \( q(x) \) are polynomials, it is only necessary to consider the terms of lowest degree in \( p(x) \) and \( q(x) \). Thus

\[
\lim_{x \to 0} \frac{2x + 3x^2 + 4x^3}{3x^2 + 5x^3} = \lim_{x \to 0} \frac{2x}{3x} = 2/3.
\]

Suppose we try the following limit problem:

\[
\lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = ?
\]

Using TAYLOR POLYNOMIALS 3.17, we write \( \sin(x) - x = -x^{3/6} \) and \( \tan(x) - x = x^{3/3} \). These identities are only approximate, but are very good for small values of \( x \). As we are looking at limit, the values of \( x \) will be small. Thus we have

\[
\lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = \lim_{x \to 0} \frac{-x^{3/6}}{x^{3/3}} = -1/2.
\]
The following exercises will give you practice in graphing and limits.

3.20 **EXERCISES**

1. Find the following limits:
   
   (a) \[ \lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} = \lim_{x \to -2} (x + 3) = 1 \]
   
   (b) \[ \lim_{x \to 9} \frac{x^2 - 81}{2(x - 9)} = \lim_{x \to 9} (x + 9)/2 = 9 \]

2. Find the following limits:
   
   (a) \[ \lim_{x \to 0} \frac{\sin x}{x} \quad \text{Use L'Hopital's rule or } \sin x = x - x^3/6. \]
   
   (b) \[ \lim_{x \to 0} \frac{\sin 2x}{\sin x} \quad \text{Use L'Hopital's rule.} \]

3. Find the following limits:
   
   (a) \[ \lim_{x \to \infty} \frac{x + \sin x}{2x} = \frac{1}{2} \quad \lim_{x \to \infty} \frac{\sin x}{x} = \frac{1}{2} \]
   
   (b) \[ \lim_{x \to \infty} \frac{\ln x}{x^a}, \ a > 0 \quad \lim_{x \to \infty} \frac{\ln(x)}{x^a} = 0 \quad \text{by L'Hopital’s rule} \]

4. Graph and find all local maxima, minima, and points of inflection for the following functions:
   
   (a) \[ y = -2x^2 + 3x + 5 \quad \text{max. } x = 3/4 \]
   
   (b) \[ y = 12(1-x)/x^2 \quad \text{min. } x = 2, \text{ inf. pt. } x = 3 \]

5. Graph (for selected values of \( \alpha \)) and find all local maxima, minima, and points of inflection in terms of the parameter \( \alpha > 0 \).

   (a) \[ f_\alpha(x) = 2\alpha x - 3\alpha \quad f_\alpha'(x) = 4\alpha x - 6\alpha^2 = 0 \quad \text{at } x = 3\alpha/2 \quad \text{(min.)} \]
   
   (b) \[ f_\alpha(x) = 1/x^2 - \alpha/x. \quad f_\alpha'(x) = -2x^{-3} + \alpha x^{-2} \]

3.21 **VARIATIONS ON EXERCISE 3.20**

*check solutions by graphing*

1. Find the following limits:

   (a) \[ \lim_{x \to 0} \frac{2x^2 + 3x}{x - 1} \]
(b) \( \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \)

(2) Find the following limits:

(a) \( \lim_{x \to 0} \frac{e^x - 1}{\tan x} \)

(b) \( \lim_{x \to 1} \frac{2 \ln x}{\sin(\pi x)} \)

(3) Find the following limits:

(a) \( \lim_{x \to 0} \frac{\ln(1 + x)}{\sin x} \)

(b) \( \lim_{x \to 0^+} x^a \ln x, \ a > 0 \)

(4) Graph and find all local maxima, minima, and points of inflection for the following functions:

(a) \( y = e^x - x \)

(b) \( y = (x^2 + 1)/x \)

(5) Graph (for selected values of \( \alpha \)) and find all local maxima, minima, and points of inflection in terms of the parameter \( \alpha > 0 \).

(a) \( f_\alpha(x) = e^{\alpha x} - e^{-\alpha x} \)

(b) \( f_\alpha(x) = x^3 - 3\alpha x^2 + 3x + 2 \)

3.22 **VARIATIONS ON EXERCISE 3.20** (SOLUTIONS end of chapter)

(1) Find the following limits:

(a) \( \lim_{x \to 3} \frac{3x^3 + 4x^2 - x + 1}{2x^2 + 4x - 5} \)

(b) \( \lim_{x \to -1} \frac{x^2 + 1}{x^2 - 1} \)

(2) Find the following limits:

(a) \( \lim_{x \to 0^+} \frac{\ln(\tan x)}{x^3} \)

(b) \( \lim_{x \to 0^+} \frac{\sin x}{\tan \sqrt{x}} \)
Find the following limits:

(a) \( \lim_{x \to 0^+} \frac{e^x - e \sqrt{x}}{\sqrt{x}} \)

(b) \( \lim_{x \to \pi/2} \frac{\sin(x)}{\cos(x)} \)

Graph and find all local maxima, minima, and points of inflection for the following functions:

(a) \( y = x \ln|x| \)

(b) \( y = \frac{(x^2 + 1)}{(x^2 - 1)} \)

Graph (for selected values of \( \alpha \)) and find all local maxima, minima, and points of inflection in terms of the parameter \( \alpha > 0 \).

(a) \( f_\alpha(x) = x - \frac{\alpha}{x} \)

(b) \( f_\alpha(x) = x^\alpha - x^{\alpha/2}, x > 0 \)

3.23 VARIATIONS ON EXERCISE 3.20

Find the following limits:

(a) \( \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{x}} \)

(b) \( \lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x - 1} \)

Find the following limits:

(a) \( \lim_{x \to \pi/2} \frac{\pi \cos(x) - 2x}{\cos(x)} \) (\( \pi + 2 \)) Let \( z = x - \pi/2 \) so \( \cos(x) = \cos(z + \pi/2) = -\sin(z) \).

(b) \( \lim_{x \to 0} \frac{\ln|\sec(x) + \tan(x)|}{\tan(x)} \) (1) l’Hopital

Find the following limits:

(a) \( \lim_{x \to 0} \frac{\cos(x) - 1}{x} \) (0) l’Hopital or \( \cos(x) \approx 1 - x^2/2 \)

(b) \( \lim_{x \to 2} \frac{\ln(\ln|x|)}{\ln x - 1} \) (1) \( z = \ln(x) - 1, \lim_{z \to 0} \frac{\ln(1 + z)}{z} \)

Graph and find all local maxima, minima, and points of inflection for the following functions: (Use online graphers to check work.)
(a) \( y = x + \sin(x) \)  
\[ D_{01} = \{ x | y'(x) = 0 \} = \{ x | \cos(x) = -1 \} \]
\[ D_{02} = \{ x | y''(x) = 0 \} = \{ x | \sin(x) = 0 \} \]
\( D_{01} \subset D_{02} \)

(b) \( y = \ln|x^3 - 3x| \)
replace by \( \ln|x+\ln|x^2-3| \), \( D_{01} = \{ -1, +1 \} \), \( x^2 \) helps for \( y'' \)

(5) Graph (for selected values of \( \alpha \)) and find all local maxima, minima, and points of inflection in terms of the parameter \( \alpha > 0 \).

(a) \( f_\alpha(x) = x|\ln|\alpha x| \)
\[ f'_\alpha(x) = \ln|\alpha x| + 1, \ f''_\alpha(x) = 0 \text{ if } x = \pm (\alpha e)^{-1}; \ f''_\alpha(x) = 1/x, \ x \neq 0 \]

(b) \( f_\alpha(x) = (x^2 + \alpha)/(x - \alpha) \)
\[ z = x - \alpha, \ f_\alpha(z) = \frac{(z + \alpha)^2 + \alpha}{z} = z + 2\alpha + (\alpha^2 + \alpha)z^{-1} \]
\[ f'_\alpha(z) = 1 + (\alpha^2 + \alpha)(-1)z^{-2}, \ f''_\alpha(z) = 0 \text{ if } z = \pm (\alpha^2 + \alpha)^{1/2} \]

---

**Now The Classical Max–Min And Related Rate Problems**

### 3.24 CLASSICAL APPLICATIONS OF DIFFERENTIATION

We now look at some typical classical applications of differentiation. Remember, when you work these problems, think about whether or not you really need calculus to solve them. Would drawing a graph do just as well? With this attitude, you will make more intelligent use of calculus and gain more insight into the real nature of these exercises.

### 3.25 EXERCISES

(1) A cylindrical beer can is to be made with 50 in\(^2\) of aluminum. What should the dimension of the can be in order to maximize the volume of the can? Sketch the optimum can.

(2) A person is at point A on the bank of a straight river and wants to get to point B on the opposite shore as quickly as possible. If he can run 10 miles per hour and ROW 5 miles per hour, if the river is 1 mile wide, and point B is 6 miles down the river from point A, what path should he take to get there as quickly as possible? This is a slowly flowing river, so ignore the motion of the water.

(3) A 10 foot wide hallway leads into a 5 foot wide hallway. What is the longest length of straight board that can be pushed along the floor around the corner joining the two hallways?

(4) A 6 foot tall man is walking away from a 20 foot high lamppost at the rate of 3 feet per second. When he is 10 feet from the lamppost, at what rate is the length of his shadow changing?

(5) If an object weighs \( w \) lbs. on the surface of the earth, then it weighs \( w(1 + .00025r)^{-2} \) lbs. when it is \( r \) miles above the surface of the earth. What is the rate of decrease of weight of a 10,000 lb. rocket 500 miles above the earth and traveling away at a velocity of 3 miles per second.
(6) A toy rocket is fired vertically with an initial velocity of 200 feet per second. Find the maximum height of the rocket and find its velocity when it strikes the ground.

(7) The parametric equations of the curve traveled by a particle are \( x = 2 - 3\cos(t) \) and \( y = 3 + 2\sin(t) \), where \( t \) is the time in seconds. Find the rate at which the tangent to the curve is changing as a function of \( t \). Sketch the curve, showing some points corresponding to specific values of \( t \).

(8) The parametric equations of the curve traveled by a particle are \( x = -1 + \cos(t) \) and \( y = 1 + 2\sin(t) \). Find the rate at which the particle is moving along the curve as a function of time.

We now give the solutions to EXERCISE 3.25. As previously, you should try to work EXERCISE 3.25 without looking at the solutions. If you get stuck, take a peek at the solution and try again. If you still get stuck, study the solution carefully and then make up and work your own variation to the problem, however minor it may be.

3.26 SOLUTIONS TO EXERCISE 3.25

(1) The surface area is fixed and given as 50 in\(^2\). The surface area is twice the area of the base \((2\pi r^2)\) plus the area of the lateral surface \((2\pi rh)\). Thus, if \( S \) denotes the surface area, \( S = 2\pi r^2 + 2\pi rh = 50 \text{ in}^2 \). There are two unknowns, \( r \) and \( h \). Solving for \( h \) we get, \( h = \frac{25/\pi r}{r} - r \). We wish to maximize the volume, \( V = \pi r^2 h \). We haven’t discussed maximizing functions of two variables (that comes in more advanced courses). By substituting the expression for \( h \) in terms of \( r \) into the expression for \( V \), we obtain \( V(r) = 25r - \pi r^3 \). This expresses \( V \) as a function of only one variable. So far, no calculus! Here are a couple of basic programs that tell us that \( V(r) \) is maximized for \( r = 1.65 \) and \( h = 3.30 \).

(a)

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(b)

10 FOR R = 1.2 TO 2 STEP .05
20 PRINT R, 25 * R - 3.14 * R^3
30 NEXT R

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We still haven’t used any calculus! To use calculus, we should compute \( \frac{d}{dr} V(r) = V'(r) = 25 - 3\pi r^2 \). This gives, by setting \( V'(r) = 0 \) and solving for \( r \), \( r = (25/3\pi)^{1/2} = 1.63 \) with \( h = 3.26 \). It seems that the optimal beer can has its diameter the same as its height.
(2) Figure 3.27 is a sketch of the situation:

From Figure 3.27, we see that the time, \( T(r) \), to get from \( A \) to \( B \), given that the person runs along the bank a distance of \( r \) miles before starting to row straight towards \( B \), is given by

\[
T(r) = \frac{r}{10} + \frac{(1 + (6 - r)^2)^{1/2}}{5}.
\]

**FIGURE 3.27** A Straight, Slow River

Here are a couple of simple BASIC programs that generate values of \( T(r) \). It is evident from these programs that \( r = 5.42 \) miles (approximately) gives the minimum time.

```
10 FOR R = 0 TO 6 STEP .2
20 PRINT R,(R/10)+((1+(6-R)^2)^.5)/5
30 NEXT R
```

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(b)

10 FOR R = 5.3 TO 5.5 STEP .02
20 PRINT R, (R/10) + ((1 + (6 - R)^2)^.5)/5
30 NEXT R

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This requires no calculus. To use calculus on this problem, we compute $T'(r)$ to get

$$T'(r) = \frac{1}{10} - \frac{6 - r}{5(1 + (6 - r)^2)^{1/2}}.$$ 

Setting $T'(r) = 0$ gives $(1 + (6 - r)^2)^{1/2} = 2(6 - r)$. Squaring both sides and solving for $6 - r$ gives $6 - r = (1/3)^{1/2}$ or $r = 5.423$. This agrees with the BASIC program given above.

(3) Note in FIGURE 3.28 that the "longest board" is the shortest line segment touching the interior corner and opposite sides of the two hallways. Evidently, the length of such a line segment is

$$L(t) = \frac{10}{\cos(t)} + \frac{5}{\sin(t)}.$$ 

Here are a couple of BASIC programs that compute $L(t)$ for selected values of $t$, measured in radians. Note that the minimum segment occurs when $t = .67$ radians or 38.4 degrees. The largest board length is 20.8097 feet.
FIGURE 3.28 Board Sliding Around Corner

(a)

10 FOR T = .5 TO 1.3 STEP .05
20 PRINT T, (10/COS(T)) + (5/SIN(T))
30 NEXT T

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(b)

10 FOR T = .6 TO .7 STEP .005
20 LPRINT T, (10/COS(T)) + (5/SIN(T))
30 NEXT T

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To use calculus on this problem, we compute \( L'(t) \) and solve for \( t \) such that \( L'(t) = 0 \). We calculate that

\[
L'(t) = \frac{10 \sin(t)}{\cos^2(t)} - \frac{5 \cos(t)}{\sin^2(t)}. 
\]

Thus, \( L'(t) = 0 \) when \( 10 \sin^3(t) = 5 \cos^3(t) \). In other words, \( \tan^3(t) = 1/2 \) or \( \tan(t) = 0.79 \). But, \( \arctan(0.79) = 0.67 \) radians or 38.4 degrees, which is no surprise as we knew that already from the BASIC program above.

(4) From FIGURE 3.29 we see by similar triangles that

\[
\frac{s}{6} = \frac{s + d}{20}. 
\]

**FIGURE 3.29** Man Walking Away from Lamppost
From this relationship, we conclude that \( s = (3/7)d \). The man is walking at 3 feet per second, so \( d = 3t \) is a function of time. Thus, \( s(t) = (9/7)t \) and \( s'(t) = 9/7 \) feet per second. Are you surprised that the rate of change of the shadow is a constant? Does it seem intuitive to you that the rate of change of the shadow should be the same if the man is very close to the lamppost or very far away?

(5) Let \( W(r) \) denote the given function that expresses weight as a function of \( r \). By the CHAIN RULE, \( \frac{dW}{dt} = \frac{dW}{dr} \frac{dr}{dt} \) gives the rate of change of weight as a function of time. But \( \frac{dW}{dr} = w(-2)(1 + .00025r)^{-3}(.00025) = -w(5 \times 10^{-4})(1 + (2.5 \times 10^{-4})r)^{-3} \). Multiplying this expression, with \( r = 500 \) miles, times \( \frac{dr}{dt} = 3 \) miles per second gives \( -10.5 \) lbs. per second as the rate of decrease of weight.

(6) This problem requires a little bit of knowledge of elementary physics. Let \( s(t) \) denote the distance of an object, as a function of time, from the surface of the earth or, alternatively, from any other fixed point of reference (50 miles up, for example). Usually, we measure \( s(t) \) as positive going up from the point of reference and negative going down. This is not always the case, however. Let’s assume for now that up is positive. The velocity, \( v(t) \), of the object as a function of time is defined to be \( \frac{ds}{dt} = s'(t) \). If the object is moving up, its velocity is positive and moving down its velocity is negative. The acceleration, \( a(t) \), of the object is, by definition, \( \frac{dv}{dt} = v'(t) \). The “speed” of an object at a time \( t \) is the absolute value of the velocity \( v(t) \). In our model, an object has positive acceleration if it is heading toward the surface of the earth with decreasing speed or heading away from the earth with increasing speed. Otherwise (away with decreasing speed or toward with increasing speed), the acceleration is negative. Long ago, people discovered that an object dropped freely, without any push, fell toward the earth with increasing speed. Such an object has negative acceleration based on our assumption that \( s(t) \) is positive directed away from the earth’s surface.

But there was a surprise about objects falling toward the surface of the earth awaiting the ancients! The acceleration didn’t seem to depend on the “weight” of the object. A small coin would accelerate at the same rate as a heavier coin. This acceleration is about \( -32 \) feet per second per second or
−32 ft/sec². In metric units, the acceleration at the earth’s surface is −9.8 meters per second per second. Two basic laws of physics lie behind this observation. The first is the famous \( F = ma \) formula, which says that the force acting on an object is the product of the object’s mass and its acceleration. The second is the law of gravitation, which states that the force between two objects of mass \( M \) and \( m \) is given by \( F = GMm/r^2 \) where \( r \) is the distance between the two objects and \( G \) is a constant. Just how this distance should be measured is itself a good calculus problem. For spheres of uniform density, the distance is measured between their centers. If \( m \) is the mass of a small marble that you are about to drop from one foot above your desk, then \( F = (GM/r^2)m \) where \( M \) is the mass of the earth and \( r \) is the distance from the center of the earth to the center of the marble. This means that the acceleration acting on the marble is \( GM/r^2 \). If you now drop the marble from one foot above the floor, the distance \( r \) is a few feet less so, technically, the acceleration is a little less. The difference between taking \( r \) in the expression \( GM/r^2 \) to be the distance from the center of the earth to the top of the desk or the center of the earth to the floor is so minute that \( GM/r^2 \) may be treated as constant. For all points near the surface of the earth, \( GM/r^2 \) is usually \( g = 32\text{ft}/\text{s}^2 \) or \( g = 9.8\text{m}/\text{s}^2 \). Notice that the acceleration \( GM/r^2 \) does not depend on \( m \). This explains why heavy objects fall as fast as light ones. We are ignoring air resistance here. Obviously, a feather or a bubble will not fall as fast as a dime in the air currents in your room.

To get back to the problem at hand, problem (6), the rocket, we assume, is launched from the ground, which we take to be at time zero and height zero. Thus, \( s(0) = 0 \). The acceleration is −32 ft/sec². This means that the velocity must be \( v(t) = −32t + v_0 \), where \( v_0 = 200 \text{ ft/sec} \) is the initial velocity, \( v(0) \). The distance \( s(t) \) must satisfy \( s'(t) = v(t) \) and \( s(0) = 0 \) and hence must look like \( s(t) = −16t^2 + 200t \). We picture the rocket shooting straight upwards until the time \( t \) when \( v(t) = 0 \). This occurs when \( −32t + 200 = 0 \) or \( t = 25/4 \) seconds. The height of the rocket at this time is \( s(25/4) = 625 \text{ ft} \). As we are ignoring air resistance, you will guess, if you have had a little physics, that the velocity when the rocket hits the ground will be exactly −200 ft/sec. The minus is because the falling object is headed toward the earth. You should show, by starting with the fact that the acceleration is −32 ft/sec², that any object dropped from 625 feet will strike the ground at −200 ft/sec.

(7) FIGURE 3.30 shows the curve and a basic program to generate points on the curve.
FIGURE 3.30 Rotation of Tangent to Ellipse

10 PI = 3.14159
20 FOR T = 0 TO 2*PI STEP 2*PI/20
30 PRINT T, 2 - 3*COS(T), 3 + 2*SIN(T)
40 NEXT T

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We are asked to find "the rate at which the tangent is changing as a function of t." There are two ways that the tangent is changing. The point of contact of the tangent line with the curve changes with t in the obvious way. The slope of the tangent line also changes. The slope of the tangent line is tan(ϕ(t)) for some angle ϕ(t) which changes with t. The most interesting measure of
change of the tangent line is \( \frac{d\Phi}{dt} \), which measures the rate of rotation of the tangent line with respect to \( t \). To compute \( \Phi'(t) \), we first compute \( \frac{dx}{dt} = 3\sin(t) \) and \( \frac{dy}{dt} = 2\cos(t) \). By the CHAIN RULE, we have

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \quad \text{or} \quad 2\cos(t) = \frac{dy}{dx} 3\sin(t).
\]

Solving, we obtain \( \frac{dy}{dx} = \frac{2}{3\cot(t)} \).

Thus, \( \Phi(t) = \arctan((2/3\cot(t)) \) and

\[
\frac{d\Phi}{dt} = \frac{-6\csc^2(t)}{9 + 4\cot^2(t)}.
\]

This gives the rate of rotation of the tangent with respect to \( t \) (see DIFFERENTIATION RULES 2.41 for \( \frac{d}{dx} \arctan x \)).

(8) The curve is an ellipse. You should write a short program to generate points on this ellipse and sketch the curve as was done in problem (7). The distance at time \( t \) that a point on the curve has traveled along the curve from a fixed reference point, say \((x(0), y(0))\), is called the "arclength at time \( t \)" and is denoted by \( L(t) \). The basic fact we need about arclength is that

\[
\frac{dL}{dt} = ((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)^{1/2} = (\sin^2(t) + 4\cos^2(t))^{1/2} = (1 + 3\cos^2(t))^{1/2}.
\]

We now look at some VARIATIONS on EXERCISE 3.25. Our first VARIATIONS involve exactly the same problems but stated more generally in terms of parameters. Here you will find computational methods less of a help and calculus much more important as a tool in finding solutions in terms of parameters. Even in dealing with problems formulated in terms of parameters, it is a good idea to assign values to the parameters and do some computational studies. We have already done this in SOLUTIONS TO EXERCISE 3.25, so in VARIATIONS 3.31 you can concentrate on the calculus.

3.31 VARIATIONS ON EXERCISE 3.25

(1) A cylindrical beer can is to be made with \( s \) in\(^2\) of aluminum. What should the dimensions of the can be, as a function of \( s \), in order to maximize the volume of the can?
(2) A person is at point A on the bank of a straight river and wants to get to a point B on the opposite shore as quickly as possible. If he can run \( \rho \) miles per hour and row \( \sigma \) miles per hour, if the river is \( \omega \) miles wide, and point B is \( \beta \) miles down the river from point A, what path should he take to get there as quickly as possible? This is a slowly flowing river, so ignore the motion of the water.

(3) A \( \beta \) foot wide hallway leads into a \( \lambda \) foot wide hallway. What is the longest length of a straight board that can be pushed along the floor around the corner joining the two hallways?

(4) A man \( \tau \) feet tall is walking away from a \( \lambda \) foot high lamppost at the rate of \( \rho \) feet per second. When he is \( \delta \) feet from the lamppost, at what rate is the length of his shadow changing?

(5) If an object weighs \( \omega \) lbs. on the surface of the earth, then it weighs \( \omega (1 + .00025r)^{-2} \) lbs. when it is \( r \) miles above the surface of the earth. What is the rate of decrease of weight of an \( R \) lb. rocket \( M \) miles above the surface of the earth and traveling away at a velocity of \( v \) miles per second?

(6) A toy rocket is fired vertically with an initial velocity of \( V_0 \) feet per second. Find the maximum height of the rocket and find its velocity when it strikes the ground.

(7) The parametric equations of a curve traveled by a particle are \( x = a + b \cos(t) \) and \( y = c + d \sin(t) \), where \( t \) is the time in seconds. Find the rate at which the tangent to the curve is changing as a function of \( t, a, b, c, \) and \( d \).

(8) The parametric equations of the curve traveled by a particle are \( x = a + b \cos(t) \) and \( y = c + d \sin(t) \). Find the rate at which the particle is moving along the curve as a function of \( t, a, b, c, \) and \( d \).

### 3.32 Variations on Exercise 3.25

(1) A cylindrical beer can has radius \( r \) and height \( h \) and is made of aluminum. Suppose the cost per in\(^2\) of aluminum is 1 cent. Let \( V \) denote the volume of the can. In terms of \( V \) and \( r \), give a formula for the cost of the least expensive can.

(2) Consider the parabola \( y = (x - 1)^2 + 4 \). Find the point on it such that the distance from this point to the origin is minimal.
(3) A sport field has the pictured geometric form. If the perimeter of the field is 1 km, what should x and y be such that the surface area is maximal? What if the field is a rectangle with only one half circle attached?

(4) A 6 foot tall man stands at the point A. A light source is located at the point C and moves upwards at the rate of 3 feet per second. When the light source is 12 feet over the earth at the point D, at what rate is the length of the man’s shadow changing?

(5) Find two positive numbers whose sum is 20 such that their product is maximal. Find two positive numbers whose product is 20 such that their sum is minimal.

(6) A toy rocket, 10 meters from the ground, is fired at an angle of 60° with an initial velocity of 200 feet per second. Find the maximum height of the rocket, its velocity when it strikes the ground, and the distance from A to B.

(7) A stone is dropped into water and causes concentric circles propagating at a rate of 5 m/s. After 10 seconds, at what rate is the area enclosed by the first concentric circle expanding?

(8) The parametric equations of the curve traveled by a particle are \( x = t + 1 \) and \( y = -t^2 + 8t + 9 \) where \( t \geq 0 \) is the time. At what time is the absolute value of the tangential velocity (rate of change of arclength) minimal?

Now go back over VARIATIONS 3.32 and introduce parameters into the problems, just as was done in VARIATIONS 3.31 relative to EXERCISES 3.25. For example, in 3.32(4), the man can be T feet tall, the light source can be moving upwards with a velocity of v feet per second, etc. Do the same to the remaining VARIATIONS ON EXERCISES 3.25 after you have solved the problems as stated.
3.33 VARIATIONS ON EXERCISE 3.25

(1) A container with a square bottom is made with \( S \) in\(^2 \) of wood. What should the dimensions of the container be in order to maximize its volume? Assume the container is a parallelepiped.

(2) Suppose a natural gas supplier is located at A and plans a pipeline to a location at B. If the cost for the pipeline in the river is four times that on the earth, find the point C such that the cost for the pipeline is minimal. How can this problem and 3.25(2) be combined into one "general idea"? (The pipe will go straight from A to C along land and straight from C to B under the river.)

(3) We are considering a half elliptical hallway 1 meter wide and want to slide a straight board along the floor entering at A and leaving at B. What is the longest board that can be brought through this hall? Prove your answer is correct.

(4) A man standing at the point A observes a car moving at the rate of 50 km/h. When the car is 200 m at the point B, at what rate is the distance between the man and the car changing?
(5) We consider a right triangle with dimensions 3, 4, and 5 m. Find the dimensions of the largest rectangle included in it as pictured.

(6) A 6 foot tall man is walking away from a light at a rate of \( \frac{70}{15 + x} \) ft/sec where \( x \) is the horizontal distance from the light. The light starts at height of 20 ft when \( x = 0 \) and is moving upwards at a rate of 10 ft/sec. How fast is the man's shadow changing when \( x = 20 \) ft?

(7) The parametric equations of the curve traveled by a particle are \( x = -1 + 3 \cos^2 t \) and \( y = 1 - 3 \sin(2t) \). Find the rate at which the particle is moving along the curve as a function of time. Sketch the curve.

(8) A light source located at A is reflected through a mirror and is caught at the point B. Find the point C such that the sum of the distances AC and CB is minimal. Show that for this C, the angle \( \alpha \) equals the angle \( \beta \). Does this result depend on the distances AP, PQ, and BQ? Explain.

3.34 VARIATIONS ON EXERCISE 3.25

(1) A cylindrical can has a fixed volume of \( v \) in\(^3\) and is made of aluminum. The cost for the top and bottom is half of the cost of the other parts of the can (which costs more because of decorations for advertising). Find the height \( h \) and the radius \( r \) of the can such that the cost for aluminum is minimal.
(2) A pond is the shape of an equilateral triangle, 40 meters on each side. A man at A, the midpoint of one side, wants to get to B, the midpoint of another side. If he can run 8m/sec and swim 2m/sec what route should he follow to minimize the time to get to B? Replace 8m/sec by \(v_r\) and 2m/sec by \(v_s\) and analyze the strategy. Could there ever be an advantage not to run or swim in a straight line?

(3) Given the curve \(y = 1/x\), find the shortest distance between the point (0, 1) and this curve.

(4) An ellipse with semi-major axis \(a = 3\) meters and semi-minor axis \(b = 2\) meters starts to expand. If \(a\) expands at a rate of 8m/sec and \(b\) at a rate of 4m/sec, what is the rate of increase of the area enclosed by the ellipse when \(b = 12\) meters? The area of an ellipse is \(\pi ab\).

(5) Find the dimensions of a rectangle included in a half circle of radius \(r\) such that the area of the rectangle is maximal.
(6) A circular island of radius 30m is surrounded by a canal of radius 10m. A man at A wants to get to B as quickly as possible. Once on the island, he must run to B along the shore of the canal (not in a straight line). If he can run 6m/sec and swim 3m/sec (he's wearing swim fins!), what is his best route to get from A to B? Replace 6m/sec by \( v_r \) and 3m/sec by \( v_w \) and analyze the strategy. Is it ever to his advantage to run first, then swim? Is it ever to his advantage to not swim in a straight line? Don't be afraid to use a computer to get a feeling for this problem. What happens if he is allowed to run in a straight line on the island?

(7) The parametric equations of the curve traveled by a particle are \( x = \sin^2 t \) and \( y = \cos^2 t \). Find the rate at which the particle is moving along the curve as a function of time \( t \). Sketch the curve for \( 0 \leq t \leq \pi/2 \).

(8) We consider a family of straight lines in the xy-plane passing through the point \((2,2)\). Suppose they intersect the x-axis at A and the y-axis at B; find the slope of the line such that the shaded area is minimal. A and B depend on the slope of the line.
SOLUTIONS to 3.22 (variations on exercise 3.20)

(1)
(a) \(\lim_{x \to 3} \frac{3x^3 + 4x^2 - x + 1}{2x^2 + 4x - 5} = \frac{3 \cdot 3^3 + 4 \cdot 3^2 - 3 + 1}{2 \cdot 3^2 + 4 \cdot 3 - 5}\)
(b) \(\lim_{x \to -1} \frac{x^2 + 1}{x^2 - 1} = +\infty\)

(2)
(a) \(\lim_{x \to 0^+} \frac{\ln(\tan(x))}{x^3} = \left(\lim_{x \to 0^+} \frac{\ln(\tan(x))}{x}\right)\left(\lim_{x \to 0^+} \frac{1}{x^2}\right) = -\infty\)
(b) \(\lim_{x \to 0^+} \frac{\sin(x)}{2\tan(\sqrt{x})} = \lim_{x \to 0^+} \frac{x - x^3/6}{2(x^{1/2} + x^{3/2}/3)} = 0 \) (TP’s)

or \(\lim_{x \to 0^+} \frac{\sin(x)}{2\tan(\sqrt{x})} = \lim_{x \to 0^+} \frac{\cos(x)}{2 \sec^2(\sqrt{x})(1/2)x^{-1/2}} = \lim_{x \to 0^+} x^{1/2} \cos(x) \cos^2(\sqrt{x}) = 0 \) (L’Hospital)

(3)
(a) \(\lim_{x \to 0^+} \frac{e^x - e^{x^{1/2}}}{x^{1/2}} = -e\)
(b) \(\lim_{x \to \pi/2} \frac{\sin(\cos(x))}{\cos(x)} = \lim_{z \to 0} \frac{\sin(z)}{z} = 1 \) (\(z = \cos(x)\))

(4)
(a) We have \(y = x \ln|x|\), \(y'(x) = \ln|x| + 1\) and \(y''(x) = 1/x\).
Thus, \(y(x) = 0\) when \(x = \pm 1\), \(y'(x) = 0\) when \(x = \pm e\).
The maximum occurs at \(x = -e\), the minimum at \(x = +e\):

(b) \(y = (x^2 + 1)/(x^2 - 1),\ y' = -4x/(x^2 - 1)^2,\ y'' = -4\).
A maximum occurs at \(x = 0\). There are no inflection points.
\( \alpha = 3, \beta = 1.5 \)

\[ f_\alpha(x) = x - \alpha/x, \quad \alpha > 0, \quad f'(x) = 1 + \alpha/x^2, \quad f''(x) = -2\alpha/x^3. \]

There are no maxima, minima or points of inflection.

(b) Let \( f(x) = x^\alpha - x^\beta, \quad \alpha > \beta > 0, \quad f'(x) = \alpha x^{\alpha - 1} - \beta x^{\beta - 1} \)
\[ f''(x) = \alpha(\alpha - 1)x^{\alpha - 2} - \beta(\beta - 1)x^{\beta - 2}. \quad f'(x) = 0 \text{ when } x^{\alpha - \beta} = \beta/\alpha, \quad x = (\beta/\alpha)^{1/(\alpha - \beta)}. \]
\[ f''(x) = \alpha(\alpha - 1)x^{\alpha - 2} - \beta(\beta - 1)x^{\beta - 2} = 0 \]
when \( x = \left( \frac{\beta(\beta - 1)}{\alpha(\alpha - 1)} \right)^{1/(\alpha - \beta)} \) (take \( \beta = \alpha/2 \) for the problem).

SOLUTIONS to 3.31 (variations on exercise 3.25)

(1) **Optimal can with fixed surface area** \( s \): Let \( x \) be the radius of the base, \( h \) the height. Then \( s = 2\pi x^2 + 2\pi xh, h = (s - 2\pi x^2)/2\pi x, V = \pi x^2 h. \) Thus,
\[ V = \pi x^2 \left( \frac{s - 2\pi x^2}{2\pi x} \right) = (x/2)(s - 2\pi x^2) = (s/2)x - \pi x^3 \]
and \( V'(x) = s/2 - 3\pi x^2. \) Thus, \( V'(x) = 0 \) when \( x^2 = s/6\pi. \) The can of area \( s \) and maximal volume has radius \( r = (s/6\pi)^{1/2} \) and
\[ h = \frac{s}{2\pi} \cdot \frac{1}{r} - r = 3 \left( \frac{s}{6\pi} \right) \frac{1}{r} - r = 3 \left( \frac{r^2}{r} \right) \frac{1}{r} - r = 2r : = d. \]

(2) **Running and rowing to cross river**: See solutions 3.26, Figure 3.27 where we have \( \beta = 6, \rho = 10, \sigma = 5, w = 1. \) Here, we have
\[ T(r) = \frac{r}{\rho} + \frac{(w^2 + (\beta - r)^2)^{1/2}}{\sigma}. \]
\[ T'(r) = \frac{1}{\rho} - \frac{\beta - r}{\sigma(w^2 + (\beta - r)^2)^{1/2}} = 0. \]
if \( w^2 + (\beta - r)^2 = \rho^2(\beta - r)^2/\sigma^2 \) or
\[ (\beta - r) = \frac{w}{(\rho^2/\sigma^2 - 1)^{1/2}}. \]
SOLUTIONS to 3.31 continued (variations on exercise 3.25)

(2) Crossing river - alternative solution (see rules 2.24):
The variable is the angle $\theta$ between the rowing direction and the bank.

$$T(\theta) = \frac{\beta - w \cot(\theta)}{\rho} + \frac{w \csc(\theta)}{\sigma}$$

$$T'(\theta) = \frac{w \csc^2(\theta)}{\rho} - \frac{w \csc(\theta) \cot(\theta)}{\sigma} = 0$$

$$\frac{\sigma}{\rho} = \frac{\csc(\theta) \cot(\theta)}{\csc^2(\theta)} = \cos(\theta).$$

(3) Longest board problem: For fixed $\lambda$, $\beta$, the length of the board is

$$L(t) = \beta \sec(t) + \lambda \csc(t).$$

Thus, $L'(t) = \beta \sec(t) \tan(t) - \lambda \csc(t) \cot(t)$. Setting $L'(t) = 0$, we get

$$\frac{\lambda}{\beta} = \frac{\sec(t) \tan(t)}{\csc(t) \cot(t)} = \tan^3(t).$$

Thus, $t_m = \arctan(\gamma)$, $\gamma := (\lambda/\beta)^{1/3}$, minimizes $L(t)$. The longest board that slides between the two hallways has length $L^\lambda = L^{\beta} = \beta \sec(\arctan(\gamma)) + \lambda \csc(\arctan(\gamma))$. Note

$$\arctan(\gamma) = \arccsc \left( (1 + \gamma^2)^{1/2} \right)$$

$$\arctan(\gamma) = \arccsc \left( (1 + y^2)^{1/2} / y \right) = \arccsc \left( (1 + y^{-2})^{1/2} \right)$$

so

$$L^{\beta\lambda} = L^{\lambda\beta} = \beta \left[ 1 + \left( \frac{\lambda}{\beta} \right)^{2/3} \right]^{1/2} + \lambda \left[ 1 + \left( \frac{\beta}{\lambda} \right)^{2/3} \right]^{1/2}. $$

(4) Changing shadow problem: By similar triangles,

$$\frac{x}{\lambda - \tau} = \frac{s}{\tau} \text{ so } \dot{s} = \left( \frac{\tau}{\lambda - \tau} \right) \dot{x}.$$
SOLUTIONS to 3.31 continued (variations on exercise 3.25)

(5) **Weight loss with height:** As a function of height \( r \) above the surface, the weight of an object weighing \( \omega \) pounds on the surface is \( W(r) = \omega(1 + .00025r)^{-2} \). We use \( \dot{z} := \frac{dz}{dt} \). Then, \( \dot{W}(r) = -2\omega(1 + .00025r)^{-3} \cdot .00025r \) and set \( \omega = R \).

(6) **Vertical motion and gravity:** Recall that \( g = 32 \text{ft/s}^2 \) or \( g = 9.8 \text{m/s}^2 \) and \( h(t) = -(1/2)gt^2 + V_0t \). Then, \( h'(t) = 0 \) implies \(-gt + V_0 = 0 \) or \( t = V_0/g \). Thus, \( h_{\text{max}} = (-1/2)g(V_0/g)^2 + V_0^2/g \) and \( h(t) = 0 \) when \( t = 0 \) or \( t = 2V_0/g \). Also, \( h'(t) = -V_0 \) when \( t = 2V_0/g \).

(7) **Rotating tangent:** Review Figure 3.30 and the discussion there. Recall differentiation rules 2.41. We have \( (x, y, \text{alt}) = (a + b \cos(t), c + d \sin(t)) \) and

\[
\frac{dy}{dx} = \left(\frac{c + d \sin(t)}{a + b \cos(t)}\right)' = \frac{d \cos(t)}{-b \sin(t)} = -\frac{d}{b} \cot(t).
\]

Thus,

\[
\phi(t) = \arctan \left( -\frac{d}{b} \cot(t) \right)
\]

and

\[
\phi'(t) = \frac{1}{1 + \left(\frac{d}{b} \cot(t)\right)^2} \left( -\frac{d}{b} \right) \left( -\csc^2(t) \right) = \frac{bd \csc^2(t)}{b^2 + d^2 \cot^2(t)}.
\]

(8) **Arclength:** The curve \( (x(t), y(t)) = (a + b \cos(t), c + d \sin(t)) \) is an ellipse. Referring to 3.26 solution (8), let \( L(t) \) be the arclength and

\[
\frac{dL}{dt} = \left((x'(t))^2 + (y'(t))^2\right)^{1/2} = (b^2 \sin^2(t) + d^2 \cos^2(t))^{1/2}.
\]
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