Tensor Algebras
Marcus Seminar Notes UCSB 1964 - 1969

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3.1 The Mixed Graded Tensor Algebra

We recall some of the language in the general theory of algebras. If $R$ is a commutative ring and $\mathcal{U}$ is a module over $R$ with a multiplication defined between the elements of $\mathcal{U}$, which satisfies the laws

$$r(u+v) = ru + rv,$$

$$(r+s)u = ru + su,$$

$$(ru)(sv) = (rs)(uv),$$

$u,v \in \mathcal{U}, \ r,s \in R$, then $\mathcal{U}$ is called an algebra over $R$. The algebra $\mathcal{U}$ is associative or commutative when the multiplication between elements of $\mathcal{U}$ has the corresponding property. If $\mathcal{U}$ is a division ring (i.e., a skew field), then $\mathcal{U}$ is called a division algebra. If $\mathcal{S}$ is a subspace of $\mathcal{U}$, which is closed under the multiplication in $\mathcal{U}$, it is called a subalgebra. If $I$ is a subspace of $\mathcal{U}$ and $\mathcal{U}I = \{ax | a \in \mathcal{U}, x \in I \} \subseteq I$, then $I$ is a left ideal in $\mathcal{U}$. If $I\mathcal{U} \subseteq I$, then $I$ is a right ideal. If $I$ is both a left and a right ideal, it is called a two-sided ideal or simply an ideal in $\mathcal{U}$.

In all our considerations the ring $R$ will always be a field of characteristic $0$ and the module $\mathcal{U}$ will always be a vector
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space over \( R \). If \( \mathcal{U} \) is an associative algebra over \( R \) and \( X \subseteq \mathcal{U} \), then \( \mathcal{U} \) is generated by \( X \) (i.e., \( X \) is a generating set for \( \mathcal{U} \)) if any element of \( \mathcal{U} \) is a finite linear combination of finite products of elements in \( X \). If \( \mathcal{U} \) and \( \mathcal{V} \) are algebras over \( R \) and \( h : \mathcal{U} \to \mathcal{V} \) is a linear map satisfying

\[
h(a_1 a_2) = h(a_1) h(a_2)
\]

for all \( a_1, a_2 \in \mathcal{U} \), then \( h \) is called an algebra homomorphism. If \( l_\mathcal{U} \) and \( l_\mathcal{V} \) are multiplicative identities in \( \mathcal{U} \) and \( \mathcal{V} \) and \( h(l_\mathcal{U}) = l_\mathcal{V} \), then \( h \) is an identity preserving algebra homomorphism. If \( h \) is bijective, then \( \mathcal{U} \) and \( \mathcal{V} \) are isomorphic. The algebras \( \mathcal{U} \) occurring here will be either finite dimensional as vector spaces over \( R \) or there will exist a basis \( X \subseteq \mathcal{U} \) with the following properties: Any element of \( \mathcal{U} \) is a linear combination of a finite subset of \( X \), and any nonzero finite subset of \( X \) is l.i..

Example 1.1(a) \( M_n(R) \) is an associative algebra with a multiplicative identity, the total matrix algebra over \( R \).

(b) Let \( T_n(R) \) be the totality of upper triangular matrices in \( R \). Then \( T_n(R) \) is a subalgebra but clearly neither a right nor a left ideal in \( M_n(R) \).

(c) Any right or left ideal in \( \mathcal{U} \) is a subalgebra of \( \mathcal{U} \).

(d) If

\[
\mathcal{U}^2 = \left\{ \sum_{i=1}^{k} x_i y_i, \ x_i, y_i \in \mathcal{U}, \ k=1,2, \ldots \right\}
\]

then \( \mathcal{U}^2 \) is obviously a two-sided ideal in \( \mathcal{U} \) and hence a subalgebra (it is called the derived algebra).
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A certain class of algebras are of particular importance in multilinear algebra.

**Definition 1.1 (Graded vector space and graded algebra)** Let G be an additive abelian group and assume that for each \( g \in G \), \( V(g) \) is a vector space over \( R \). The totality of functions \( f \) defined on \( G \) which satisfy

\[
f(g) \in V(g), \quad g \in G,
\]

where \( f(g) \neq 0 \) obtains for at most a finite number of elements of \( G \), constitute an object \( \mathcal{U} \) called a **G-graded vector space** over \( R \). The addition and scalar multiplication in \( \mathcal{U} \) are defined by

\[
(f_1 + f_2)(g) = f_1(g) + f_2(g),
\]

\[
(rf)(g) = rf(g),
\]

\( g \in G, \ r \in R \). If \( \mathcal{U} \) is also an algebra over \( R \) which satisfies

\[
V(g_1)V(g_2) \subseteq V(g_1 + g_2)
\]

for all \( g_1, g_2 \in G \), then \( \mathcal{U} \) is called a **G-graded algebra**. The meaning of the inclusion (4) will be explained below.

In general some remarks about this definition are in order. The G-graded vector space \( \mathcal{U} \) is sometimes written as

\[
\mathcal{U} = \sum_{g \in G} V(g)
\]

and a typical element \( f \in \mathcal{U} \) as

\[
f = \sum_{g \in G} f(g).
\]
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Both (5) and (6) are obvious abuses of notation. In (5), \( V(g) \) is confused with the set \( \{ f | f \in \mathcal{M}, f(h) = 0, h \neq g \} \), i.e., the set of all functions that are 0 for all \( h \in G \) other than \( g \)—any such function is called homogeneous of degree \( g \). In (6), \( f(g) \) is confused with the function that is homogeneous of degree \( g \) and whose value at \( g \) is \( f(g) \). Thus, the inclusion (4) means: If \( f_i \) is a function that is homogeneous of degree \( g_i \), \( i = 1, 2 \), then \( f_1 f_2 \) is homogeneous of degree \( g_1 + g_2 \). If \( f \) is homogeneous of degree \( g \), we write

\[
\text{deg } f = g. \tag{7}
\]

The zero element is regarded as being homogeneous of any degree.

The inclusion (4) then can be succinctly described as follows: If \( \text{deg } f_1 = g_1 \) and \( \text{deg } f_2 = g_2 \), then \( \text{deg } f_1 f_2 = g_1 + g_2 \), i.e.,

\[
\text{deg } f_1 f_2 = \text{deg } f_1 + \text{deg } f_2. \tag{8}
\]

Once again, the multiplication \( f_1 f_2 \) is the multiplication in the algebra \( \mathcal{M} \).

If \( \mathcal{B} \) is a subspace of the \( G \)-graded space \( \mathcal{M} \) and \( \mathcal{B} \) is spanned by homogeneous elements, then \( \mathcal{B} \) is called a homogeneous subspace. If \( \mathcal{B} \) is an ideal in \( \mathcal{M} \) and as a subalgebra of \( \mathcal{M} \) is generated by a set of homogeneous elements, then \( \mathcal{B} \) is called a homogeneous ideal.

**Definition 1.2 (Homogeneous linear transformation)** Let

\[
V = \sum_{g \in G} V(g) \quad \text{and} \quad U = \sum_{g \in G} U(g)
\]

be two \( G \)-graded vector spaces. If \( \varphi \in \mathcal{L}(V, U) \) and there exists a \( k \in G \) such that for every \( g \in G \), \( \varphi(V(g)) \subseteq U(g+k) \), then \( \varphi \) is homogeneous of degree \( k \).
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In the following example we exhibit some \( G \)-graded spaces and algebras and show how to construct others.

Example 1.2 (a) Let \( G = \mathbb{Z} \), the additive group of integers. Let \( x \) be an indeterminate over \( R \) and for each nonnegative integer \( n \) let \( V(n) \) be the set of all monomials \( a_n x^n \), \( a_n \in \mathbb{R} \).

If \( n < 0 \), let \( V(n) \) be the set consisting of 0. Then the polynomial ring \( R[x] \) is a \( \mathbb{Z} \)-graded vector space:

\[
R[x] = \sum_{n \in \mathbb{Z}} V(n). \tag{9}
\]

For, a polynomial \( p(x) = \sum_{n=0}^{\infty} a_n x^n \) can be thought of as a function on \( \mathbb{Z} \) whose value at each \( n \) is the term of degree \( n \), namely, \( a_n x^n \in V(n) \). The formulas (2) and (3) then translate into

\[
\sum_{n=0} a_n x^n + \sum_{n=0} b_n x^n = \sum_{n=0} (a_n + b_n) x^n
\]

and

\[
r \sum_{n=0} a_n x^n = \sum_{n=0} (ra_n) x^n.
\]

(b) The polynomial ring \( R[x] \) is also a \( \mathbb{Z} \)-graded algebra using the usual definition for polynomial multiplication. The inclusion (4) translates into the statement

\[
(a_n x^n)(b_m x^m) = a_n b_m x^{n+m} \in V(n+m)
\]

and the "degree" of a homogeneous element \( a_n x^n \) coincides with the ordinary meaning for the degree of a polynomial.
(c) Let \( k \) be a fixed positive integer and define \( \varphi \in L(R[x], R[x]) \) by

\[
\varphi(p(x)) = x^k p(x).
\]

Obviously \( \varphi \) is linear and \( \deg \varphi(p(x)) = k + \deg p(x) \). Hence

\[
\varphi(V(n)) \subseteq V(n+k)
\]

so that \( \varphi \) is homogeneous of degree \( k \).

(d) Let \( D \in L(R[x], R[x]) \) be differentiation (with respect to \( x \)). Then if \( \deg p(x) > 0 \), \( \deg D(p(x)) = \deg p(x) - 1 \). If \( \deg p(x) \leq 0 \), i.e., \( p(x) \) is a constant \([0 \text{ in case } \deg p(x) < 0]\), then \( D(p(x)) = 0 \) and \( 0 \) is certainly of degree \( \deg p(x) - 1 \) trivially since \( V(n) = \{0\} \) for \( n < 0 \). Thus \( D \) is homogeneous of degree \(-1\).

(e) If \( \mathcal{B} \) is a homogeneous subspace of the \( G \)-graded space \( \mathcal{H} = \sum_{g \in G} V(g) \), then \( \mathcal{B} = \sum_{g \in G} \mathcal{B} \cap V(g) \), and hence \( \mathcal{B} \) is a \( G \)-graded space as well. For, let \( S \) be a spanning set of \( \mathcal{B} \) consisting of homogeneous elements and let \( S(g) = S \cap V(g) \). Clearly any finite linear combination of elements of \( S \) can be written as a linear combination of elements from a finite number of the \( S(g) \). Moreover \( \langle S(g) \rangle \subseteq V(g) \). Thus \( \mathcal{B} = \sum_{g \in G} \langle S(g) \rangle \) and obviously

\[
\langle S(g) \rangle = \mathcal{B} \cap V(g).
\]

(f) If \( \mathcal{B} \) is a homogeneous ideal in the \( G \)-graded algebra \( \mathcal{H} \), then \( \mathcal{B} = \sum_{g \in G} \mathcal{B} \cap V(g) \) and hence \( \mathcal{B} \) is a \( G \)-graded algebra as well. For, let \( S \) be a generating set of \( \mathcal{B} \) consisting of homogeneous elements. Recall that a generating set for \( \mathcal{B} \) has the property that everything in \( \mathcal{B} \) can be achieved as a finite linear
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combination of finite products of elements of $\mathcal{S}$. Products of homogeneous elements are homogeneous so that $\mathcal{B}$ is in fact spanned as a subspace by homogeneous elements and we can use (e) above to conclude that $\mathcal{B} = \sum_{g \in G} \mathcal{B} \cap V(g)$.

(g) Let $\mathcal{U} = \sum_{g \in G} V(g)$ be a $G$-graded space and $\mathcal{B} = \sum_{g \in G} W(g)$ a homogeneous (and hence $G$-graded) subspace ($W(g) = \mathcal{B} \cap V(g)$).

Let $q: \mathcal{U} \rightarrow \mathcal{U}/\mathcal{B}$ be the quotient map. Then $q(V(g))$ is naturally isomorphic to $V(g)/W(g)$ in such a way that a $\mathcal{B}$ coset corresponds to a $W(g)$ coset. Moreover,

$$\mathcal{U}/\mathcal{B} = \sum_{g \in G} q(V(g)) \text{ so that } \mathcal{U}/\mathcal{B} = \sum_{g \in G} V(g)/W(g)$$

($\Leftrightarrow$ means isomorphic). To see this, suppose $x_g \in V(g)$. Then $q(x_g) = x_g + \mathcal{B}$. Now set

$$i_g(x_g + \mathcal{B}) = x_g + W(g) \in V(g)/W(g).$$

If $x_g - y_g \in \mathcal{B}$, then since $\mathcal{B}$ is $G$-graded, $x_g - y_g \in W(g)$. Thus $x_g + W(g) = y_g + W(g)$ so that $i_g$ is well defined. Also

$$i_g(x_g + \mathcal{B}) = i_g(x'_g + \mathcal{B}), \text{ iff } x_g + W(g) = x'_g + W(g), \text{ iff } x_g - x'_g \in W(g) \subset \mathcal{B}, \text{ iff } x_g + \mathcal{B} = x'_g + \mathcal{B}. \text{ Hence } i_g \text{ is a bijection and } q(V(g)) \cong V(g)/W(g).$$

Clearly, if $f = \sum_{g \in G} x_g$, then $q(f) = \sum_{g \in G} q(x_g)$ (i.e., $x_g$ is the homogeneous element of degree $g$ whose value at $g$ is $x_g$). Thus $\mathcal{U}/\mathcal{B} = \sum_{g \in G} q(V(g))$. Suppose $\sum_{g \in G} q(x_g) = 0$. Then $\sum_{g \in G} (x_g + \mathcal{B}) = \sum_{g \in G} x_g + \mathcal{B} = \mathcal{B}$ so that $x_g \in \mathcal{B}$. But then since $\mathcal{B} = \sum_{g \in G} W(g)$ it follows that $x_g \in W(g) \subset \mathcal{B}$ and $q(x_g) = 0$. Thus the sum $\sum_{g \in G} q(V(g))$ is
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direct. The isomorphism between \( \mathcal{U}/\mathcal{I} \) and \( \bigoplus_{g \in \mathcal{G}} V(g)/\mathcal{W}(g) \) is the one that sends \( \bigoplus_{g \in \mathcal{G}} (x_g + \mathcal{W}) \) into \( \bigoplus_{g \in \mathcal{G}} (x_g + \mathcal{W}(g)) \). Observe that the map \( \varphi: \mathcal{U} \to \bigoplus_{g \in \mathcal{G}} q(V(g)) \) is homogeneous of degree 0.

We have the following standard construction for making a quotient space into an algebra.

**Theorem 1.1(a)** Let \( \mathcal{U} \) be an algebra over \( \mathbb{R} \) and \( \mathcal{I} \) an ideal (two-sided) in \( \mathcal{U} \). Let \( \varphi: \mathcal{U} \to \mathcal{U}/\mathcal{I} \) be the quotient map into the factor space. Then

\[
q(x)q(y) = q(xy)
\]

(10)
defines a multiplication in \( \mathcal{U}/\mathcal{I} \), which makes the factor space into an algebra, called the quotient algebra. The pair \( (\mathcal{U}/\mathcal{I}, q) \) is universal in the following sense: if \( \mathcal{B} \) is an algebra over \( \mathbb{R} \), and \( \varphi: \mathcal{U} \to \mathcal{B} \) is an algebra homomorphism with \( \mathcal{I} \subset \text{ker} \varphi \), then there exists a unique algebra homomorphism \( h: \mathcal{U}/\mathcal{I} \to \mathcal{B} \) such that \( \varphi = h\varphi \).

(b) If \( \mathcal{U} = \bigoplus_{g \in \mathcal{G}} V(g) \) is a \( \mathcal{G} \)-graded algebra and \( \mathcal{B} = \bigoplus_{g \in \mathcal{G}} W(g) \) is a homogeneous ideal in \( \mathcal{U} \), then the quotient algebra \( \mathcal{U}/\mathcal{B} \) is a \( \mathcal{G} \)-graded algebra.

**Proof:** (a) To prove the multiplication in (10) is well defined let \( x = x_1 + i_1, \ y = y_1 + i_2, \) and \( i_1, i_2 \in \mathcal{I} \). Then

\[
xy = x_1 y_1 + (x_i_2 + i_1 y_1 + i_1 i_2)
\]

and the second term on the right is in \( \mathcal{I} \). Thus

\[
q(x)q(y) = q(xy) = q(x_1 y_1) = q(x_1)q(y_1).
\]
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It is easy to confirm that \( \mathfrak{A}/I \) satisfies the axioms for an algebra over \( \mathbb{R} \) (see Exercise 2).

If \( \phi: \mathfrak{A} \to \mathfrak{B} \) is an algebra homomorphism, \( I \subseteq \ker \phi \), then \( \phi(x) = \phi(x_1) \) implies that \( x - x_1 = i_1 \in I \) and hence

\[
\phi(x) = \phi(x_1 + i_1) = \phi(x_1) + \phi(i_1) = \phi(x_1)
\]

(i.e., \( i_1 \in I \subseteq \ker \phi \)). Define \( h: \mathfrak{A}/I \to \mathfrak{B} \) by \( h(q(x)) = \phi(x) \).
Then \( h \) is obviously an algebra homomorphism completely determined by \( \phi \).

(b) From Example 1.2 (g), \( \mathfrak{A}/I \) is a \( G \)-graded vector space, and in fact we verified that

\[
\mathfrak{A}/I = \bigoplus_{g \in G} q(V(g)).
\]

Now suppose that \( q(x_{i_1}) \in q(V(g_1)), i = 1, 2 \). Then from (10), \( q(x_{g_1})q(x_{g_2}) = q(x_{g_1}x_{g_2}) \) and \( x_{g_1}^* x_{g_2}^* \in V(g_1 + g_2) \). Thus

\[
q(V(g_1))q(V(g_2)) \subset q(V(g_1 + g_2)).
\]

Definition 1.3 (Mixed graded tensor algebra) Let \( G \) be the additive group consisting of all pairs of integers \( (p, q) \) with addition defined by

\[
(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2).
\]

For \( p \) and \( q \) positive or exactly one of \( p \) and \( q \) zero define

\[
V^p \otimes V^q = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*;
\]

if \( p = q = 0 \) define \( V^0 = \mathbb{R} \), and if either \( p < 0 \) or \( q < 0 \) define \( V^p \) to consist of the \( 0 \) vector only (see Definition 4.1, Chapter 1). The \( G \)-graded space

\[
T(V) = \sum_{(p, q) \in \mathbb{G}} V^p
\]

(11)
is called the space of mixed tensors over $V$. For each pair of elements $(p_i, q_i) \in G$, $i = 1, 2$, let

$$
\nu: V_{p_1}^{q_1} \times V_{p_2}^{q_2} \rightarrow V_{p_1 + p_2}^{q_1 + q_2}
$$

be a bilinear map that satisfies

$$
\nu(x_1 \otimes \cdots \otimes x_{p_1} \otimes f_1 \otimes \cdots \otimes f_{q_1}, y_1 \otimes \cdots \otimes y_{p_2} \otimes g_1 \otimes \cdots \otimes g_{q_2})
= x_1 \otimes \cdots \otimes x_{p_1} \otimes y_1 \otimes \cdots \otimes y_{p_2} \otimes f_1 \otimes \cdots \otimes f_{q_1} \otimes g_1 \otimes \cdots \otimes g_{q_2}
$$

for all $x_i, y_i \in V$, $f_i, g_i \in V^*$ (see Exercise 3). If $(p_1, q_1) = (0, 0)$, set $\nu(v, w) = vw$ for $v \in V_{p_2}^{q_2}$ and similarly $\nu(z, r) = rz$ if $(p_2, q_2) = (0, 0)$ and $z \in V_{q_1}^{p_1}$. Next, let $\tau: T(V) \times T(V) \rightarrow T(V)$ be the bilinear function defined by

$$
\tau(F_1, F_2)((m, n)) = \sum_{\alpha, \beta \in G} \nu(F_1(\alpha), F_2(\beta)), \; (m, n) \in G.
$$

Then the $G$-graded space $T(V)$ together with the operation $\tau$ is called the mixed graded tensor algebra over $V$.

Some remarks concerning this definition are in order. In (13) there are in fact a number of functions $\nu$, one for each pair of elements $\alpha$ and $\beta$ in $G$. (Of course, we can define a single bilinear $\nu: T(V) \times T(V) \rightarrow T(V)$ by prescribing its action on each pair of homogeneous subspaces.) The elements $F_1$ and $F_2$ in (13) are functions defined on $G$, i.e., elements in the $G$-graded space $T(V)$, and the function $\tau$ produces a well-defined function in $T(V)$ whose value at any $(m, n) \in G$ is given by the right-hand
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side of (13). For, we need only confirm that \( \tau(F_1, F_2) \) takes on at most a finite number of nonzero values. But obviously
\[
\tau(F_1, F_2)((m,n)) \neq 0 \quad \text{only if} \quad F_1(\alpha) \neq 0 \quad \text{and} \quad F_2(\beta) \neq 0, \quad \alpha + \beta = (m,n);
\]
there are only a finite number of such pairs \( \alpha \) and \( \beta \) because \( F_1 \) and \( F_2 \) are in \( T(V) \). We also note that if \( z_i \in V_{q_i}^{p_i}, \ i = 1, 2, 3, \) then
\[
\nu(z_1, \nu(z_2, z_3)) = \nu(\nu(z_1, z_2), z_3). \tag{14}
\]

It suffices to verify (14) for decomposable elements (the \( \nu \) are bilinear), and thus (14) is an immediate consequence of (12). Also note that \( \tau(1, F) = \tau(F, 1) = F \) (see Exercise 4). Also, if \( F_i \) is homogeneous of degree \( (p_i, q_i), \ i = 1, 2, \) then \( F_i(\alpha) = 0 \) unless \( \alpha = (p_i, q_i) \) and \( F_i(\beta) = 0 \) unless \( \beta = (p_i, q_i) \) and hence
\[
\nu(F_1(\alpha), F_2(\beta)) = 0 \quad \text{unless} \quad \alpha = (p_1, q_1) \quad \text{and} \quad \beta = (p_2, q_2). \quad \text{But}
\]
then \( \nu(F_1(\alpha), F_2(\beta)) \in V_{q_1 + q_2}^{p_1 + p_2} \) and we conclude that
\[
\tau(V_{q_1}^{p_1}, V_{q_2}^{p_2}) \subset V_{q_1 + q_2}^{p_1 + p_2}. \tag{15}
\]

It is also elementary to see that \( \tau \) is associative, i.e.,
\[
\tau(F_1, \tau(F_2, F_3)) = \tau(\tau(F_1, F_2), F_3), \tag{16}
\]
which need be confirmed for homogeneous elements \( F_1, F_2, \) and \( F_3 \) only, and this, in turn, is an immediate consequence of (14) (see Exercise 5). The distributive laws,
\[
\tau(F_1, F_2 + F_3) = \tau(F_1, F_2) + \tau(F_1, F_3), \tag{17}
\]
\[
\tau(F_1 + F_2, F_3) = \tau(F_1, F_3) + \tau(F_2, F_3), \tag{18}
\]
and the remaining axiom,

\[ \tau(r_1 F_1, r_2 F_2) = r_1 r_2 \tau(F_1, F_2), \]  

(19)

are simple to confirm (see Exercise 6). We have thus established

the following elementary but basic result.

**Theorem 1.2**  The space \( T(V) \) is an associative \( G \)-graded

algebra with identity \( 1 \in R \).

For decomposable elements we have from (12) and (13)

\[
\tau(x_1 \otimes \cdots \otimes x_{p_1} \otimes f_1 \otimes \cdots \otimes f_{q_1}, y_1 \otimes \cdots \otimes y_{p_2} \otimes g_1 \otimes \cdots \otimes g_{q_2}) \\
= x_1 \otimes \cdots \otimes x_{p_1} \otimes y_1 \otimes \cdots \otimes y_{p_2} \otimes f_1 \otimes \cdots \otimes f_{q_1} \otimes g_1 \otimes \cdots \otimes g_{q_2},
\]

and hence it is sensible to refer to \( \tau \) as tensor multiplication

and to write \( \tau(F_1, F_2) \) as

\[ \tau(F_1, F_2) = F_1 \otimes F_2. \]  

(20)

If we combine this notational convention with the abuse of notation

indicated in (6), then the following formula is perfectly intelli-

gible:

\[
\begin{align*}
(2 + x_1 \otimes f_1 + x_2 \otimes x_3 \otimes f_2) \otimes (x_4 \otimes x_5 \otimes f_3 + f_4 \otimes f_5) \\
= 2x_4 \otimes x_5 \otimes f_3 + 2f_4 \otimes f_5 + x_1 \otimes x_4 \otimes x_5 \otimes f_3 \\
+ x_1 \otimes f_1 \otimes f_4 \otimes f_5 + x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes f_2 \otimes f_3 \\
+ x_2 \otimes x_3 \otimes f_2 \otimes f_4 \otimes f_5, \quad x_i \in V, \quad f_i \in V^*.
\end{align*}
\]

**Example 1.3 (a)** Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and

\( \{f_1, \ldots, f_n\} \) a basis of \( V^* \). Let \( q \) be fixed and suppose that for
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Each \( \omega \in \Gamma_n^p \), \( F_\omega \in \mathcal{V}_q^0 = \bigotimes_{l=1}^q \mathcal{V}_l^* \). If

\[
\sum_{\omega \in \Gamma_n^p} e_\omega \otimes F_\omega = 0, \tag{21}
\]

then \( F_\omega = 0 \) for every \( \omega \in \Gamma_n^p \). To see this write

\[
F_\omega = \sum_{\beta \in \Gamma_n^q} a_{\omega \beta} f_\beta^\otimes.
\]

Then (21) becomes

\[
0 = \sum_{\omega \in \Gamma_n^p} e_\omega \otimes \sum_{\beta \in \Gamma_n^q} a_{\omega \beta} f_\beta^\otimes = \sum_{\omega \in \Gamma_n^p, \beta \in \Gamma_n^q} a_{\omega \beta} e_\omega \otimes f_\beta^\otimes.
\]

However,

\[
e_\omega \otimes f_\beta^\otimes = e_{\omega(1)} \otimes \cdots \otimes e_{\omega(p)} \otimes f_{\beta(1)} \otimes \cdots \otimes f_{\beta(q)}
\]

and as \( \omega \) and \( \beta \) vary we obtain a basis of \( \mathcal{V}_q^p \). Hence \( a_{\omega \beta} = 0 \) for all \( \omega \) and \( \beta \).

(b) If \( \deg F^q = (0, q) \) and \( \deg F^r = (0, r) \) and \( F^q \otimes F^r = 0 \), then either \( F^q = 0 \) or \( F^r = 0 \). For, let

\[
F^q = \sum_{\alpha \in \Gamma_n^q} a_{\alpha} f_\alpha^\otimes, \text{ and } F^r = \sum_{\gamma \in \Gamma_n^r} b_{\gamma} f_\gamma^\otimes
\]

so that

\[
F^q \otimes F^r = \sum_{\alpha \in \Gamma_n^q, \gamma \in \Gamma_n^r} a_{\alpha} b_{\gamma} f_\alpha^\otimes \otimes f_\gamma^\otimes, \tag{22}
\]
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If neither \( r^u \) nor \( r^v \) is 0, let \( \alpha' \in \Gamma_n^u \) and \( \gamma' \in \Gamma_n^v \) be the highest sequences in lexicographic order in their respective sets for which \( a'_{\alpha'} b_{\gamma'} \neq 0 \). But obviously each basis element \( f_{\omega}^\otimes \), \( \omega \in \Gamma_n^{q+r} \) is obtained precisely once in the sum (22) as \( \alpha \) runs over \( \Gamma_n^q \) and \( \gamma \) runs over \( \Gamma_n^r \). Hence if \( r^q \otimes r^r = 0 \), we conclude that every coefficient in the sum in (22) must be 0, contradicting \( a'_{\alpha'} b_{\gamma'} \neq 0 \).

(c) If \( \deg z_{pq} = (p,q) \) and \( \deg w_{rs} = (r,s) \) and \( z_{pq} \otimes w_{rs} = 0 \), then \( z_{pq} = 0 \) or \( w_{rs} = 0 \). To see this let

\[
z_{pq} = \sum_{\alpha \in \Gamma_n^p, \beta \in \Gamma_n^q} c_{\alpha \beta} e_{\alpha}^\otimes \otimes f_{\beta}^\otimes,
\]

\[
w_{rs} = \sum_{\gamma \in \Gamma_n^r, \delta \in \Gamma_n^s} d_{\gamma \delta} e_{\gamma}^\otimes \otimes f_{\delta}^\otimes.
\]

Then

\[
z_{pq} = \sum_{\alpha \in \Gamma_n^p} e_{\alpha}^\otimes \otimes f_{\alpha}^q,
\]

(23)

\[
w_{rs} = \sum_{\gamma \in \Gamma_n^r} e_{\gamma}^\otimes \otimes f_{\gamma}^s,
\]

(24)

where

\[
f_{\alpha}^q = \sum_{\beta \in \Gamma_n^q} c_{\alpha \beta} f_{\beta}^\otimes , \quad f_{\gamma}^s = \sum_{\delta \in \Gamma_n^s} d_{\gamma \delta} f_{\delta}^\otimes .
\]

Hence

\[
z_{pq} \otimes w_{rs} = \sum_{\alpha \in \Gamma_n^p, \gamma \in \Gamma_n^r} e_{\alpha}^\otimes \otimes e_{\gamma}^\otimes \otimes f_{\alpha}^q \otimes f_{\gamma}^s,
\]

(25)
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For each \( w \in \Gamma_{n}^{p+r} \) there is precisely one \( \alpha \in \Gamma_{n}^{p} \) and one \( \gamma \in \Gamma_{n}^{r} \) for which \( e_{\alpha} \otimes e_{\gamma} = e_{w} \). Thus if we apply the result of part (a) of this example to (25), we can conclude that \( F_{\alpha}^{q} \otimes F_{\gamma}^{s} = 0 \) for all \( \alpha \in \Gamma_{n}^{p}, \gamma \in \Gamma_{n}^{r} \). But then from part (b), \( F_{\alpha}^{q} = 0 \) or \( F_{\gamma}^{s} = 0 \).

However, unless \( z_{pq} = 0 \), we see from (23) that there must be some \( F_{\alpha}^{q} \neq 0 \), and similarly unless \( w_{rs} = 0 \) there must be some \( F_{\gamma}^{s} \neq 0 \).

(d) \( T(V) \) contains no zero-divisors, i.e., if \( F_{1} \otimes F_{2} = 0 \) then \( F_{1} = 0 \) or \( F_{2} = 0 \). To see this we first order the pairs \( (p,q) \) in \( G \) lexicographically. Now if neither \( F_{1} \) nor \( F_{2} \) is 0, then write

\[
F_{1} = z_{pq} + \text{(terms of higher degree)}
\]

and

\[
F_{2} = w_{rs} + \text{(terms of higher degree)},
\]

\( z_{pq} \neq 0 \) and \( w_{rs} \neq 0 \). Let \( m = p+r \) and \( n = q+s \). From (c) we know that \( z_{pq} \otimes w_{rs} \neq 0 \), and thus if we show that \( z_{pq} \otimes w_{rs} \) is the only term of degree \((m,n)\) in \( F_{1} \otimes F_{2} \), we will be done.

Suppose then that \( \deg(z_{ij} \otimes w_{kl}) = (m,n) \), \((i,j) \geq (p,q)\), \((k,l) \geq (r,s)\) (lexicographic ordering). Now \( i \geq p \), and since \( i+k = m = p+r \), it follows that \( k \leq r \). But \((k,l) \geq (r,s)\) implies that \( k \geq r \). Hence \( k = r \) and thus \( i = p \) i.e., \((p,j) = (i,j) \geq (j,q)\) and \((r,l) = (k,l) \geq (r,s)\). Hence \( j \geq q \), \( l \geq s \). But \( j + k = n = q+s \), so neither inequality \( j \geq q \) nor \( l \geq s \) can be strict, i.e., \( j = q \), \( l = s \). In other words \((i,j) = (p,q)\), \((k,l) = (r,s)\).

**Definition 1.4** (Contravariant and covariant tensor algebras)

The \( Z \)-graded algebras
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\[ T_0(V) = \sum_{p \in \mathbb{Z}} \mathbb{V}_0^p \]

and

\[ T^0(V) = \sum_{q \in \mathbb{Z}} \mathbb{V}_q^0 \]

in which the multiplication is the tensor multiplication \( \otimes \) in \( T(V) \) are called the contravariant and covariant tensor algebras, respectively.

(The additive group of integers \( \mathbb{Z} \) can obviously be regarded as a subgroup of \( G \).)

**Example 1.4 (a)** Let \( I \) be the (two-sided) ideal in \( T_0(V) \) generated by all elements of the form \( x \otimes x, \ x \in V \). That is, \( I \) consists of all finite sums of the form

\[ \sum_{x, u, v} u \otimes x \otimes x \otimes v \otimes . \quad (26) \]

Then clearly \( I \) is a homogeneous ideal, i.e., it is spanned by homogeneous elements, and thus according to Example 1.2 (f), \( I \) is a \( \mathbb{Z} \)-graded algebra as well;

\[ I = \sum_{p \in \mathbb{Z}} I \cap \mathbb{V}_0^p. \quad (27) \]

Hence according to Theorem 1.1 (b) the quotient algebra

\[ T_0(V)/I = \sum_{p \in \mathbb{Z}} q(\mathbb{V}_0^p) \quad (28) \]

is a \( \mathbb{Z} \)-graded algebra (\( q \) is the quotient map). From (26) we see that the homogeneous elements in \( I \) are all of degree at least 2.
and hence 1 is not all of $T_0(V)$. In fact, it is clear that $q$ is injective on both $R$ and $V$, and so we identify $q(R)$ with $R$ and $q(V)$ with $V$, i.e., $q(r) = r$ and $q(v) = v$ for all $r \in R$ and $v \in V$. Since $q: T_0(V) \to T_0(V)/I$ is linear, for a fixed $m$ we define the multilinear function

$$
u(v_1, \ldots, v_m) = q(v_1 \otimes \cdots \otimes v_m)$$

and observe from (28) that $\langle \text{Im } \nu \rangle = q(V^m)$, the set of homogeneous elements in $T_0(V)/I$ of degree $m$. For any $x$ and $y$ in $V$ and arbitrary $u^\otimes$, $v^\otimes$ we know that $u^\otimes \otimes (x+y) \otimes (x+y) \otimes v^\otimes \in I$ and hence

$$0 = q(u^\otimes \otimes (x+y) \otimes (x+y) \otimes v^\otimes)$$

$$= q(u^\otimes \otimes x \otimes x \otimes v^\otimes) + q(u^\otimes \otimes x \otimes y \otimes v^\otimes)$$

$$+ q(u^\otimes \otimes y \otimes x \otimes v^\otimes) + q(u^\otimes \otimes y \otimes y \otimes v^\otimes). \quad (29)$$

The first and last summands in (29) are 0 and hence the second and third summands are negatives of one another. Hence, in particular we have

$$\nu(v_1, \ldots, v_i, v_{i+1}, \ldots, v_m) = q(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_m)$$

$$= -q(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_m)$$

$$= -\nu(v_1, \ldots, v_{i+1}, v_i, \ldots, v_m). \quad (30)$$

Since any permutation $\sigma \in S_m$ is a product of adjacent interchanges and the number of such interchanges has precisely the same parity as $\epsilon(\sigma)$, we see that

$$\nu(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \epsilon(\sigma) \nu(v_1, \ldots, v_m) \quad (31)$$
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\( \nu \in M_m(V, q(V^\perp)), S_m, \varepsilon \). It is also obvious that

\( I \cap V^m_0 = \ker S_\varepsilon \), where \( S_\varepsilon \) is the symmetry operator associated with \( S_m \) and \( \varepsilon \). In fact, (31) states that \( q(\nu^\otimes_\sigma) = \varepsilon(\sigma)q(\nu^\otimes) \)

and hence \( \nu^\otimes_\sigma - \varepsilon(\sigma)\nu^\otimes \in I \cap V^m_0 \). But \( \ker S_\varepsilon = \Im(I - S_\varepsilon) \),

\[
(I - S_\varepsilon)\nu^\otimes = -\frac{1}{n!} \sum_{\sigma \in S_m} \varepsilon(\sigma)(\nu^\otimes_\sigma - \varepsilon(\sigma)\nu^\otimes)
\]

and each of the summands is in \( I \cap V^m_0 \). But then \( \ker S_\varepsilon = \Im(I - S_\varepsilon) \subset I \cap V^m_0 \). The inclusion in the other direction is trivial because every element of \( I \cap V^m_0 \) is a sum of decomposable elements each of which involves a vector in \( V \) at least twice and hence is in \( \ker S_\varepsilon \). Finally, if \( \{e_1, ..., e_n\} \) is a basis of \( V \), then the elements \( \nu(e_{\omega(1)}, ..., e_{\omega(m)}) = q(e^\otimes_\omega), \omega \in Q_{m,n}, \) are l.i. For, if

\[
\sum_{\omega \in Q_{m,n}} c_\omega q(e^\otimes_\omega) = 0,
\]

then \( \sum_{\omega \in Q_{m,n}} c_\omega e^\otimes_\omega \in \ker q = I \), and since \( \sum_{\omega \in Q_{m,n}} c_\omega e^\otimes_\omega \in V^m_0 \), we conclude \( \sum_{\omega \in Q_{m,n}} c_\omega e^\otimes_\omega \in I \cap V^m_0 = \ker S_\varepsilon \). But then

\[
0 = S_\varepsilon(\sum_{\omega \in Q_{m,n}} c_\omega e^\otimes_\omega)
\]

\[
= \sum_{\omega \in Q_{m,n}} c_\omega e^{^\wedge}_\omega.
\]

It follows that since \( e^{^\wedge}_\omega, \omega \in Q_{m,n} \), is a basis of \( ^m \wedge V \), \( c_\omega = 0 \), \( \omega \in Q_{m,n} \). Thus we have proved: \( \langle \Im \nu \rangle = q(V^m_0) \),
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\[ v \in M_n(V, q(v^m), S_m, e), \text{ and finally that } \dim v \text{ contains } \binom{m}{m} \text{ l.i. elements. Theorem 3.3, Chapter 2 then implies that } q(v^m_0) \text{ is a symmetry class associated with } S_m \text{ and } e, \text{ i.e.,} \]

\[ q(v^m_0) = \bigwedge^m V. \quad (32) \]

[If \( m > n = \dim V \), then of course \( q(v^m_0) = \bigwedge^m V = 0 \).] From (28) and (32) we have

\[ \dim T_0(V)/I = \sum_{p=0}^{n} \binom{n}{p} = 2^n. \]

Now the multiplication in the quotient algebra is given in general [see Theorem 1.1(a)] by \( q(zw) = q(z)q(w) \). In the event \( z = v_1 \otimes \cdots \otimes v_p \) and \( w = v_{p+1} \otimes \cdots \otimes v_{p+q} \), we have

\[
q(v_1 \otimes \cdots \otimes v_p \otimes v_{p+1} \otimes \cdots \otimes v_{p+q}) \\
= q((v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q})) \\
= q(v_1 \otimes \cdots \otimes v_p)q(v_{p+1} \otimes \cdots \otimes v_{p+q}). \quad (33)
\]

Since \( v(v_1, \ldots, v_m) = q(v^\otimes) \) and since \( q(v^m_0) = \bigwedge^m V \), we write \( v_1 \wedge \cdots \wedge v_m \) for \( q(v^\otimes) \) and we use the symbol \( \wedge \) to denote the multiplication in \( T_0(V)/I \) rather than juxtaposition. Thus (33) becomes

\[
(v_1 \wedge \cdots \wedge v_p) \wedge (v_{p+1} \wedge \cdots \wedge v_{p+q}) = v_1 \wedge \cdots \wedge v_{p+q}. \quad (34)
\]

If we set \( T_0(V)/I = \bigwedge V \), then \( \bigwedge V \) is called the Grassmann or exterior or skew-symmetric algebra over \( V \). We have from (28) and (32)
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\[ \wedge V = \sum_{p=0}^{n} P V, \]

\[ \dim \wedge V = 2^n. \]  \hspace{1cm} (35)

(b) In this example take I to be the (two-sided) ideal in \( T_0(V) \) generated by all elements of the form \( x \otimes y - y \otimes x, \) \( x, y \in V. \) That is, I consists of all finite sums of the form

\[ \sum_{x, y, u, v} u^\otimes \otimes (x \otimes y - y \otimes x) \otimes v^\otimes. \]

Again, I is a homogeneous ideal and hence I is \( \mathbb{Z} \)-graded:

\[ I = \sum_{p \in \mathbb{Z}}^* I \cap V_0^p. \]

The quotient algebra \( T_0(V)/I \) is also \( \mathbb{Z} \)-graded and

\[ T_0(V)/I = \sum_{p \in \mathbb{Z}}^* q(V_0^p), \] \hspace{1cm} (36)

where \( q \) is the quotient map [again, \( I \cap (V \cup R) = 0 \) and hence \( q \) is injective on \( R \) and \( V; \) we identify \( q(R) \) with \( R \) and \( q(V) \) with \( V \)]. Now define a multilinear function \( \nu: x V \to q(V_0) \)

by

\[ \nu(v_1, \ldots, v_m) = q(v_1 \otimes \cdots \otimes v_m), \]

\[ \langle \text{Im } \nu \rangle = q(V_0^m). \]  We have
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\[ \nu(v_1, \ldots, v_i, v_{i+1}, \ldots, v_m) - \nu(v_1, \ldots, v_{i+1}, v_i, \ldots, v_m) = q((v_1 \otimes \cdots \otimes v_{i-1}) \otimes v_{i+1} \otimes (v_{i+2} \otimes \cdots \otimes v_m)) \]
\[ - q((v_i \otimes \cdots \otimes v_{i-1}) \otimes v_{i+1} \otimes v_1 \otimes (v_{i+2} \otimes \cdots \otimes v_m)) \]
\[ = q(u \otimes (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) \otimes v) \]
\[ = 0, \]

where \( u = v_1 \otimes \cdots \otimes v_{i-1}, \ v = v_{i+2} \otimes \cdots \otimes v_m \). In other words, the interchange of two adjacent vectors in \( \nu(v_1, \ldots, v_m) \) leaves the value of the function unaltered, and it follows easily that \( v \in \ker(S_m, S_{m-1}) \). We also assert that \( I \cap v_m = \ker(S_1) \), where \( S_1 \) is the symmetrizer associated with \( S_m \) and \( \chi \neq 1 \). For,

\[ q(v_\sigma \otimes v) = q(v_\sigma) - q(v_\sigma - v) \]
\[ = \nu(v_\sigma(1), \ldots, v_\sigma(m)) - \nu(v_1, \ldots, v_m) = 0 \]

so that \( v_\sigma \otimes v \in I \cap v_m \). But \( \ker(S_1) = \text{Im}(I - S_1) \),

\[ (I - S_1)v = \frac{1}{m!} \sum_{\sigma \in S_m} (v_\sigma \otimes v) \],

and each summand is in \( I \cap v_m \). Thus \( \ker(S_1) = \text{Im}(I - S_1) \subset I \cap v_m \).

The inclusion in the other direction is equally easy since every element of \( I \cap v_m \) is a sum of elements of the form

\[ u_1 \otimes \cdots \otimes u_k \otimes x \otimes y \otimes v_1 \otimes \cdots \otimes v_p \]
\[- u_1 \otimes \cdots \otimes u_k \otimes y \otimes x \otimes v_1 \otimes \cdots \otimes v_p, \]

which \( S_1 \) clearly maps into 0. If \( \{e_1, \ldots, e_n\} \) is a basis of \( V \),
then the elements \( \nu(e_{w(1)}, \ldots, e_{w(m)}) = q(e_w^\otimes), \ w \in G_{m,n} \) are l.i. For, if

\[
\sum_{\omega \in G_{m,n}} c_\omega q(e_\omega^\otimes) = 0,
\]

then

\[
\sum_{\omega \in G_{m,n}} c_\omega e_\omega^\otimes \in \ker q = I
\]

so that

\[
\sum_{\omega \in G_{m,n}} c_\omega e_\omega^\otimes \in I \cap V_0^m = \ker S_1.
\]

Hence

\[
0 = S_1 \sum_{\omega \in G_{m,n}} c_\omega e_\omega^\otimes = \sum_{\omega \in G_{m,n}} c_\omega e_\omega^\ast.
\]

But the \( e_\omega^\ast, \ \omega \in G_{m,n} \), form a basis of \( V^{(m)} \), the symmetry class associated with \( S_m \) and \( \chi \equiv 1 \). Hence \( c_\omega = 0, \ \omega \in G_{m,n} \), and the \( q(e_\omega^\otimes) \) are l.i. We have proved \( \langle \text{Im } \nu \rangle = q(V_0^m) \), 

\( \nu \in M_m(V, q(V_0^m), S_m, 1) \), and \( \text{Im } \nu \) contains \( \binom{n+m-1}{m} \) l.i. elements. From Theorem 3.3, Chapter 2 we conclude that \( q(V_0^m) \) is a symmetry class associated with \( S_m \) and \( \chi \equiv 1 \), i.e.,

\[
q(V_0^m) = V^{(m)}.
\]

Thus from (36)

\[
T_0(V)/I = \sum_{m=0}^{\infty} V^{(m)}
\]

and the multiplication in \( T_0(V)/I \) implies that
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\[ q(v_1 \otimes \cdots \otimes v_p \otimes v_{p+1} \otimes \cdots \otimes v_{p+q}) \]
\[ = q((v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q})) \]
\[ = q(v_1 \otimes \cdots \otimes v_p)q(v_{p+1} \otimes \cdots \otimes v_{p+q}). \quad (37) \]

Now \( v(v_1, \ldots, v_m) = q(v^\otimes) \), and since \( q(v^m) = v^{(m)} \) we write \( v_1 \cdots v_m \) for \( q(v^\otimes) \) [see formula (37), Section 2.4] and use a dot instead of juxtaposition for the multiplication in \( T_0(V)/I \).

Then (37) becomes

\[ (v_1 \cdots v_p)(v_{p+1} \cdots v_{p+q}) = v_1 \cdots v_{p+q}. \quad (38) \]

We denote \( T_0(V)/I \) by \( V^* \); it is called the graded symmetric algebra over \( V \). It is of course an infinite dimensional algebra because

\[ V^* = \sum_{m=0}^{\infty} v^{(m)} \]

and

\[ \dim v^{(m)} = \binom{n+m-1}{m}. \]

The tensor algebra \( T_0(V) \) has a certain important universal property with respect to linear maps of \( V \) into an associative algebra.

**Theorem 1.3** Let \( V \) be a vector space over \( R \) and \( \mathfrak{A} \) an associative algebra over \( R \) with multiplicative identity \( 1_{\mathfrak{A}} \). Let \( h: V \rightarrow \mathfrak{A} \) be a linear map. Then there exists a unique identity preserving algebra homomorphism \( \overline{h}: T_0(V) \rightarrow \mathfrak{A} \) such that \( \overline{h}|V = h \).

**Proof:** Define \( h_0: R \rightarrow R \) to be the identity, and define
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\[ h_p : \otimes^p V \to \mathcal{U} \] so that \[ h_p(\otimes^i v_i) = \prod_{i=1}^p h(v_i) \]. That such an \( h_p \) exists follows immediately from the universal factorization property of \( \otimes^p V \). Then define \( \overline{h} : T_0(V) \to \mathcal{U} \) by

\[ \overline{h} \left( \sum_{p=0}^{m} z_p \right) = \sum_{p=0}^{m} h_p(z_p), \]

where \( z_p \) is homogeneous of degree \( p \). Obviously, \( \overline{h}|V = h \) and

\[ \overline{h}(x \otimes y) = \overline{h}(x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q) \]
\[ = h_{p+q}(x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q) \]
\[ = \prod_{j=1}^{p} h(x_j) \prod_{j=1}^{q} h(y_j) \]
\[ = h_p(x \otimes) h_q(y \otimes) \]
\[ = \overline{h}(x \otimes) \overline{h}(y \otimes). \]

Thus \( \overline{h} \) is multiplicative on decomposable elements, and since these generate \( T_0(V) \), it is an algebra homomorphism. Also \( \overline{h} \) is uniquely specified since the values of an algebra homomorphism are determined by the values on a generating set and \( \{1\} \cup V \) is a generating set of \( T_0(V) \). (See Exercise 7.)

**Definition 1.5 (Universal tensor algebra property)** Let \( V \) be a vector space over \( R \) and \( \mathcal{U} \) an associative algebra over \( R \) with identity \( 1_{\mathcal{U}} \). Suppose \( \nu : V \to \mathcal{U} \) is linear and satisfies the following two conditions:

(i) \( \mathcal{U} \) is generated by \( I_m \nu \) and \( 1_{\mathcal{U}} \);
(ii) if \( \varphi: V \to \mathcal{B} \) is any linear map into an associative algebra \( \mathcal{B} \) over \( \mathbb{R} \) with identity \( 1_\mathcal{B} \), then there exists an algebra homomorphism \( h: \mathcal{U} \to \mathcal{B} \) such that \( h(1_\mathcal{U}) = 1_\mathcal{B} \) and \( \varphi = hv \), i.e., the diagram

\[
\begin{array}{c}
V \\
\downarrow \varphi \\
\mathcal{U} \\
\downarrow h \\
\mathcal{B}
\end{array}
\]

is commutative. We then say that the pair \( (\mathcal{U}, v) \) has the universal tensor algebra property. Observe that \( h \) is uniquely determined by \( \varphi \) since \( h(1_\mathcal{U}) = 1_\mathcal{B} \), \( h(\varphi(v)) = \varphi(v) \), and \( \mathcal{U} \) is generated by \( 1_\mathcal{U} \) and \( \text{Im } v \).

We first remark that if the pairs \( (\mathcal{U}, v) \) and \( (\mathcal{U}', v') \) both have the universal tensor algebra property with respect to \( V \), then there is a unique identity preserving algebra isomorphism \( h: \mathcal{U} \to \mathcal{U}' \) such that \( h(\varphi(v)) = v'(v) \) for all \( v \in V \) (see Exercise 8). Our next result tells us that \( T_0(V) \) has the universal tensor algebra property. More precisely, we have

**Theorem 1.4** Let \( v: V \to T_0(V) \) denote the map that sends each vector in \( V \) into the corresponding homogeneous element of \( T_0(V) \), i.e., \( v(v) = v \). Then the pair \( (T_0(V), v) \) has the universal tensor algebra property.

**Proof:** Obviously (i) of Definition 1.4 is satisfied. To prove (ii), suppose \( \varphi: V \to \mathcal{B} \) [as in (ii)]. By Theorem 1.3, extend \( \varphi \) uniquely to an algebra homomorphism \( \overline{\varphi}: T_0(V) \to \mathcal{B} \), \( \overline{\varphi}(1) = 1_\mathcal{B} \). But
then for $v \in V$

$$\varphi(v(v)) = \varphi(v) = \varphi(v)$$

so that $\varphi$ plays the role of $h$ in (ii).

In much the same way, we can show that the Grassmann algebra over $V$ has an appropriate universal property.

**Theorem 1.5** There exists a fixed linear map $c: V \to \wedge V$ which satisfies the following. If $\varphi \in L(V, \mathbb{U})$, where $\mathbb{U}$ is an associative algebra with identity $1_{\mathbb{U}}$ and if $\varphi(v)^2 = 0$ for all $v \in V$, then there exists a unique algebra homomorphism $h: \wedge V \to \mathbb{U}$ such that $\varphi = hc$, $h(1) = 1_{\mathbb{U}}$, and $h(z \wedge w) = h(z) h(w)$ for all $z, w$ in $\wedge V$.

**Proof:** As we saw in Example 1.4(a), $\mathcal{T}_0(V)/I = \wedge V$ where $I$ is the ideal defined in (26). We define the mapping $c$ to be the quotient map restricted to $V$, i.e., $c = q|V$. By Theorem 1.3, extend $\varphi$ to an algebra homomorphism $\varphi: \mathcal{T}_0(V) \to \mathbb{U}$ such that $\varphi(1) = 1_{\mathbb{U}}$. Observe that $\varphi(x \otimes x) = \varphi(x)^2 = \varphi(x)^2 = 0$ for any $x \in V$, and hence $\varphi$ vanishes on all of the ideal $I$ [see (26)]. In other words, $I \subseteq \ker \varphi$. But then Theorem 1.1(a) implies that there exists a unique algebra homomorphism $h: \mathcal{T}_0(V)/I \to \mathbb{U}$ such that $\varphi = hq$. Thus for any $v \in V$, $\varphi(v) = \varphi(v) = h(q(v)) = h(c(v))$, i.e., $\varphi = hc$. Also, $h(1) = h(q(1)) = \varphi(1) = 1_{\mathbb{U}}$. [We recall from
Example 1.4(a) that the quotient map \( q \) is injective on \( R \) and \( V \) and we identified \( q(R) \) with \( R \) and \( q(V) \) with \( V \).]

In precisely the same way, we can show that the symmetric algebra \( V^* \) is universal for mappings of \( V \) into associative algebras in which the values in the range of \( \varphi \) commute. To be precise, we have

**Theorem 1.6** There exists a fixed linear map \( c: V \to V^* \) that satisfies the following. If \( \varphi \in L(V, U) \), where \( U \) is an associative algebra with identity \( 1_U \), and if \( \varphi(x)\varphi(y) = \varphi(y)\varphi(x) \) for all \( x, y \in V \), then there exists a unique algebra homomorphism \( h: V^* \to U \) such that \( \varphi = hc \), \( h(1) = 1_U \), and \( h(zw) = h(z)h(w) \) for all \( z, w \) in \( V^* \).

**Proof:** As in Example 1.4(b), \( T_0(V)/I = V^* \), where \( I \) is the homogeneous ideal generated by elements of the form \( x \otimes y - y \otimes x \).

Define the mapping \( c \) to be the quotient map \( q|V \). By Theorem 1.3 extend \( \varphi \) to an algebra homomorphism \( \overline{\varphi}: T_0(V) \to U \) such that \( \overline{\varphi}(1) = 1_U \). Then

\[
\overline{\varphi}(x \otimes y - y \otimes x) = \overline{\varphi}(x \otimes y) - \overline{\varphi}(y \otimes x) = \overline{\varphi}(x)\overline{\varphi}(y) - \overline{\varphi}(y)\overline{\varphi}(x) = \overline{\varphi}(x)\varphi(y) - \varphi(y)\varphi(x) = 0.
\]

Hence \( \overline{\varphi} \) vanishes on the ideal; i.e., \( I \subseteq \ker \overline{\varphi} \). Applying Theorem 1.1(a) we can then conclude that there exists a unique algebra homomorphism \( h: T_0(V)/I \to U \) such that \( \overline{\varphi} = h\overline{q} \). Then if \( v \in V \), \( \overline{\varphi}(v) = \overline{\varphi}(v) = h(q(v)) = h(c(v)) \), i.e., \( \varphi = hc \). Also, \( h(1) = h(q(1)) = \overline{\varphi}(1) = 1_U \). Once again, we recall from
Example 1.4(b) that the quotient map \( q \) is injective on \( R \cup V \) and we identify \( q(R) \) with \( R \) and \( q(V) \) with \( V \). 

**Example 1.5(a)** Let \( \xi_1, \ldots, \xi_n \) be independent indeterminates over the field \( R \) and let \( R[\xi_1, \ldots, \xi_n] \) be the polynomial algebra over \( R \) in these indeterminates. Clearly, any polynomial in \( R[\xi_1, \ldots, \xi_n] \) can be written in the form

\[
p(\xi_1, \ldots, \xi_n) = \sum_w a_w \xi_{i_1}^{m_{i_1}(w)} \cdots \xi_{i_n}^{m_{i_n}(w)}
\]

in which \( w \) runs over an appropriate union of sets \( \Gamma_n^n \) and where \( m_t(w) \) is the number of times \( t \) occurs in \( \mathrm{Im} \, w \). We define the degree of the polynomial in (39) to be the largest integer \( r \) for which there is a \( w \in \Gamma_n^n \) such that \( a_w \neq 0 \): \( \deg p(\xi_1, \ldots, \xi_n) = r \). Next define \( R_r[\xi_1, \ldots, \xi_n] \) to be the set of all polynomials (39) in \( R[\xi_1, \ldots, \xi_n] \) for which each of the nonzero summands on the right in (39) has degree \( r \). Then clearly by grouping terms any polynomial can be written as a unique sum of such polynomials. Thus

\[
R[\xi_1, \ldots, \xi_n] = \bigoplus_{r=0}^\infty R_r[\xi_1, \ldots, \xi_n],
\]

i.e., \( R[\xi_1, \ldots, \xi_n] \) is a \( Z \)-graded algebra. Moreover, \( R[\xi_1, \ldots, \xi_n] \) is commutative and associative and has \( 1 \) as multiplicative identity.

Next let \( V \) be a vector space over \( R \), \( \dim V = n \), and let \( e_1, \ldots, e_n \) be a basis of \( V \). Define \( \varphi: V \to R[\xi_1, \ldots, \xi_n] \) by \( \varphi(e_i) = \xi_i, \, i = 1, \ldots, n \). Then according to Theorem 1.6 there exists a unique algebra homomorphism \( h: V^* \to R[\xi_1, \ldots, \xi_n] \) such that \( h(e_i) = \varphi(e_i) = \xi_i, \, i = 1, \ldots, m \). If \( w \in G_{m,n} \), then
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\[ h(e^*_w) = \prod_{t=1}^{n} m_t(w) \]

and

\[ \sum_{t=1}^{n} m_t(w) = m = \deg \prod_{t=1}^{n} e_t(w) \]

It is also clear that \( R_m[\xi_1, \ldots, \xi_n] \) has as a basis the polynomials

\[ \prod_{t=1}^{n} e_t(w), \quad w \in \mathcal{O}_{m, n} \]

(see Exercise 9). Thus \( h|\nu^{(m)} : R_m[\xi_1, \ldots, \xi_n] \to V^n \) is a bijection. It follows that \( h \) is an algebra isomorphism, i.e., the symmetric algebra \( V^* \) is isomorphic with the algebra of polynomials \( R[\xi_1, \ldots, \xi_n] \).

(b) The exterior algebra over \( V, \wedge V \), always has zero divisors. For \( v \wedge v = 0 \) for any \( v \in V \). However, the symmetric algebra over \( V \) has no zero divisors. This can be easily seen by using the isomorphism in (a) above. For, \( R[\xi_1, \ldots, \xi_n] \) clearly has no zero-divisors.

(c) In this example we use some simple properties of the exterior algebra to derive Cramer's rule for solving linear equations. Let \( V = V_n(R) \) and consider \( Ax = b \), where \( A \) is nonsingular in \( M_n(R) \) and \( x \) and \( b \) are in \( V \). If \( A^{(j)} \) denotes the \( j \)th column of \( A \), let

\[ F_t = A^{(1)} \wedge \cdots \wedge A^{(t-1)} \wedge A^{(t+1)} \wedge \cdots \wedge A^{(n)}, \quad t = 1, \ldots, n. \]

Then \( F_t \wedge A^{(j)} = 0 \) for \( t \neq j \). Write \( Ax = b \) as \( \sum_{j=1}^{n} x_j A^{(j)} = b \).
and multiply both sides by $F_t$ to obtain

$$F_t \wedge A(t) x_t = F_t \wedge b.$$  \hspace{1cm} (41)

Now

$$F_t \wedge A(t) = A(1) \wedge \cdots \wedge A(t-1) \wedge A(t+1) \wedge \cdots \wedge A(n) \wedge A(t)$$

$$= (-1)^{n-t} A(1) \wedge \cdots \wedge A(n)$$

$$= (-1)^{n-t} (\det A) e_1 \wedge \cdots \wedge e_n,$$

where $e_i = (\delta_{i1}, \ldots, \delta_{in})$, $i = 1, \ldots, n$. Similarly

$$F_t \wedge b = (-1)^{n-t} (\det A_t) e_1 \wedge \cdots \wedge e_n,$$

where $A_t$ is the matrix whose $n$ columns in order are

$$A(1), \ldots, A(t-1), A(t+1), \ldots, A(n).$$

Thus from (41) $x_t = \det A_t/\det A$, $t = 1, \ldots, n$.

(d) The *Laplace expansion theorem* can also be readily proved.

Using the notation of the preceding example (here $A(t)$ is the $t^{th}$ row of $A$)

$$(\det A) e_1 \wedge \cdots \wedge e_n$$

$$= A(1) \wedge \cdots \wedge A(p) \wedge A(p+1) \wedge \cdots \wedge A(n)$$

$$= (A(1) \wedge \cdots \wedge A(p)) \wedge (A(p+1) \wedge \cdots \wedge A(n))$$

$$= \sum_{w \in Q_{p,n}} \det A[1, \ldots, p, w] e_w ^\wedge \sum_{\gamma \in Q_{n-p,n}} \det A[p+1, \ldots, n | \gamma] e_\gamma ^\wedge$$

$$= \sum_{w \in Q_{p,n}} \det A[1, \ldots, p, w] \det A[p+1, \ldots, n | \gamma] e_w ^\wedge e_\gamma ^\wedge.$$  \hspace{1cm} (42)
Now \( e_\gamma \wedge e_\gamma = 0 \) unless \( \gamma = w' \), the complementary sequence to \( w \) in \( Q_{n-p,n} \), and then it is easy to check that (see Exercise 14)

\[
e_\gamma \wedge e_\gamma' = \varepsilon \left( \begin{array}{cccc}
1 & \cdots & p & p+1 \cdots n \\
w(1) & \cdots & w(p) & w'(1) \cdots w'(n-p)
\end{array} \right) e_1 \wedge \cdots \wedge e_n
\]

\[
= (-1)^{s(w) + p(p+1)/2} e_1 \wedge \cdots \wedge e_n ,
\]

(43)

where \( s(w) = w(1) + \cdots + w(p) \). Replacing (43) in (42) and equating the coefficients of \( e_1 \wedge \cdots \wedge e_n \) on either side yields the formula

\[
det A = \sum_{w \in Q_{p,n}} \det A[l, \ldots, p \mid w] \det A(l, \ldots, p \mid w)(-1)^{s(w) + p(p+1)/2} ,
\]

the Laplace expansion of \( A \) "by the first \( p \) rows" [in general, if \( \alpha \) and \( \beta \) are both sequences of distinct integers then \( A(\alpha \mid \beta) \) denotes the submatrix obtained from \( A \) by deleting rows numbered \( \alpha \) and columns numbered \( \beta \)].

(e) We show that if \( u_r \) and \( u_s \) are homogeneous elements in \( \wedge V \) of degrees \( r \) and \( s \), respectively, then \( u_r \wedge u_s = (-1)^{rs} u_s \wedge u_r \). We can assume \( u_r \) and \( u_s \) are decomposable. But then \( u_r \wedge u_s = x_1 \wedge \cdots \wedge x_r \wedge x_{r+1} \wedge \cdots \wedge x_{r+s} \). Now, move \( x_{r+s} \) to the first position. This requires \( r+s-1 \) interchanges. Then move \( x_{r+s-1} \) to the first position, requiring \( r+s-1 \) interchanges again. Continue for a total of \( s(r+s-1) \) interchanges to obtain

\[
u_s \wedge u_r = x_{r+1} \wedge \cdots \wedge x_{r+s} \wedge x_1 \wedge \cdots \wedge x_r \].

But

\[
(-1)^{s(r+s-1)} = (-1)^{rs} .
\]

Several additional results of this kind are to be found in Exercises 15 and 16.
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The isomorphism between the completely symmetric algebra $V^*$ and the algebra of polynomials in $n$ indeterminates $R[\xi_1, \ldots, \xi_n]$ exhibited in Example 1.5 (a) can be used to prove a theorem on the structure of spanning sets of $V^*$, which is useful in the representation theory of the full linear group.

**Theorem 1.7** Let $\dim V = n$ and let $\{e_1, \ldots, e_n\}$ be a basis of $V$. Let $e(w) = \sum_{t=1}^{n} m_t(w)e_t$, $w \in G_{m,n}$, and set

$$e^*(w) = e(w) \cdots e(w) \in V^{(m)}.$$ 

Then $\{e^*(w), w \in G_{m,n}\}$ is a basis of $V^{(m)}$. Thus $V^{(m)}$ is spanned by elements of the form $x \cdots x$, $x \in V$.

**Proof:** Let $h: V^* \to R[\xi_1, \ldots, \xi_n]$ be the algebra isomorphism described in Example 1.5 (a). Then

$$h(e^*(w)) = h(e(w))^m$$

$$= \left( \sum_{t=1}^{n} m_t(w) h(e_t) \right)^m$$

$$= \left( \sum_{t=1}^{n} m_t(w) \xi_t \right)^m \quad (44)$$

Since $\dim V^{(m)} = |G_{m,n}|$, it follows that we need only show that the polynomials on the right in (44) are linearly independent in $R[\xi_1, \ldots, \xi_n]$. If we set $a_t = m_t(w)$ (as a dummy summing index), then a linear dependence relation among these polynomials can be expressed as

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\[ S = \sum_{a_1 + \cdots + a_n = m} d(a_1, \ldots, a_n) \left( \sum_{t=1}^{n} a_t \xi_t \right)^m = 0, \quad (45) \]

where the summation is over all partitions of \( m \) into \( n \) parts and \( d(a_1, \ldots, a_n) \in R \). We prove that these coefficients are all 0 by induction on \( m+n \). If \( m+n=2 \), then \( m=n=1 \) and \( d\xi_1 = 0 \), and hence \( d = 0 \). Suppose then that \( m+n=k \), and write

\[ S = S_0 + S_1, \quad \text{where} \quad S_0 \quad \text{is the sum of all terms in (45) for which} \quad a_n = 0 \quad \text{and} \quad S_1 \quad \text{is the sum of all terms for which} \quad a_n \geq 1. \]

Then \( \partial S_0 / \partial \xi_n = 0 \), and since \( S = 0 \), we have \( \partial S_1 / \partial \xi_n = 0 \). We compute that

\[ \frac{\partial S_1}{\partial \xi_n} \]

\[ = m \sum_{a_1 + \cdots + a_n = m} a_n d(a_1, \ldots, a_n) \left( \sum_{t=1}^{n} a_t \xi_t \right)^{m-1} \]

\[ \frac{a_1 + \cdots + a_n = m}{a_1 \geq 0, a_n \geq 1} \]

\[ = m \sum_{a_1 + \cdots + a_n = m-1} (1+a_n)d(a_1, \ldots, a_{n-1}, 1+a_n)(a_1 \xi_1 + \cdots + a_n \xi_n + \xi_n)^{m-1}. \quad (46) \]

[The second equality in (46) results from replacing the dummy index \( a_n \geq 1 \) by \( a_n + 1 \) and summing for \( a_n \geq 0 \).] Replace each \( \xi_1 \) by \( \xi_1 + z \), where \( z = -\xi_n / m \) in (46). Then the term involving the indeterminates in the summand on the right in (46) becomes
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\[ a_1(\xi_1 + z) + \cdots + a_n(\xi_n + z) + \xi_n + z \]
\[ = a_1 \xi_1 + \cdots + a_n \xi_n + z(a_1 + \cdots + a_n + 1) + \xi_n \]
\[ = a_1 \xi_1 + \cdots + a_n \xi_n - \frac{n}{m} (m-1+1) + \xi_n \]
\[ = a_1 \xi_1 + \cdots + a_n \xi_n. \]

Hence, since \( \partial S_1 / \partial \xi_n \) is the zero polynomial, we have from (46),

\[ \sum_{a_1 + \cdots + a_n = m-1}^{m} (1+a_n) d(a_1, \ldots, a_{n-1}, 1+a_n)(a_1 \xi_1 + \cdots + a_n \xi_n)^{m-1} = 0. \]

By induction, it follows that

\[ (1+a_n) d(a_1, \ldots, a_{n-1}, 1+a_n) = 0 \]

for \( a_1 + \cdots + a_n = m-1 \), i.e.,

\[ d(a_1, \ldots, a_{n-1}, a_n) = 0 \]

for \( a_1 + \cdots + a_n = m \) and \( a_n \geq 1 \). In other words, every coefficient that appears in \( S_1 \) is zero. But then \( S_0 \) is also the zero polynomial, and \( S_0 \) is given by

\[ S_0 = \sum_{a_1 + \cdots + a_{n-1} = m} d(a_1, \ldots, a_{n-1}, 0)(a_1 \xi_1 + \cdots + a_{n-1} \xi_{n-1})^m. \]

Once again by induction, \( d(a_1, \ldots, a_{n-1}, 0) = 0 \) for all \( a_1 + \cdots + a_{n-1} = m \). Hence all the coefficients in \( S_0 \) are also zero and thus all the coefficients in \( S \) are zero.

Exercises

1. Show that any vector space \( V \) is isomorphic to a \( G \)-graded space.
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Hint: Define \( V(0) = V \) and \( V(g) = \{0\} \), \( g \neq 0 \) (0 is the identity in \( g \)).

2. In Theorem 1.1(a) confirm that \( \mathcal{U}/I \) satisfies the axioms for an algebra over \( R \).

3. Show that a bilinear map \( \nu \) satisfying (12) exists.

Hint: In general if \( V_1, \ldots, V_{p+q} \) are vector spaces over \( R \) and \( \sigma \in S_{p+q} \), then there exists a bilinear

\[
\nu: \left( \bigotimes_{1}^{p} V_{1} \right) \times \left( \bigotimes_{p+1}^{p+q} V_{1} \right) \rightarrow \bigotimes_{1}^{p+q} V_{\sigma(1)}
\]

such that

\[
\nu(v_1 \otimes \cdots \otimes v_{p}, v_{p+1} \otimes \cdots \otimes v_{p+q}) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p+q)}.
\]

For, let

\[
h: \left( \bigotimes_{1}^{p} V_{1} \right) \otimes \left( \bigotimes_{p+1}^{p+q} V_{1} \right) \rightarrow \bigotimes_{1}^{p+q} V_{\sigma(1)}
\]

be the isomorphism which satisfies

\[
h((v_1 \otimes \cdots \otimes v_{p}) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q})) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p+q)}
\]

and let

\[
\beta: \bigotimes_{1}^{p} V_{1} \times \bigotimes_{p+1}^{p+q} V_{1} \rightarrow \left( \bigotimes_{1}^{p} V_{1} \right) \otimes \left( \bigotimes_{p+1}^{p+q} V_{1} \right)
\]

be the tensor map, i.e., \( \beta(z,w) = z \otimes w \). Then set \( \nu = h\beta \).

4. Prove that \( \tau(1,F) = \tau(F,1) = F \) (see Theorem 1.2).

5. Verify (16).

6. Verify (17), (18), and (19).
7. Let $\mathcal{U}$ and $\mathcal{B}$ be associative algebras over $\mathbb{R}$ and assume that $X$ is a generating set for $\mathcal{U}$. Suppose that $f: X \to \mathcal{B}$ is a function. Then show that there is at most one algebra homomorphism $\varphi: \mathcal{U} \to \mathcal{B}$ such that $\varphi(x) = f(x)$ for all $x \in X$.

Hint: First observe that no such $\varphi$ need exist. For, if $X$ is a spanning set for $\mathcal{U}$ and $X$ is not l.i., then of course it is not possible to assign arbitrary values to a linear function on $X$. Now, a typical element $y$ of $\mathcal{U}$ can be expressed as a finite linear combination of finite products of elements of $X$:

$$y = \sum_{\omega \in \Gamma} \sum_{t=1}^{r} c_{w(t)},$$

$$\Gamma = \bigcup_{r=1}^{n} \Gamma^{r}.$$ Then if $\varphi$ is a homomorphism, we immediately have

$$\varphi(y) = \sum_{\omega \in \Gamma} \sum_{t=1}^{r} \varphi(x_{w(t)})$$

$$= \sum_{\omega \in \Gamma} \sum_{t=1}^{r} f(x_{w(t)}).$$

8. Show that if $(\mathcal{U},\nu)$ and $(\mathcal{U}',\nu')$ both have the universal tensor algebra property with respect to $V$, then there is a unique identity preserving algebra isomorphism $h: \mathcal{U} \to \mathcal{U}'$ such that $h(\nu(v)) = \nu'(v)$ for all $v \in V$.

Hint: By the universal tensor algebra property of $(\mathcal{U},\nu)$ there exists an algebra homomorphism $h: \mathcal{U} \to \mathcal{U}'$ such that $h(1_{\mathcal{U}}) = 1_{\mathcal{U}'}$ and $hv = \nu'$. Similarly, there exists $h': \mathcal{U} \to \mathcal{U}$ such
that $h'(l\nu') = l\nu$ and $h'\nu' = \nu$. Hence $hh'\nu' = \nu'$ and $hh'(l\nu') = l\nu'$. Since $hh'$ is the identity on $\text{Im} \nu'$ and $l\nu'$, it follows that $hh' = l\nu'$, and similarly $h'h = l\nu$.

9. Show that if $R_x[\xi_1, \ldots, \xi_n]$ is the set of all polynomials in (39) in which each of the terms has degree $r$, then the polynomials

$$
\prod_{t=1}^{n} m_t(\omega), \quad \omega \in G_{r,n'}
$$

form a basis of $R_x[\xi_1, \ldots, \xi_n]$.

Hint: Any monomial

$$
\xi_1^{a_1} \xi_2^{a_2} \cdots \xi_n^{a_n},
$$

$a_1 + a_2 + \cdots + a_n = r$, can be so expressed by choosing $\omega$ to be the sequence in $G_{r,n}$ for which $m_t(\omega) = a_t$, $t = 1, \ldots, n$.

10. Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras over $R$ and let 

$\psi: \mathcal{A} \to \mathcal{B}$ be a linear transformation. Show that if $X$ is a basis of $\mathcal{A}$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x,y \in X$, then $\phi$ is an algebra homomorphism.

Hint: Suppose that $a = \Sigma a_x x$ and $b = \Sigma b_x x$ are finite linear combinations of elements of $X$. Then
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\[ q(ab) = q\left( \sum_c c_x x \sum_d d_x x \right) \]
\[ = q\left( \sum_{x,y} c_x d_y x y \right) \]
\[ = \sum_{x,y} c_x d_y q(x y) \]
\[ = \sum_{x,y} c_x d_y q(x) q(y) \]
\[ = \sum_{x} c_x q(x) \sum_{y} d_y q(y) \]
\[ = q(a) q(b). \]

11. Show that if \( \mathcal{U} = \sum_{g \in G} V(g) \) is a \( G \)-graded algebra, then \( l_{\mathcal{U}} \)
    is homogeneous of degree \( 0 \), where \( 0 \) is the identity in \( G \).

Hint: Let \( l_{\mathcal{U}} = \sum_{g \in G} e_g \). If \( a \in \mathcal{U} \), then

\[ a = a l_{\mathcal{U}} \]
\[ = a \sum_{g \in G} e_g \]
\[ = \sum_{g \in G} ae_g. \]

Now suppose \( \deg a = k \). Then \( ae_g \in V(k+g) \) so that \( ae_g = 0 \)
    unless \( g = 0 \). Thus \( ae_0 = a \). Since \( ae_0 = a \) is true for all
    homogeneous elements \( a \), it is true of all elements of \( \mathcal{U} \), and
    hence \( l_{\mathcal{U}} = l_{\mathcal{U}} e_0 = e_0 \), i.e., \( l_{\mathcal{U}} \) is homogeneous of degree \( 0 \).
12. Let $\mathcal{U}$ be a $G$-graded algebra and suppose that $a \in \mathcal{U}$ and $a^{-1}$ exists. Prove that if $a$ is homogeneous, then $a^{-1}$ is homogeneous.

**Hint:** Let $a^{-1} = \Sigma b_g$. Then $l_{\mathcal{U}} = aa^{-1} = \Sigma ab_g$ and $\deg(ab_g) = g + k$, where $k = \deg a$. Since $\deg l_{\mathcal{U}} = 0$ it follows that $ab_g = 0$ unless $g = -k$ and $ab_{-k} = l_{\mathcal{U}}$. But $0 = a^{-1}(ab_g) = b_g$, and hence $a^{-1} = b_{-k}$. Thus $a^{-1}$ is homogeneous of degree $-k$.

13. Prove that if $\mathcal{U} = \Sigma^G V(g)$ is a $G$-graded algebra over $R$, then the elements $rl_{\mathcal{U}} \in V(0)$, i.e., "scalars," are homogeneous of degree 0.

**Hint:** The result is clearly true for $r = 0$. Let $0 \neq a \in \mathcal{U}$ be homogeneous of degree $g$. Then if $r \neq 0$, $(rl_{\mathcal{U}})a = l_{\mathcal{U}}(ra) = ra \in V(g)$. Hence $\deg((rl_{\mathcal{U}})a) = g$. On the other hand,

$$g = \deg((rl_{\mathcal{U}})a) = \deg(rl_{\mathcal{U}}) + \deg(a) = \deg(rl_{\mathcal{U}}) + g.$$

It follows that $\deg(rl_{\mathcal{U}}) = 0$.

14. Let $w \in \mathbb{Q}_{p,n}$. Show that the sign of the permutation

$$\begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & n \\ w(1) & \ldots & w(p) & w'(1) & \ldots & w'(n-p) \end{pmatrix}$$

is $(-1)^{s(w) + p(p+1)/2}$, where $s(w) = w(1) + \cdots + w(p)$ [see Example 1.5 (d)].
15. Show that it is not necessarily the case that $u \wedge u = 0$ for all $u \in \wedge V$, which involve no terms of degree zero.

Hint: For, let $\{e_1, \ldots, e_n\}$ be a basis of $V$, $n \geq 3$, and let $u = e_1 + e_2 \wedge e_3$. Then

$$u \wedge u = 2e_1 \wedge e_2 \wedge e_3.$$ 

16. Show that if $u \in \wedge V$, $\dim V = n$, and $u$ has no terms of degree zero, then

$$u \wedge \cdots \wedge u = 0.$$ 

Hint: Let $u = \sum_{r=1}^{n} u_r$, where $u_r$ is homogeneous of degree $r$, $r = 1, \ldots, n$. Then $u^{n+1}$ is a sum of Grassmann products, each of length $n+1$. Since $\dim V = n$, each one of these products is zero.

17. Let $\mathcal{U}$ be a vector space over $R$ and $f \in L(\mathcal{U} \otimes \mathcal{U}, \mathcal{U})$. Define a multiplication in $\mathcal{U}$ by $xy = f(x \otimes y)$. Show that $\mathcal{U}$ is an algebra with respect to this multiplication.

Hint: Since $\mathcal{U}$ is a vector space, only the distributive laws and the formula $(r_1 x)(r_2 y) = r_1 r_2 (xy)$ need be verified. But,

$$(r_1 x)(r_2 y) = f(r_1 x \otimes r_2 y) = f(r_1 r_2 (x \otimes y))$$

$$= r_1 r_2 f(x \otimes y) = r_1 r_2 (xy).$$

Also,

$$x(y_1 + y_2) = f(x \otimes (y_1 + y_2)) = f(x \otimes y_1 + x \otimes y_1)$$

$$= f(x \otimes y_1) + f(x \otimes y_2) = xy_1 + xy_2.$$
18. Show that if $\mathcal{U}$ is an algebra over $\mathbb{R}$, then there exists a linear map $f \in L(\mathcal{U} \otimes \mathcal{U}, \mathcal{U})$ such that $f(x \otimes y) = xy$ for all $x, y \in \mathcal{U}$.

Hint: This is obvious from the universal factorization property since the product $xy$ is bilinear.

19. Let $\mathcal{U}$ and $\mathcal{B}$ be algebras over $\mathbb{R}$. Then show that there exists a unique multiplication defined on $\mathcal{U} \otimes \mathcal{B}$ that satisfies

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2.$$ 

Hint: Let $\nu: (\mathcal{U} \otimes \mathcal{B}) \otimes (\mathcal{U} \otimes \mathcal{B}) \to (\mathcal{U} \otimes \mathcal{U}) \otimes (\mathcal{B} \otimes \mathcal{B})$ be the canonical isomorphism which satisfies

$$\nu((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = (a_1 \otimes a_2) \otimes (b_1 \otimes b_2).$$

Let $f_\mathcal{U}: \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ and $f_\mathcal{B}: \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ be linear maps for which $f_\mathcal{U}(a_1 \otimes a_2) = a_1a_2$ and $f_\mathcal{B}(b_1 \otimes b_2) = b_1b_2$. Define the multiplication in $\mathcal{U} \otimes \mathcal{B}$ by $uv = f_\mathcal{U} \otimes f_\mathcal{B} \nu(u \otimes v)$. The multiplication is unique because the decomposable elements span $\mathcal{U} \otimes \mathcal{B}$.

20. Show that if $\mathcal{U}$ and $\mathcal{B}$ in Exercise 19 are associative algebras, then so is $\mathcal{U} \otimes \mathcal{B}$.

21. Let $\mathcal{U}$, $\mathcal{B}$, and $\mathcal{C}$ be algebras over $\mathbb{R}$ and $\varphi: \mathcal{U} \times \mathcal{B} \to \mathcal{C}$ be a bilinear map satisfying $\varphi(a_1a_2, b_1b_2) = \varphi(a_1b_1)\varphi(a_2b_2)$. Then show there exists a unique algebra homomorphism $h: \mathcal{U} \otimes \mathcal{B} \to \mathcal{C}$ such that $\varphi(x, y) = h(x \otimes y)$.

Hint: Obtain $h$ by universal factorization. To show $h$ is an algebra homomorphism, it suffices to prove that $h$ is
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multiplicative on products of decomposable elements. But

\[ h((a_1 \otimes b_1)(a_2 \otimes b_2)) = h(a_1 a_2 \otimes b_1 b_2) \]
\[ = \varphi(a_1 a_2, b_1 b_2) \]
\[ = \varphi(a_1, b_1)\varphi(a_2, b_2) \]
\[ = h(a_1 \otimes b_1)h(a_2 \otimes b_2). \]

22. Let \( U \) be a \( G \)-graded vector space over \( \mathbb{C} \), \( U = \bigoplus_{g \in G} V(g) \), and suppose that each \( V(g) \) is a unitary space with inner product \( \langle \cdot, \cdot \rangle_g \). Show that \( U \) is a unitary space in which the inner product is given by \( \langle f_1, f_2 \rangle = \sum_{g \in G} \langle f_1(g), f_2(g) \rangle_g \).

Hint: The conjugate bilinearity is obvious, and

\[ \langle f, f \rangle = \sum_{g \in G} \langle f(g), f(g) \rangle_g = \sum_{g \in G} \|f(g)\|_g^2. \]

Hence \( \langle f, f \rangle \geq 0 \) with equality, iff \( f(g) = 0 \) for all \( g \in G \), i.e., iff \( f = 0 \).

23. Let \( e_1 \) be a unit vector in the unitary space \( V \). Define \( T: V^* \to V^* \) by \( Tz = e_1 \cdot z \), i.e., multiplication by \( e_1 \) in the symmetric algebra. Let \( h: V^{(r)} \to V^{(r+1)} \) be the restriction \( h = T|_{V^{(r)}} \). Compute the singular values of \( h \).

(Recall that the singular values of \( h \) are the non-negative square roots of the eigenvalues of \( h^*h \), where \( h^* \) is the conjugate dual of \( h \) with respect to the inner product in \( V^* \), i.e., the inner product in the \( Z \)-graded algebra \( V^* = \bigoplus_{r=0}^{\infty} V^{(r)} \) described in Exercise 22.)
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Hint: It is important in understanding this problem to keep in mind the definition of the inner product \((z,w)\) in \(V'\): If \(z = x' = x_1 \cdots x_r\) and \(w = y' = y_1 \cdots y_r\) are in \(V^{(r)}\) and are decomposable, then \((z,w) = (x',y') = (x',y')_r = \frac{1}{r!} \text{per}(x_i y_j)\) (see Theorem 4.4, Chapter 2). Thus
\[
(h^* h z, w) = (h^* h z, w) = (h z, h w) = (h z, h w)_{r+1}
\]
so that in particular
\[
(h^* h x', y')_r = (h x', h y')_{r+1}
\]
\[
= (e_1 \cdot x_1 \cdots x_r, e_1 \cdot y_1 \cdots y_r)
\]
\[
= \frac{1}{(r+1)!} \text{per}
\[
\begin{bmatrix}
(e_1, e_1) & (e_1, y_1) & \cdots & (e_1, y_r) \\
(x_1, e_1) & (x_1, y_1) & \cdots & (x_1, y_r) \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & \vdots \\
(x_r, e_1) & (x_r, y_1) & \cdots & (x_r, y_r)
\end{bmatrix}
\]

Let \(H = h^* h\) and recall [see formula (38) in Section 4, Chapter 2] that
\[
\sqrt{\frac{r!}{\nu(w)}} e_{\omega', w} \quad w \in G_{r,n},
\]
is an o.n. basis of \(V^{(r)}\), where \(\{e_1, e_2, \ldots, e_n\}\) is a completion of \(e_1\) to an o.n. basis of \(V\). Then for \(w\) and \(\tau\) in \(G_{r,n}\)
\[
(He_{\omega'}, e_\tau')_r = (h e_{\omega'}, h e_\tau')_{r+1}
\]
\[
= (e_1 \cdot e_{\omega'}, e_1 \cdot e_\tau')_{r+1}
\]
\[
= (e_{1:}\omega', e_{1:\tau'})_{r+1}.
\]
where \( 1:w = (1, w(1), \ldots, w(r)) \in \mathcal{G}_{r+1,n} \) and similarly for \( 1:\tau \) (obviously \( w = \tau \), iff \( 1:w = 1:\tau \)). Then

\[
\langle He^*_w, e^*_\tau \rangle_\mathcal{I} = \left( \sqrt{\frac{(r+1)!}{v(1:w)}} e^*_1 w, \sqrt{\frac{(r+1)!}{v(1:\tau)}} e^*_1 \tau \right) \frac{\sqrt{v(1:w)v(1:\tau)}}{(r+1)!} \\
\quad = \delta_{1:w, 1:\tau} \frac{\sqrt{v(1:w)v(1:\tau)}}{(r+1)!} \\
\quad = \delta_{w, \tau} \frac{\sqrt{v(1:w)v(1:\tau)}}{(r+1)!}.
\]

Hence

\[
(H \sqrt{\frac{\tau}{v(\omega)}} e^*_w, \sqrt{\frac{\tau}{v(\tau)}} e^*_\tau) = \delta_{w, \tau} \frac{\tau}{(r+1)!} \frac{\sqrt{v(1:w)v(1:\tau)}}{v(w)v(\tau)}.
\]

It follows that the e.v.'s of \( H \) are the numbers

\[
\frac{1}{r+1} \frac{v(1:w)}{v(w)} = \frac{1}{r+1} \frac{m_1(1:w)!}{m_1(w)!} = \frac{1}{r+1} \frac{(m_1(w)+1)!}{m_1(w)!} = \frac{1}{r+1} (m_1(w)+1), \quad w \in \mathcal{G}_{r,n}.
\]

Thus the distinct singular values of \( h \) are the numbers

\[
\sqrt{\frac{k}{r+1}}, \quad k = 1, \ldots, r+1.
\]

Also observe that minimum e.v. of \( H \) is \( \frac{1}{r+1} \) and the set of corresponding o.n. eigenvectors is

\[
\left\{ \sqrt{\frac{\tau}{v(\omega)}} e^*_w, \; w \in \mathcal{G}_{r,n}, \; m_1(w) = 0 \right\}.
\]

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Similarly, the maximum e.v. of $H$ is $1$ and the corresponding eigenvector is $e_0^*, w_0 = (1, ..., 1)$.

24. Let $e_1, ..., e_n$ be a basis of $V$ and suppose that $v_1, ..., v_m$ are nonzero vectors in $V$. If

$$v_1 \cdots v_m \in \langle e_1^*, w \in G_{m,n}, m_1(w) = 0 \rangle,$$

then $v_i \in \langle e_2, ..., e_n \rangle$, $i = 1, ..., m$.

Hint: Suppose on the contrary that some of the $v_i$ have nonzero components on $e_1^*$, which by the symmetry of $v^*$ we can assume to be $v_1, ..., v_r$. Now $v_1 \cdots v_m \neq 0$ and thus $v_{r+1} \cdots v_m \neq 0$. Using the dual product space model of $V \otimes V$, it follows that there exist $f_{r+1}, ..., f_m$ in $V^*$ such that $v_{r+1} \cdots v_m f_{r+1} \cdots f_m \neq 0$ (i.e., products of linear functionals span $V \otimes V$, see Exercise 6, Section 1.3) and hence $0 \neq v_{r+1} \cdots v_m (f_{r+1} \cdots f_m) = \frac{1}{(m-r)!} \text{per} [f_j(v_i)],$

$$i, j = r+1, ..., m.$$

Now choose $f_1 \in V^*$ such that $f_1(e_1) = 1$ and $f_1(e_j) = 0$, $j = 2, ..., n$, and let $f_1 = \cdots = f_r$. Then

$$e_j^*(f_1 \cdots f_m) = \frac{1}{m! \text{ per}} \begin{bmatrix} f_1(e_j) & \cdots \\ f_1(e_1) & \cdots \\ \vdots \\ f_1(e_m) & \cdots \end{bmatrix},$$

and if $m_1(w) = 0$ the first column of this matrix is $0$ so
that \( e^*_w(f_1 f_2 \cdots f_m) = 0 \). Hence \( v_1 v_2 \cdots v_m f_1 f_2 \cdots f_m = 0 \).

However,

\[
\begin{align*}
& v_1 \cdots v_m (f_1 \cdots f_m) \\
& \quad = \frac{1}{m!} \cdot \text{per} \left[ \begin{array}{cccc}
 f_1(v_1) & f_1(v_2) & \cdots & f_1(v_r) \\
 f_1(v_2) & f_1(v_2) & \cdots & f_1(v_2) \\
 \vdots & \vdots & \ddots & \vdots \\
 f_1(v_r) & f_1(v_r) & \cdots & f_1(v_r) \\
 0 & \cdots & \cdots & 0 \\
 \end{array} \right] \\
& \quad = \frac{1}{m!} r! \prod_{t=1}^{r} f_1(v_t) \text{ per}(f_j(v_1)) \neq 0 \quad (i,j = r+1, \ldots, m).
\end{align*}
\]

25. Let \( X = [x_i, x_j], \ i,j = 1, \ldots, r+1, \) be a Gram matrix based on the nonzero vectors \( x_1, \ldots, x_{r+1} \). Prove that

\[
\|x_1\|^2 \text{ per } X(1|1) \leq \text{ per } X \leq (r+1)\|x_1\|^2 \text{ per } X(1|1).
\]

Show that the lower equality can hold, iff \( (x_j, x_1) = 0, \ j = 2, \ldots, r+1 \) and the upper equality can hold, iff

\[
x_j = c_j \frac{x_j}{\|x_1\|}, \quad j = 2, \ldots, r+1, \quad \prod_{j=2}^{r+1} c_j = 1.
\]

Recall that \( X(1|1) \) is the submatrix obtained from \( X \) by deleting row and column 1.
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Hint: Consider the transformation $h$ of Exercise 23 in which

$$e_1 = \frac{x_1}{\|x_1\|}.$$ Then

$$\|hx_2 \otimes \cdots \otimes x_{r+1}\|_{r+1}^2 = (hx_2 \otimes \cdots \otimes x_{r+1}, hx_2 \otimes \cdots \otimes x_{r+1})_{r+1}$$

$$= (e_1 \otimes x_2 \otimes \cdots \otimes x_{r+1}, e_1 \otimes x_2 \otimes \cdots \otimes x_{r+1})_{r+1}$$

$$= \frac{1}{\|x_1\|^2} (x_1 \otimes x_2 \otimes \cdots \otimes x_{r+1}, x_1 \otimes x_2 \otimes \cdots \otimes x_{r+1})_{r+1}$$

$$= \frac{1}{\|x_1\|^2} \frac{1}{(r+1)!} \text{per} [(x_i, x_j)].$$

Also

$$\|hx_2 \otimes \cdots \otimes x_{r+1}\|_{r+1}^2 = (h^* hx_2 \otimes \cdots \otimes x_{r+1}, x_2 \otimes \cdots \otimes x_{r+1})_r$$

and from Exercise 23

$$\frac{1}{r+1} \|x_2 \otimes \cdots \otimes x_{r+1}\|_r^2 \leq (h^* hx_2 \otimes \cdots \otimes x_{r+1}, x_2 \otimes \cdots \otimes x_{r+1})_r \leq \|x_2 \otimes \cdots \otimes x_{r+1}\|_r^2$$

or

$$\frac{1}{r+1} \frac{1}{r!} \text{per} \left[ \begin{array}{ccc} (x_2, x_2) & \cdots & (x_2, x_{r+1}) \\ (x_{r+1}, x_2) & \cdots & (x_{r+1}, x_{r+1}) \end{array} \right] = \frac{1}{\|x_1\|^2} \frac{1}{(r+1)!} \text{per} [(x_i, x_j)]$$

$$\leq \frac{1}{r!} \text{per} \left[ \begin{array}{ccc} (x_2, x_2) & \cdots & (x_2, x_{r+1}) \\ (x_{r+1}, x_2) & \cdots & (x_{r+1}, x_{r+1}) \end{array} \right]$$

which simplifies to the required inequality. From Exercise 23
equality holds in the lower inequality iff \( x_2 \cdots x_{r+1} \in \langle e'_n \mid w, \in \mathcal{G}_{r,n}, m_w(w) = 0 \rangle \). From Exercise 24 this can happen iff \( \langle x_j, x_1 \rangle = 0, \ j = 2, \ldots, r+1 \). Similarly from Exercise 23 equality holds in the upper inequality iff \( x_2 \cdots x_{r+1} = c e_1 \cdots e_1 \). But this latter equality holds iff \( x_j = c_j \frac{x_1}{\|x_1\|^r}, j = 2, \ldots, r+1, \) and \( \prod_{j=2}^{r+1} c_j = c \) [see Theorem 4.5(c), Chapter 2].

26. Let \( A \) be an \( r+1 \)-square p.s.d. matrix. Prove that

\[
a_{11} \per A(1 \| 1) \leq \per A \leq (r+1)a_{11} \per A(1 \| 1).
\]

If some main diagonal entry of \( A \) is 0, then both inequalities are equality. If no main diagonal entry is 0, then the lower inequality is equality iff \( A = a_{11} + A(1 \| 1) \); the upper inequality is equality iff \( A \) has rank 1.

Hint: This is an immediate consequence of Exercise 25 since any p.s.d. \( A \) is a Gram matrix.

27. State and prove an analog for the permanent function of the Hadamard determinant theorem.

Hint: Let \( A \) be a p.s.d. \( n \)-square matrix. Then

\[
\per(A) \geq \prod_{i=1}^{n} a_{ii}. \] Equality holds iff some \( a_{ii} = 0 \) or \( A \) is a diagonal matrix. This follows immediately from Exercise 26 and the fact that if \( A \) is p.s.d. and \( a_{ii} = 0 \), then \( A(i) = A^{(i)} = 0 \).

28. Let \( A \) be \( n \)-square p.s.d. with every row and column sum equal to 1, i.e., \( A \) is doubly stochastic (d.s.). Show that the p.s.d. square root of \( A \sqrt{A} \) is also doubly stochastic.
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Hint: It is easy to see that $A^{1/n}$ is a polynomial in $A$. For suppose $U^*AU = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq 0$. Let $p(\lambda) = \sum_{k=0}^{m} c_k \lambda^k$ be an interpolating polynomial, which satisfies $p(\lambda_i) = \lambda_i^{1/2}$, $i = 1, \ldots, n$. Then $p(A) = A^{1/2}$. Now, for $A$ to be d.s. is equivalent to the assertion that $J_n A_n = J_n A = J_n$, where $J_n$ is the matrix with every entry $1/n$. Then

$$J_n A_n^{1/2} = J_n p(A) = J_n \sum_{k=0}^{m} c_k J_n A_n^k = \left( \sum_{k=0}^{m} c_k \right) J_n = c J_n.$$

Similarly, $A^{1/2} J_n \neq c J_n$, and since $A$ is p.s.d., $c \geq 0$ (why?). But $J_n A_n^{1/2} = c J_n$ implies $J_n A = c J_n A_n^{1/2} = c^2 J_n$ so that $J_n = c^2 J_n$ and $c^2 = 1$. Thus $c = 1$ and $A^{1/2}$ is d.s.

29. Use Exercise 28 to prove the van der Waerden conjecture [see Example 4.6(d), Chapter 2].

Hint: Let $A^{1/2}$ be the p.s.d. d.s. square root of $A$. Then from the inequality (41) in Section 2.4,

$$|d(J_n)|^2 = |d(J_n A_n^{1/2})|^2 \leq d(J_n^2) d((A_n^{1/2})^2) = d(J_n) d(A).$$

Now

$$d(J_n) = \frac{1}{n} \sum_{\sigma \in S_n} \chi(\sigma),$$

so (in order to have a nontrivial result) we take $\chi \equiv 1$. Then $d(A) \leq \frac{1}{n} |H|$. If $H = S_n$, $\text{per}(A) \geq \frac{n!}{n^n}$. Now equality can hold iff $A_n^{1/2} (1) \cdots A_n^{1/2} (n) = J_n (1) \cdots J_n (n)$ [see Example 4.5(b), Chapter 2], which from Theorem 4.5(c) implies each $A_j$ is a multiple of $(1, \ldots, 1)$. But then $A = J_n$. 49
3.2 Derivations

In this section we study an important class of linear transformations on symmetry classes and graded algebras, the derivations. These maps and their generalizations are useful (among other places) in the study of Clifford algebras, in invariant theory, and in the theory of inequalities.

Definition 2.1 (Derivation) Let $\mathcal{U} = \sum_{n \in \mathbb{Z}} V(n)$ be a $\mathbb{Z}$-graded algebra over $\mathbb{R}$ [$V(n) = \{0\}$, $n < 0$, $V(0) = \mathbb{R}$]. The main involution in $\mathcal{U}$ is the linear map $J: \mathcal{U} \to \mathcal{U}$ defined for any $a_n \in V(n)$ by

$$J a_n = \begin{cases} a_n, & \text{if } n \text{ is even} \\ -a_n, & \text{if } n \text{ is odd.} \end{cases}$$

Let $D \in L(\mathcal{U}, \mathcal{U})$ be homogeneous of degree $p$ and assume that for every $a$ and $b$ in $\mathcal{U}$,

$$D(ab) = D(a)b + J^p(a)D(b).$$

Then $D$ is a derivation of $\mathcal{U}$ of degree $p$.

Example 2.1(a) Let $\mathcal{U}$ be the algebra of all real valued functions, which are infinitely differentiable everywhere on the real line. We can assume $\mathcal{U}$ is a $\mathbb{Z}$-graded algebra trivially by defining $V(0) = \mathbb{R}$ and $V(n) = \{0\}$ for all $n \neq 0$. Let $D$ be the differentiation operator. Then $D$ is homogeneous of degree $0$.

(b) Let $A \in L(V, V)$. By the usual argument (see Exercise 1) there exists a unique linear map $h_m: \bigotimes_{1}^{m} V \to \bigotimes_{1}^{m} V$ satisfying

$$h_m(x \otimes) = \sum_{i=1}^{m} x_1 \otimes \cdots \otimes x_{i-1} \otimes A x_i \otimes x_{i+1} \otimes \cdots \otimes x_m.$$
Now define $D(A) : T^0_0(V) \to T^0_0(V)$ by $D(A)|_{V^m_O} = h^m_m$, $m \geq 2$, $D(A)|_V = A$, $D(A)|_R = 0$. Then $D(A)$ is clearly a homogeneous linear map of degree 0 on the contravariant tensor algebra. Also if $a = x_1 \otimes \cdots \otimes x_m \in V^m_O$ and $b = x_{m+1} \otimes \cdots \otimes x_{m+n} \in V^n$, then

$$D(A)a \otimes b = \sum_{i=1}^{m+n} x_1 \otimes \cdots \otimes A x_i \otimes \cdots \otimes x_{m+n}$$

$$= \left( \sum_{i=1}^{m} x_1 \otimes \cdots \otimes A x_i \otimes \cdots \otimes x_{m} \right) \otimes b$$

$$+ a \otimes \left( \sum_{i=m+1}^{m+n} x_{m+1} \otimes \cdots \otimes A x_i \otimes \cdots \otimes x_{m+n} \right)$$

$$= (D(A)a) \otimes b + a \otimes (D(A)b).$$

Thus $D(A)$ is a derivation of degree 0. Observe that

$$D(A)|_{V^m_O} = A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I$$

$$+ \cdots + I \otimes \cdots \otimes I \otimes A.$$ 

The derivation $D(A)$ is called the derivation induced by $A$ on $T^0_0(V)$. 

(c) Let $W = L(V, V)$ be regarded as a $Z$-graded algebra with $V(n) = \{0\}$, $n \neq 0$, $V(0) = L(V, V)$. Let $A \in L(V, V)$ and define the "Ad" operator by $Ad A(X) = AX -XA$. Then

$$Ad A(X)\cdot Y + X\cdot Ad A(Y) = (AX -XA)Y + X(AY -YA)$$

$$= AXY - XYA$$

$$= Ad A(XY).$$
Hence \( \text{Ad}A \) is a derivation (of degree 0).

The following result shows that any l.t. on \( V \) can be extended to a derivation of \( T_{0}(V) \).

**Theorem 2.1** If \( D: V \rightarrow T_{0}(V) \) is linear and \( \text{Im} \, D \subset V_{0}^{p+1} \), \( p \geq -1 \), then \( D \) can be extended uniquely to a derivation of degree \( p \) on \( T_{0}(V) \).

**Proof:** Assume first that \( p \) is odd, define \( h_{m}: V_{0}^{m} \rightarrow V_{0}^{m+p} \) by

\[
h_{m}(x^{\otimes}) = \sum_{k=1}^{m} (-1)^{k-1} x_{1} \otimes \cdots \otimes x_{k-1} \otimes Dx_{k} \otimes x_{k+1} \otimes \cdots \otimes x_{m}, \quad m \geq 1, \quad (2)
\]

\( h_{0} = 0 \), and let \( h: T_{0}(V) \rightarrow T_{0}(V) \) satisfy \( h|V_{0}^{m} = h_{m} \). Obviously \( h \) is homogeneous of degree \( p \), \( h|V = D \), and thus we need only prove that for arbitrary \( u \) and \( v \) in \( T_{0}(V) \),

\[
h(u \otimes v) = h(u) \otimes v + (J^{p}(u)) \otimes h(v). \quad (3)
\]

Moreover, (3) need only be verified for homogeneous elements. If either \( \deg u = 0 \) or \( \deg v = 0 \), then (3) is obvious (see Exercise 2). Next note that if \( \deg a_{m} = m \), then in general

\[
J^{p}a_{m} = (-1)^{mp}a_{m} \quad (4)
\]

so that

\[
J^{p}x^{\otimes} = \begin{cases} 
 x^{\otimes} & \text{if } p \text{ is even} \\
 (-1)^{mp}x^{\otimes} & \text{if } p \text{ is odd}
\end{cases} \quad (5)
\]

(see Exercise 3). Thus if \( u = x_{1} \otimes \cdots \otimes x_{m} \) and \( v = x_{m+1} \otimes \cdots \otimes x_{m+q} \), we compute (using the fact that \( p \) is odd) that
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\[ h(u) \otimes v + (R^u) \otimes h(v) \]

\[ = \sum_{k=1}^{m} (-1)^{k-1} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_{m+q} \]

\[ + \sum_{k=1}^{q} (-1)^{k-1} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_{m+k} \otimes \cdots \otimes x_{m+q} \]

\[ = \sum_{k=1}^{m+q} (-1)^{k-1} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_m \]

\[ = h(x_1 \otimes \cdots \otimes x_{m+q}) = h(u \otimes v). \]

If \( p \) is even, then the definition of \( h_m \) is changed to

\[ h_m(x^\otimes) = \sum_{k=1}^{m} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_{m-1} \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_m, \]

\( h_0 = 0 \) and the proof proceeds as above (see Exercise 4). The uniqueness of \( h \) follows from the fact that a derivation is completely determined by its values on the homogeneous elements of degree 1 (see Exercise 5).

**Example 2.2** Let \( g \in V^* \) and let \( g \) denote the extension of \( g \) to a derivation on \( \tau_0(V) \) of degree \(-1\). Then from (2), if \( \deg x^\otimes = m \),

\[ g(x^\otimes) = \sum_{k=1}^{m} (-1)^{k-1} g(x_k) x_1 \otimes \cdots \otimes x_{k-1} \otimes x_{k+1} \otimes \cdots \otimes x_m. \]  

(6)

We shall be considering some important properties that these "\( g \) derivations" have. In order to abbreviate writing the values
\[ x_k^\otimes = x_k \otimes \cdots \otimes x_{k-1} \otimes x_{k+1} \otimes \cdots \otimes x_m \]

(or in a general symmetry class \[ x_k^\ast = x_k \ast \cdots \ast x_{k-1} \ast x_{k+1} \ast \cdots \ast x_m \])

so that (6) becomes

\[ \delta^*_g(x^\otimes) = \sum_{k=1}^{\ast m} (-1)^{k-1} g(x_k) x_k^\otimes. \]

Our next result exhibits certain algebraic combinations of two derivations as a derivation.

**Theorem 2.2** Let \( D_1 \) and \( D_2 \) be derivations of \( \mathcal{U} \) of degrees \( p_1 \) and \( p_2 \), respectively. Then

\[ D = D_1 D_2 - (-1)^{p_1 p_2} D_2 D_1 \quad (7) \]

is a derivation of degree \( p_1 + p_2 \).

**Proof:** First note that if \( \deg a = n \), then

\[ J^{p_1} D_2(a) - (-1)^{p_1 p_2} D_2 J^{p_1}(a) = 0, \quad (8) \]

and

\[ D_1 J^{p_2}(a) - (-1)^{p_1 p_2} J^{p_2} D_1(a) = 0. \quad (9) \]

To verify (8), for example, we use (1) as follows: \( \deg D_2(a) = n + p_2 \) and hence \( J^{p_1} D_2(a) = (-1)^{(n+p_2)p_1} D_2(a) \), \( J^{p_1}(a) = (-1)^{np_1} a \), so that \( D_2 J^{p_1}(a) = (-1)^{np_1} D_2 a \). Thus the left side of (8) becomes

\[ ((-1)^{(n+p_2)p_1} - (-1)^{p_1 p_2} (-1)^{np_1}) D_2(a) = 0, \]

similarly for (9). Now for arbitrary \( a \) and \( b \) in \( \mathcal{U} \) we compute
that
\[ D(ab) = D_1(D_2(ab)) - (-1)^{P_1 P_2} D_2(D_1(ab)) \]
\[ = D_1(D_2(a) b + j^{P_2} D_2(b)) - (-1)^{P_1 P_2} D_2[D_1(a) b + j^{P_2} D_1(b)] \]
\[ = (D_1 D_2(a)) b + (j^{P_2} D_2(a)) D_1(b) \]
\[ + ((D_1 j^{P_2} D_2(a)) b + j^{P_2} (j^{P_2} D_2(a)) D_1(b)) \]
\[ - (-1)^{P_1 P_2} ((D_1 j^{P_2} D_2(a)) b + (j^{P_2} D_2(a)) D_1(b)) \]
\[ - (-1)^{P_1 P_2} ((D_2 j^{P_1} D_1(a)) b + j^{P_1} (j^{P_1} D_1(a)) D_2(b)) \].

Now regroup as follows: Put together the first terms in the first and third curly brackets; and put together the second terms in the second and fourth curly brackets. We then have
\[ D(ab) = [D_1 D_2(a) - (-1)^{P_1 P_2} D_2 D_1(a)] b \]
\[ + j^{P_1 + P_2} (D_1 D_2(b) - (-1)^{P_1 P_2} D_2 D_1(b)) \]
\[ + [j^{P_1} D_2(a) - (-1)^{P_1 P_2} D_2 D_1(a)] D_1(b) \]
\[ + [D_1 j^{P_2} (a) - (-1)^{P_1 P_2} j^{P_2} D_2 D_1(a)] D_2(b). \]

From (8) and (9) the third and fourth square bracketed quantities in (10) are 0 and the first two are just the expression for
\[ D(a) \cdot b + j^{P_1 + P_2} (a) D(b). \]

**Example 2.3(a)** If \( D \) is an odd degree derivation of \( \mathcal{U} \), then \( D^2 \) is a derivation. For, simply choose \( D_1 = D_2 = D, \ P_1 = P_2 = P \), and the right side of (7) becomes
\[ D^2 - (-1)^{P_2} D^2 = 2D^2. \] Thus \( 2D^2 \) and hence \( D^2 \) is a derivation.
(b) If \( \deg D_1 = p_1 \), \( \deg D_2 = p_2 \) and \( p_1 p_2 \) is even, then
(7) implies that \( D_1 D_2 - D_2 D_1 \) is a derivation.

(c) For any \( g \in V^* \), \( \delta_g^2 = 0 \). For, \( \deg \delta_g = -1 \) and by (a)
\( \delta_g^2 \) is a derivation. If \( x \in V \) then \( \delta_g^2(x) = \delta_g \delta_g(x) = \delta_g(g(x)) = 0 \).

(d) In general, if \( g, h \in V^* \), then \( \delta_h \delta_g + \delta_g \delta_h = 0 \). To see
this first note that \( \delta_{g+h} = \delta_g + \delta_h \) and hence
\[
0 = \delta_{g+h}^2 = (\delta_g + \delta_h)^2 = \delta_g^2 + \delta_h^2 + \delta_g \delta_h + \delta_h \delta_g = \delta_g \delta_h + \delta_h \delta_g.
\]

(e) Let \( A \in \mathfrak{L}(V, V) \) and for \( s \in \mathbb{R} \) consider \( \otimes^m (I + sA) = \)
\[
\otimes I + s(A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A) + \]
higher degree terms in \( s \). Thus the linear term in \( s \) in \( \otimes^m (I + sA) \)
is the restriction of the derivation induced by \( A \) on \( T_0(V) \) to the homogeneous subspace \( \otimes^1 V \) [see Example 2.1(b)].

The \( \delta_g \) derivations operate on the Grassmann algebra \( \wedge V \) in
an interesting way. To be precise we have

**Theorem 2.3** If \( v_1, \ldots, v_m \) are in \( V \), then
\[
\delta_g(v^\wedge) = \sum_{k=1}^m (-1)^{k-1} g(v_k)v_k^{\wedge}.
\]

It follows that \( \delta_g \) maps nonzero decomposable elements in \( \otimes^m V \) into
nonzero decomposable elements in \( \wedge^m V \).

**Proof:** We use the dual product representation of \( \otimes^1 V \) (see
Exercise 6, Section 1.3). For notational purposes write \( g = h_m \)
and let \( h_1, \ldots, h_{m-1} \) be arbitrary elements of \( V^* \). Then using
formula (45), Section 2.4, we evaluate the right side of (11) at
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\[ h_1 \cdots h_{m-1} \text{ to obtain } \]
\[
\frac{1}{(m-1)!} \sum_{k=1}^{m} (-1)^{k-1} h_m(v_k) \det[h_j(v_i)](k|m), \tag{12}
\]

which by the Laplace expansion theorem is
\[
\frac{(-1)^{m+1}}{(m-1)!} \det[h_j(v_i)] = (-1)^{m+1} m^v (h_1 \cdots h_m). \tag{13}
\]

If we evaluate the left side of (11) at \( h_1 \cdots h_{m-1} \), we obtain
\[
\frac{1}{m!} \sum_{k=1}^{m} (-1)^{k-1} \sum_{\sigma \in S_m} e(\sigma) h_1(v_{\sigma(1)}) \cdots h_{k-1}(v_{\sigma(k-1)}) h_k(v_{\sigma(k)}) \cdots h_{m-1}(v_{\sigma(m)}). \tag{14}
\]

The inside summation in (14) is just
\[
\begin{vmatrix}
h_1(v_1) & \cdots & h_{k-1}(v_1) & h_m(v_1) & h_k(v_1) & \cdots & h_{m-1}(v_1) \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
h_1(v_m) & \cdots & h_{k-1}(v_m) & h_m(v_m) & h_k(v_m) & \cdots & h_{m-1}(v_m)
\end{vmatrix}
\]

By putting the \( k^{th} \) column in the above matrix into the \( m^{th} \) column position the determinant of the above matrix is equal to
\[
(-1)^{m-1-k+1} \det[h_j(v_i)]. \tag{15}
\]

Thus replacing the inside summation in (14) with (15) we obtain
\[
\frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-1} \det[h_j(v_i)] = (-1)^{m-1} m^v (h_1 \cdots h_m),
\]
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precisely the right side of (13).

To verify the second assertion in the theorem suppose first that $v^A = 0$, i.e., $v_1, \ldots, v_m$ are l.d. [Theorem 4.5(b), Chapter 2]. Then $\delta_g(v^A) = 0$, which is trivially decomposable. Similarly if $g$ vanishes on $\langle v_1, \ldots, v_m \rangle$, the right side of (11) is 0. So assume $v_1, \ldots, v_m$ are l.i. and $g$ is not zero on every $v_1$. Then select (see Exercise 6) a basis $u_1, \ldots, u_m$ of $\langle v_1, \ldots, v_m \rangle$ such that $g(u_m) \neq 0$, $g(u_1) = \cdots = g(u_{m-1}) = 0$ and $u^A = v^A$. Then from (11)

$$\delta_g(v^A) = \delta_g(u^A)$$

$$= \sum_{k=1}^{m} (-1)^{k-1} g(u_k)u_k^A$$

$$= (-1)^{m-1} g(u_m)u_m^A,$$

a decomposable element in $\bigwedge^m V$.

A number of important and interesting properties of the $\delta_g$ derivations will be found in the Exercises.

In Example 2.3(e) we saw that the coefficient of the linear term in $s$ in the expansion of $\otimes (I+sA)$ is the derivation induced by $A$ on $T_0(V)$ restricted to the homogeneous subspace $\otimes V$. We turn our attention now to a generalization of this idea to arbitrary symmetry classes. The following examples will help to clarify the idea.

**Example 2.4(a)** Let $D(A)$ be the derivation of degree 0 induced by $A$ on $T_0(V)$ [see Example 2.1(b)]. Let $I$ be the
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ideal in $T_0(V)$ that defines the exterior algebra, i.e.,

$T_0(V)/I = \wedge V$ [see Example 1.4 (a)]. Observe first that $D(A)I \subseteq I$.

For, let $z = u \otimes x \otimes x \otimes v$ be a decomposable element in $I$. Then

from Example 2.1 (b) we know that $D(A)z$ is a sum of terms all of

which are automatically in $I$ except

$$u \otimes Ax \otimes x \otimes v + u \otimes x \otimes Ax \otimes v.$$

But

$$u \otimes (A+I)x \otimes (A+I)x \otimes v = u \otimes Ax \otimes Ax \otimes v + u \otimes Ax \otimes x \otimes v$$

$$+ u \otimes x \otimes Ax \otimes v + u \otimes x \otimes x \otimes v.$$

The left side of this equation is in $I$ as are the first and last

terms on the right. Thus

$$u \otimes Ax \otimes x \otimes v + u \otimes x \otimes Ax \otimes v \in I.$$

It follows that if $q: T_0(V) \to T_0(V)/I = \wedge V$ is the quotient map,

then the map $\overline{D}(A): \wedge V \to \wedge V$ given by $\overline{D}(A)(q(w)) = q(D(A)w)$, is

a well-defined linear transformation on the exterior algebra $\wedge V$

(see Exercise 7). Note that if $w = x_1 \otimes \cdots \otimes x_m$, then

$q(w) = x_1 \wedge \cdots \wedge x_m$ and

$$\overline{D}(A)x_1 \wedge \cdots \wedge x_m = \overline{D}(A)(q(w))$$

$$= q(D(A)w)$$

$$= q(D(A)x_1 \otimes \cdots \otimes x_m)$$

$$= q\left( \sum_{k=1}^{m} x_1 \otimes \cdots \otimes Ax_k \otimes \cdots \otimes x_m \right)$$

$$= \sum_{k=1}^{m} x_1 \wedge \cdots \wedge Ax_k \wedge \cdots \wedge x_m. \quad (16)$$

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Also for arbitrary $a$ and $b$ in $T_0(V)$,

$$
\overline{D}(A)(q(a) \wedge q(b)) = \overline{D}(A)(q(a \otimes b))
= q(D(A)(a \otimes b))
= q(D(A)a \otimes b + a \otimes D(A)b)
= q(D(A)a \otimes b) + q(a \otimes D(A)b)
= q(D(A)a) \wedge q(b) + q(a) \wedge q(D(A)b)
= \overline{D}(A)q(a) \wedge q(b) + q(a) \wedge \overline{D}(A)q(b).
$$

(17)

In other words $\overline{D}(A) : \wedge V \to \wedge V$ is a derivation of the exterior algebra of degree 0 and $\overline{D}(A)x^\wedge$ is given by (16).

(b) Let $I$ be the ideal in $T_0(V)$ that defines the symmetric algebra [see Example 1.4(b)]. We again verify that $D(A)I \subseteq I$.

For any homogeneous element is a sum of elements of the form

$$
z = u \otimes (x \otimes y - y \otimes x) \otimes v,
$$

where $u$ and $v$ are decomposable. Then $D(A)z$ is a sum of terms all automatically in $I$ except

$$
u \otimes (Ax \otimes y - Ay \otimes x) \otimes v + u \otimes (x \otimes Ay - y \otimes Ax) \otimes v.
$$

However

$$
u \otimes [(A + I)x \otimes (A + I)y - (A + I)y \otimes (A + I)x] \otimes v
= u \otimes [Ax \otimes Ay - Ay \otimes Ax] \otimes v + u \otimes [x \otimes y - y \otimes x] \otimes v
+ u \otimes [Ax \otimes y - Ay \otimes x] \otimes v + u \otimes [x \otimes Ay - y \otimes Ax] \otimes v.
$$

(18)

The left side of (18) is in $I$ as are the first two terms on the right. Thus the sum of the last two terms is in $I$. Let

$q : T_0(V) \to T_0(V)/I = V^*$ be the quotient map and as in the preceding example define $\overline{D}(A)q(w) = q(D(A)w), \ w \in T_0(V)$, a well-defined map
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on $V^*$. If $w = x_1 \otimes \cdots \otimes x_m$, then $q(w) = x_1 \cdots x_m$ and

$$\overline{D}(A) x_1 \cdots x_m = \overline{D}(A)q(w)$$

$$= q(D(A)w)$$

$$= q(D(A)x_1 \otimes \cdots \otimes x_m)$$

$$= q\left( \sum_{k=1}^{m} x_1 \otimes \cdots \otimes Ax_k \otimes \cdots \otimes x_m \right)$$

$$= \sum_{k=1}^{m} x_1 \cdots Ax_k \cdots x_m. \tag{19}$$

Also, for arbitrary $a$ and $b$ in $\tau_0(V)$,

$$\overline{D}(A)(q(a) \cdot q(b)) = \overline{D}(A)(q(a \otimes b))$$

$$= q(D(A)(a \otimes b))$$

$$= q(D(A)a \otimes b + a \otimes D(A)b)$$

$$= q(D(A)a) \cdot q(b) + q(a) \cdot q(D(A)b)$$

$$= \overline{D}(A)q(a) \cdot q(b) + q(a) \cdot \overline{D}(A)q(b). \tag{20}$$

Hence $\overline{D}(A) : V^* \to V^*$ is a derivation of the symmetric algebra of degree 0 and $\overline{D}(A)x_1 \cdots x_m$ is given by (19).

(c) Let $P(\sigma)$ be a permutation operator on $\otimes V$ and let

$$\pi_k(A) = I \otimes \cdots \otimes A \otimes \cdots \otimes I$$

where $A$ appears in the $k^{th}$ position in the indicated tensor product of l.t.'s. We show that

$$P(\sigma) \pi_k(A) = \pi_{\sigma(k)}(A)P(\sigma). \tag{21}$$

Evaluate both sides of (21) on $v_1 \otimes \cdots \otimes v_m$, let $x_j = v_j$, $j \neq k$, $x_k = Av_k$, and we have on the left
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\[ P(\sigma) \pi_k(A) v^\otimes = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(j)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}. \] (22)

In position \( j \) of the tensor (22) \( x_{\sigma^{-1}(j)} \) appears, and since \( x_{\sigma^{-1}(j)} = v_{\sigma^{-1}(j)} \) unless \( \sigma^{-1}(j) = k \), we see that the \( j \)th vector in (22) is \( v_{\sigma^{-1}(j)} \) unless \( j = \sigma(k) \). The vector in position \( \sigma(k) \) is \( x_k = Av_k \). Evaluating the right side of (21) on \( v^\otimes \) we see that the vector in position \( j \) is \( v_{\sigma^{-1}(j)} \) unless \( j = \sigma(k) \), in which case it is \( Av_{\sigma^{-1}(\sigma(k))} = Av_k \). Thus the two sides of (21) agree on an arbitrary \( v^\otimes \) and hence are equal. Now \( D(A) = \sum_{k=1}^m \pi_k(A) \) so that from (21),

\[
P(\sigma)D(A) = \sum_{k=1}^m P(\sigma) \pi_k(A) = \sum_{k=1}^m \pi_{\sigma(k)}(A)P(\sigma).
\]

But as \( k \) runs through 1, \ldots, \( m \) so does \( \sigma(k) \) and hence

\[
P(\sigma)D(A) = D(A)P(\sigma).
\]

It follows immediately that the derivation \( D(A) \) commutes with any symmetrizer and hence any symmetry class is an invariant subspace of \( D(A) \). Moreover, if \( S \) is a symmetrizer defined by \( H \) and \( \chi \) and \( Sx^\otimes = x^\ast \), then

\[
D(A)x^\ast = D(A)Sx^\otimes
= SD(A)x^\otimes
= S \left( \sum_{k=1}^m x_1 \otimes \cdots \otimes Ax_k \otimes \cdots \otimes x_m \right)
= \sum_{k=1}^m x_1 \ast \cdots \ast Ax_k \ast \cdots \ast x_m.
\] (23)
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The formula (23) generalizes (15) and (19) to any symmetry class, and in fact \( \bar{D}(A) | V = D(A) | V \), \( \bar{D}(A) | V' = D(A) | V' \). In what follows we shall refine and extend the ideas contained in the preceding discussion.

Let \( \tau \) be a partition of \( m \) into \( p \) nonnegative integers

\[
\tau: r_1, \ldots, r_p,
\]

\[
\tau: r_1 + \cdots + r_p = m
\]

and for each \( i = 1, \ldots, p \), let \( \omega^i \in Q_{r_i}^m \). Assume also that \( \omega^1, \ldots, \omega^p \) are nonoverlapping sequences, i.e., \( \operatorname{Im} \omega^i \cap \operatorname{Im} \omega^j = \emptyset \), if \( i \neq j \). For a fixed choice of \( \omega^1, \ldots, \omega^p \) and fixed linear transformations \( T_i \in L(V, V) \), \( i = 1, \ldots, p \), define

\[
\pi_{\omega}(T_1, \ldots, T_p) = \bigotimes_{j=1}^{m} X_j,
\]

where \( X_{\omega^i(j)} = T_i \), \( j = 1, \ldots, r_i \), \( i = 1, \ldots, p \). In other words, \( \pi_{\omega}(T_1, \ldots, T_p) \) is a tensor product of \( \omega^i \)'s in which \( T_i \) appears in the positions corresponding to the sequence \( \omega^i \), \( i = 1, \ldots, p \).

Then define

\[
\delta(T_1, \ldots, T_p) = \sum_{\omega} \pi_{\omega}(T_1, \ldots, T_p), \quad (25)
\]

where the summation in (25) is over all nonoverlapping sequences \( \omega^1, \ldots, \omega^p \) (the partition \( \tau \) is fixed).

**Example 2.5 (a)** If \( p = 1 \) and \( r_1 = m \), then \( \omega^1 = (1, \ldots, m) \) and

\[
\delta(T_1) = \bigotimes_{j=1}^{m} T_1.
\]
the usual \( m^{th} \) tensor power of \( T_1 \).

(b) If \( p = 2, \ r_1 = 1, \ r_2 = m-1, \ T_1 = A, \) and \( T_2 = I, \)
then \( w^1 = \{ k \} \) and \( w^2 = (1, \ldots, k-1, k+1, \ldots, m), \ k = 1, \ldots, m, \)

\[
\pi_{w^1(T_1, T_2)} = I \otimes \cdots \otimes A \otimes \cdots \otimes I = \pi_k(A).
\]

Then

\[
\delta(A, I) = \sum_{k=1}^{m} \pi_k(A)
\]

\[
= D(A)
\]

[see Example 2.4 (c)].

(c) For any \( T_1, \ldots, T_p \),

\[
\pi_{w}(T_1, \ldots, T_p) = \prod_{i=1}^{p} \prod_{j=1}^{r_i} \pi_{w^i(j)}(T_i).
\]  \hspace{1cm} (26)

For, the product on the right places \( T_i \) in the positions indexed by \( w^i \), \( i = 1, \ldots, p. \)

(d) For any \( T_1, \ldots, T_p \) and any \( \sigma \in S_m \),

\[
\delta(T_1, \ldots, T_p)P(\sigma) = P(\sigma)\delta(T_1, \ldots, T_p).
\]  \hspace{1cm} (27)

For, from (26) and (21),

\[
P(\sigma)\pi_{w}(T_1, \ldots, T_p) = \prod_{i=1}^{p} \prod_{j=1}^{r_i} \pi_{\sigma w^i(j)}(T_i)P(\sigma)
\]

\[
= \pi_{\sigma w}(T_1, \ldots, T_p)P(\sigma),
\]

where \( \pi_{\sigma w}(T_1, \ldots, T_p) \) indicates that \( T_i \) appears in positions
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numbered $\sigma^i(i), \ldots, \sigma^i(r_p)$, $i = 1, \ldots, p$. Now for $\sigma$ fixed, $\sigma^1, \ldots, \sigma^p$ runs over all sets of nonoverlapping sequences of lengths $r_1, \ldots, r_p$, respectively, as $w^1, \ldots, w^p$ vary over all such choices. Thus

$$P(\sigma)\delta(T_1, \ldots, T_p) = P(\sigma) \sum \pi_{\sigma}(T_1, \ldots, T_p)$$

$$= \sum \pi_{\sigma}(T_1, \ldots, T_p)P(\sigma)$$

$$= \delta(T_1, \ldots, T_p)P(\sigma).$$

It follows immediately from (27) that $\delta(T_1, \ldots, T_p)$ commutes with any symmetrizer and hence that any symmetry class is an invariant subspace of $\delta(T_1, \ldots, T_p)$. We make the following definition.

**Definition 2.2 (Partial derivation)** The restriction of $\delta(T_1, \ldots, T_p)$ to a symmetry class of tensors $V_{\chi}^m(H)$ is called the partial derivation corresponding to the partition $\tau$ [see (23)] induced by $T_1, \ldots, T_p$ on the symmetry class $V_{\chi}^m(H)$. We denote the partial derivation by $D(T_1, \ldots, T_p)$.

Observe that $D(T_1, \ldots, T_p)$ depends on the partition (24) and on the particular symmetry class $V_{\chi}^m(H)$.

**Example 2.6 (a)** Let $V_{\chi}^m(H)$ be an arbitrary symmetry class, and let $p = 1$, $r_1 = m$, and $T_1 = T$. Then from Example 2.5 (a), $\delta(T) = \delta T$ and hence $D(T) = \delta T(V_{\chi}^m(H)) = K(T)$ (see Theorem 4.1, Chapter 2). Thus the induced transformation $K(T)$ is itself a special case of a partial derivation.
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Let \( \dim V = n = m \), and let the symmetry class be the 1-dimensional \( n \)-dimensional Grassmann space over \( V, \wedge V \). Take \( p = 2 \), \( r_1 = r \), \( r_2 = n - r \), \( T_1 = T \), and \( T_2 = I \). The sequence \( \omega^1 \in Q_{r, n} \) is arbitrary but then \( \omega^2 \) must be the complementary sequence so that

\[
\sum T(\omega) = \sum T(I)
\]

Then \( D(T, I) \) is a linear transformation on a 1-dimensional space and hence is simply multiplication by some scalar \( c_T \). To compute \( c_T \) first observe that

\[
D(T, I)x^\wedge = D(T, I)x^\wedge
\]

\[
= S_c \delta(T, I)x^\wedge
\]

\[
= S_c \sum_{\omega \in Q_{r, n}} T(\omega) x^\wedge
\]

\[
= S_c \sum_{\omega \in Q_{r, n}} x_1 \wedge \cdots \wedge T_\omega(1) \wedge \cdots \wedge T_\omega(r) \wedge \cdots \wedge x_n
\]

Let \( x_i = e_i \), \( i = 1, \ldots, n \), be a basis of \( V \). Then since \( D(T, I)e^\wedge = c_T e^\wedge \), we have

\[
c_T e^\wedge = \sum_{\omega \in Q_{r, n}} e_1 \wedge \cdots \wedge T_\omega(1) \wedge \cdots \wedge T_\omega(r) \wedge \cdots \wedge e_n.
\]
Let $f_1, \ldots, f_n$ be a basis of $V^*$ dual to $e_1, \ldots, e_n$. Then by the usual argument

$$c_r \det[f_j(e_i)] = \sum_{\omega \in Q_{r,n}} \det \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ f_1(Te_\omega(1)) & \cdots & f_n(Te_\omega(1)) \\ f_1(Te_\omega(r)) & f_n(Te_\omega(r)) \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

$$= \sum_{\omega \in Q_{r,n}} \det[f_{\omega(j)}(Te_\omega(i))].$$

This last equality follows from the Laplace expansion theorem applied to each of the summands. Now if $A = [T]_E^E$, it follows that

$$f_{\omega(j)}(Te_\omega(i)) = f_{\omega(j)}(\sum_{k=1}^n a_{k\omega(i)} e_k) = a_{\omega(j), \omega(i)}$$

so that

$$[f_{\omega(j)}(Te_\omega(i))] = (A[\omega | \omega])^\top.$$

Hence

$$c_r = \sum_{\omega \in Q_{r,n}} \det A[\omega | \omega];$$

i.e., the sum of all principal $r$-square subdeterminants of $A$.

Note that $(-1)^r c_r$ is the coefficient of $x^{n-r}$ in the characteristic
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polynomial of $T$.

(c) If $V$ is a unitary space and $T_1, \ldots, T_p$ are hermitian (or p.s.d. or p.d.), then so is $\delta(T_1, \ldots, T_p)$. For, from (25) it suffices to prove that each $\pi_w(T_1, \ldots, T_p)$ is hermitian and this is the content of Theorem 2.2 (b), Chapter 2. If $T_1, \ldots, T_p$ are p.s.d. (p.d.), so is $\pi_w(T_1, \ldots, T_p)$ and hence $\delta(T_1, \ldots, T_p)$ is p.s.d. (p.d.) (see Theorem 2.2 (c) and (d), Chapter 2).

(d) If $S$ and $T$ are in $L(V, V)$, then the following formula is valid for arbitrary $T_1, \ldots, T_p$:

$$K(S)D(T_1, \ldots, T_p)K(T) = D(ST_1, ST_2, \ldots, ST_p).$$  (28)

For, $K(S) = \bigotimes_{\chi} S|_V^m(H)$, $D(T_1, \ldots, T_p) = \delta(T_1, \ldots, T_p)|_V^m(H)$, and $K(T) = \bigotimes_{\chi} T|_V^m(H)$. It is obvious that $\bigotimes_{\chi} \delta(T_1, \ldots, T_p) \otimes T = \delta(ST_1, \ldots, ST_p)$, and the product of the restrictions of linear transformations to a common invariant subspace is the restriction of the product.

(e) If $V$ is a unitary space, then

$$D(T_1, \ldots, T_p)^* = D(T_1^*, \ldots, T_p^*).$$  (29)

The $^*$ on the left is the conjugate dual with respect to the inner product in $\bigotimes_{\chi} V$ [restricted to $V^m(H)$] and the $^*$ on the right is the conjugate dual with respect to the inner product in $V$. We can confirm (29) by a routine calculation analogous to the proof of Theorem 4.2 (d), Chapter 2 (see Exercise 8).

(f) It follows from (c) above that if $T_1, \ldots, T_p$ are p.s.d. or p.d., then $D(T_1, \ldots, T_p)$ has the corresponding property. For,
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\[ D(T_1, \ldots, T_p) \] is the restriction of \( \delta(T_1, \ldots, T_p) \) to the invariant subspace \( V^m_\chi(H) \).

\( (g) \) If \( v_1, \ldots, v_m \) are arbitrary vectors in \( V \), then

\[
D(T_1, \ldots, T_p)v^* = \sum_{\omega} \cdots \sum_{i} T_{i_1}w_{i_1}(1) \cdots \sum_{i} T_{i_p}w_{i_p}(r_i) \cdots \cdots . \hspace{1cm} (30)
\]

The summation on the right in \( (30) \) is over all nonoverlapping \( w^1, \ldots, w^p \) (the partition \( \tau: r_1 + \cdots + r_p = m \) is fixed and \( w^i \in \mathcal{O}_{T_i,m} \), \( i = 1, \ldots, p \)). Also, \( T_i \) appear in positions \( w^i(1), \ldots, w^i(r_i), \) \( i = 1, \ldots, p \). The formula \( (30) \) is easily confirmed using \( (25) \) and \( (26) \) and the fact that \( \delta(T_1, \ldots, T_p) \) commutes with the symmetrizer defined by \( H \) and \( \chi \) (see Exercise 9).

\( (h) \) We saw in \( (a) \) above that the induced transformation \( K(T) \) is a particular instance of a partial derivation. Conversely, the partial derivations can be recaptured from the induced transformations \( K(T) \) as follows. First, let \( \xi_1, \ldots, \xi_n \) be independent indeterminates over the field \( R \), and using Theorem 4.3, Chapter 1, we can regard the vector space \( V \) as being over the rational function field \( R(\xi_1, \ldots, \xi_n) \). Then using only the multilinearity of the symmetric product we compute that

\[
K\left( \sum_{j=1}^{p} \xi_jT_j \right)v^* = \left( \sum_{j=1}^{p} \xi_jT_j \right)v_1 \cdots \sum_{j=1}^{p} \xi_jT_jv_m \hspace{1cm} (31)
\]

\[
= \sum_{r_1 + \cdots + r_p = m} \xi_1^{r_1} \cdots \xi_p^{r_p} \sum_{\omega} \cdots \sum_{i} T_{i_1}w_{i_1}(1) \cdots \sum_{i} T_{i_p}w_{i_p}(r_i) \cdots \cdots .
\]
The inner summation is over all nonoverlapping $w^i \in Q_{i^{\tau},m}$ and the outer summation is over all partitions of $m$ into $p$ nonnegative parts. Thus from (30),

$$K \left( \sum_{j=1}^{p} \varepsilon_j \tau_j \right) = \sum_{\tau} \varepsilon_{r_1}^{T_1} \cdots \varepsilon_{r_p}^{T_p} \delta(T_1, \ldots, T_p), \quad (32)$$

where, of course, $\delta(T_1, \ldots, T_p)$, the coefficient of $\varepsilon_{r_1}^{T_1} \cdots \varepsilon_{r_p}^{T_p}$ on the right in (32), is the partial derivation that goes with the partition $\tau: r_1 + \cdots + r_p = m$.

(i) Let $X_i \in L(V \wedge V), \ i = 1, \ldots, m$, and set

$$S(x_1, \ldots, x_m) = \sum_{\varphi \in S_m} x_{\varphi(1)} \otimes \cdots \otimes x_{\varphi(m)}.$$

If $\tau: r_1 + \cdots + r_p = m$ is a fixed partition of $m$, then

$$S(T_1, \ldots, T_1, T_2, \ldots, T_2, \ldots, T_p, \ldots, T_p) = r_1! \cdots r_p! \delta(T_1, \ldots, T_p),$$

where $\delta$ corresponds to the partition $\tau$. To confirm this equality note first that every $\pi_w(T_1, \ldots, T_p)$ occurs in the sum on the left.

For, given nonoverlapping $w^t \in Q_{r_t,m}, \ t = 1, \ldots, p$, choose $\varphi \in S_m$ such that

$$\varphi^{-1} \left( \sum_{k=1}^{t-1} r_k + j \right) = w^t(j), \ j = 1, \ldots, r_t, \ t = 1, \ldots, p,$$

and then set $x_1 = \cdots = x_{r_1} = T_1, \ x_{r_1+1} = \cdots = x_{r_1+r_2} = T_2, \ etc.$

Clearly $x_{\varphi(1)} \otimes \cdots \otimes x_{\varphi(m)} = \pi_w(T_1, \ldots, T_p)$. Moreover the total number of permutations $\varphi$ satisfying the preceding equalities is
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precisely $r_1' \cdots r_p'$. Hence each $\pi_w(T_1', \ldots, T_p')$ appears
$r_1' \cdots r_p'$ times.

(j) If $D(X_1', \ldots, X_m)$ is the partial derivation corresponding
to the partition

$$\tau_0: 1 + \cdots + 1 = m$$

and $\tau: r_1 + \cdots + r_p = m$ is any other partition of $m$, then

$$D(T_1', \ldots, T_{r_1'}, \ldots, T_{r_2'}, \ldots, T_{r_p'}, \ldots, T_p') = r_1' \cdots r_p' D_\tau(T_1', \ldots, T_p'),$$

where the partial derivation on the right corresponds to the
partition $\tau$. To see this we let $\delta(X_1, \ldots, X_m)$ correspond to the
partition $\tau_0$. Then by (i) above

$$\delta(X_1, \ldots, X_m) = \delta(X_1', \ldots, X_m')$$

and hence

$$\delta(T_1', \ldots, T_{r_1'}, \ldots, T_{r_2'}, \ldots, T_{r_p'}, \ldots, T_p')$$

$$= \delta(T_1, \ldots, T_{r_1}, \ldots, T_{r_2}, \ldots, T_{r_p}, \ldots, T_p)$$

$$= r_1' \cdots r_p' \delta(T_1', \ldots, T_p'),$$

where the $\delta$ on the right corresponds to $\tau$. Since the partial
derivations are restrictions of the $\delta$'s to invariant subspaces
(i.e., the symmetry classes) we have the desired equality.

In order to discuss the structure of the eigenvalues of
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\( D(T_1, \ldots, T_p) \) we define a class of polynomials that can be specialized to the usual elementary symmetric polynomials.

**Definition 2.3 (Partition polynomial)** Let \( \tau: r_1 + \cdots + r_p = m \) be a fixed partition of \( m \) and let \( \xi_{ij}, \ i = 1, \ldots, p, \ j = 1, \ldots, m, \) be pm indeterminates over \( R. \) Let \( \Xi = [\xi_{ij}] \) denote the \( p \times m \) matrix whose \( i,j \) entry is \( \xi_{ij} \) and define the polynomial

\[
ed{\tau}(\Xi) = \sum_{\omega} \prod_{i=1}^{p} \prod_{j=1}^{r_i} \xi_{ij}^{\omega(i,j)}. \tag{33}
\]

The summation on the right in (33) is over all nonoverlapping \( \omega_i \in Q_{r_i, m}, \ i = 1, \ldots, p. \) The polynomial (33) is called the partition polynomial corresponding to \( \tau. \)

**Example 2.7 (a)** Take \( p = 2, \ r_1 = r, \) and \( r_2 = m - r. \) Then from (33),

\[
ed{\tau}(\Xi) = \sum_{\omega \in Q_{r, m}} \xi_{1\omega(1)} \cdots \xi_{1\omega(r)} \xi_{2\omega'(1)} \cdots \xi_{2\omega'(m-r)},
\]

where \( \omega' \in Q_{m-r, m} \) is the sequence whose range is complementary to the range of \( \omega. \) If we make the specialization \( \xi_{2j} = 1, \ j = 1, \ldots, m, \) and set \( \xi_{1j} = \xi_{j}, \ j = 1, \ldots, m, \) we obtain

\[
\sum_{\omega \in Q_{r, m}} \xi_{\omega(1)} \cdots \xi_{\omega(r)},
\]

the \( r^{th} \) elementary symmetric polynomial of \( \xi_1, \ldots, \xi_m \) [see Example 4.4(d), Chapter 2].
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(b) If $\xi_1, \ldots, \xi_p$ and $\xi_{1j}, \ i = 1, \ldots, p, \ j = 1, \ldots, m,$ are independent indeterminates over $R$, then

$$\prod_{j=1}^m \sum_{i=1}^p \xi_i \xi_{ij} = \sum_{\tau: r_1 + \cdots + r_p = m} \xi_1^{r_1} \cdots \xi_p^{r_p} e_\tau(\Xi). \quad (34)$$

To confirm (34) consider the array

$$\begin{array}{cccc}
\xi_1^p11 & \xi_1^p12 & \cdots & \xi_1^p1m \\
\xi_2^p21 & \xi_2^p22 & \cdots & \xi_2^p2m \\
\vdots & \vdots & \ddots & \vdots \\
\xi_p^p1 & \xi_p^p2 & \cdots & \xi_p^pm \\
\end{array} \quad (35)$$

in which the $j^{th}$ factor in the product on the left in (34) is the $j^{th}$ column sum in (35). The product in (34) is formally obtained by selecting one entry from each column in all possible ways, multiplying these together, and adding all such products. A term in which $\xi_j$ appears precisely $r_j$ times is obtained by choosing the element in row $i$ in precisely $r_j$ of the columns of (35). Thus to obtain the coefficient of $\xi_1^{r_1} \cdots \xi_p^{r_p}$ we first select $\omega^i \in Q_{r_1, m}$ in order to designate the columns $\omega^i(1), \ldots, \omega^i(r_1)$, and then in each of these columns we use the entry in row $i$ to form the product. Since we choose exactly one entry from each column, the $\omega^i$ must be chosen to be nonoverlapping and $r_1 + \cdots + r_p = m$. The resulting term is
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\[
\prod_{i=1}^{p} (\xi_1^{i}, \omega^{i}(1), \xi_1^{i}, \omega^{i}(2), \ldots, \xi_1^{i}, \omega^{i}(r_1)) = \prod_{i=1}^{p} \xi_1^{i} \prod_{j=1}^{r_1} \xi_1^{i}, \omega^{i}(j) \\
= \xi_1^{r_1} \cdots \xi_1^{r_p} \prod_{i=1}^{p} \prod_{j=1}^{r_i} \xi_1^{i}, \omega^{i}(j).
\]

To obtain the complete coefficient of \( \xi_1^{r_1} \cdots \xi_1^{r_p} \) we must sum over such nonoverlapping \( \omega^{i} \in Q_{x_i,m}, \ i = 1, \ldots, p, \) and from (33) this is precisely \( e_\tau(m) \).

**Theorem 2.4** If \( E = \{e_1, \ldots, e_n\} \) is a triangular basis for each \( T_i, \ i = 1, \ldots, p, \) then the basis \( E_\alpha = \{e_\alpha, \ \alpha \in \Lambda\} \) is a triangular basis for \( D(T_1, \ldots, T_p) \). Moreover the e.v.'s of \( D(T_1, \ldots, T_p) \) are

\[
e_\tau([\lambda, \omega(j)]), \ \alpha \in \Lambda,
\]

where \( \lambda_{i1}, \ldots, \lambda_{in} \) are the e.v.'s (in some order) of \( T_i, \ i = 1, \ldots, p, \) and \( \tau \) is the partition of \( m \) used to define the partial derivation.

**Proof:** Let \( \xi_1, \ldots, \xi_p \) be independent indeterminates over \( R \) and as in Example 2.6 (b)

\[
K\left( \sum_{j=1}^{p} \xi_j t_j \right) = \sum_{r_1 + \cdots + r_p = m} \xi_1^{r_1} \cdots \xi_p^{r_p} D(T_1, \ldots, T_p).
\]

Clearly \( E \) is a triangular basis for \( \sum_{j=1}^{p} \xi_j t_j \) and hence by Theorem 4.3, Chapter 2, \( E_\alpha \) is a triangular basis for \( K(\sum_{j=1}^{p} \xi_j t_j) \).
The \((\alpha, \alpha)\) entry in the matrix representation of this induced transformation is

\[
\prod_{t=1}^{m} \left( \sum_{j=1}^{p} \xi_{j}^{E} \alpha(t), \alpha(t) \right) = \prod_{t=1}^{m} \left( \sum_{j=1}^{p} \xi_{j}^{E} \alpha(t) \right)
\]

\[
= \sum_{\tau: r_1 + \cdots + r_p = m} \xi_{r_1}^{T_1} \cdots \xi_{r_p}^{T_p} e_{\tau}([\lambda_{1}(x)]) \quad \text{(38)}
\]

(This second equality is an application of (34).) From (37) we have

\[
\left[ K \left( \sum_{j=1}^{p} \xi_{j}^{T} \right) \right]_{E*}^{E*} = \sum_{r_1 + \cdots + r_p = m} \xi_{r_1}^{T_1} \cdots \xi_{r_p}^{T_p} [D(T_1, \ldots, T_p)]_{E*}^{E*} \quad \text{(39)}
\]

The polynomials \(\xi_{r_1}^{T_1} \cdots \xi_{r_p}^{T_p}\) are l.i. over \(R\) (see Exercise 10) so that none of the matrices \([D(T_1, \ldots, T_p)]_{E*}^{E*}\) can have a nonzero element below the main diagonal, i.e., the matrix on the left in (39) is upper triangular. Thus \(E^*\) is a triangular basis for each partial derivation \(D(T_1, \ldots, T_p)\). Moreover the coefficient of \(\xi_{r_1}^{T_1} \cdots \xi_{r_p}^{T_p}\) in the \((\alpha, \alpha)\) entry on the left in (39) is given by (38) as \(e_{\tau}([\lambda_{11}(x)])\) and hence the \((\alpha, \alpha)\) entry of \([D(T_1, \ldots, T_p)]_{E*}^{E*}\) is equal to this number.

We remark that the order in which \(\lambda_{11}, \ldots, \lambda_{in}\) are to be taken is determined by the order in which these numbers appear down the main diagonal in \([T_j]\)\(E\). It is also clear from Theorem 2.4 that if \(T_i e_{\alpha(i)} = \lambda_{i\alpha(i)} e_{\alpha(i)}\), \(i = 1, \ldots, p\), and \(e_{\alpha}^* \neq 0\), then \(e_{\alpha}^*\) is an eigenvector of \(D(T_1, \ldots, T_p)\) corresponding to the e.v. \(e_{\tau}([\lambda_{11}(x)])\). Thus if \(T_1, \ldots, T_p\) possess a basis \(E = \{e_1, \ldots, e_n\}\).
of common eigenvectors, then \( E_\alpha = \{ e_\alpha^*, \ \alpha \in \Delta \} \) is a basis of eigenvectors of \( D(T_1, \ldots, T_p) \).

**Example 2.8 (a)** Let \( \dim V = n = m \) and let the symmetry class be the 1-dimensional \( n \)-th Grassmann space over \( V \). Take \( p = 2, \ r_1 = r, \ r_2 = n - r, \ T_1 = T, \) and \( T_2 = I \). Then \( \Delta = Q_{n,n} \) contains the single sequence \( \alpha = (1, \ldots, n) \) and by Example 2.7 (a) and Theorem 2.4, \( D(T, I) \) has the single e.v.

\[
\sum_{\omega \in Q_{r,n}} \lambda_\omega(1) \cdots \lambda_\omega(r) = E_r(\lambda_1, \ldots, \lambda_n).
\]

This is in agreement with Example 2.6 (b).

(b) Let \( \dim V = n \) and let the symmetry class be either \( \wedge V \) or \( V^{(m)} \). Again take \( p = 2, \ r_1 = r, \ r_2 = m - r, \ T_1 = T, \) and \( T_2 = I \). Then \( \Delta = Q_{m,n} \) or \( G_{m,n} \) and the e.v.'s of \( D(T, I) \) are [see Example 2.7 (a)]

\[
E_r(\lambda_\alpha(1), \ldots, \lambda_\alpha(m)), \quad \alpha \in Q_{m,n} \quad \text{(for } \wedge V)\]

or

\[
E_r(\lambda_\alpha(1), \ldots, \lambda_\alpha(m)), \quad \alpha \in G_{m,n} \quad \text{(for } V^{(m)})\]

where \( \lambda_1, \ldots, \lambda_n \) are the e.v.'s of \( T \).

(c) Let \( p = 2, \ r_1 = r, \ r_2 = m - r, \ T_1 = T, \) and \( T_2 = I \) and let \( E = \{ e_1, \ldots, e_n \} \) be a basis of \( V \) (not necessarily a triangular basis for \( T \)). We compute from (30) that

\[
D(T, I)e_\alpha^* = \sum_{\omega \in Q_{r,m}} e_{\alpha(1)}^* \cdots T_{\alpha} \omega(1)^* \cdots T_{\alpha} \omega(r)^* \cdots * e_{\alpha(m)}. \quad (40)
\]
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Let \( \{f_1, \ldots, f_n\} \) be a basis of \( V^* \) dual to \( E \) and let \( h_\beta \otimes = f_\beta = f_\beta(1) \cdots f_\beta(m) \) for each \( \beta \in \Lambda \); then \( h_\beta(e_\alpha^*) = \delta_\alpha^\beta \frac{|\chi(H)|}{\nu(\beta)} \) (see Exercise 11) and hence

\[
\left\{ \frac{|H|}{\nu(\beta)} h_\beta \mid \chi(H), \ \beta \in \Lambda \right\}
\]

is dual to the basis \( e_\alpha^* \). It follows that the \((\alpha, \alpha)\) entry in the matrix \([D(T,I)]_{e*}^{E*}\) is \(\frac{|H|}{\nu(\alpha)} h_\alpha(D(T,I)e_\alpha^*)\). From (40) we compute [using formula (45) in Section 2.4]

\[
\frac{|H|}{\nu(\alpha)} h_\alpha(D(T,I)e_\alpha^*) = \frac{|H|}{\nu(\alpha)} \sum_{\omega \in Q_{r,m}} h_\alpha(e_\alpha(1) \cdots \ast T_{\omega \omega(1)} \cdots \ast T_{\omega \omega(r)} \ast \cdots \ast e_\alpha(m))
\]

Suppose now that \( \alpha \in _m n \) [so that \( \nu(\alpha) = 1 \)] and \( A = [T]_E^E \). Then abbreviating \( d^H_\chi \) to \( d \) and continuing we have

\[
(D(T,I))_{e*}^{E*} = \sum_{\omega \in Q_{r,m}} d \left( \begin{array}{ccc}
 f_\alpha(1)(e_\alpha(1)) & \cdots & f_\alpha(m)(e_\alpha(1)) \\
 \vdots & \ddots & \vdots \\
 f_\alpha(1)(T_{\omega \omega(1)}) & \cdots & f_\alpha(m)(T_{\omega \omega(1)}) \\
 f_\alpha(1)(T_{\omega \omega(r)}) & \cdots & f_\alpha(m)(T_{\omega \omega(r)}) \\
 f_\alpha(1)(e_\alpha(m)) & \cdots & f_\alpha(m)(e_\alpha(m))
\end{array} \right).
\]
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If we further assume that $\nu^m(H)$ is either $\wedge V$ or $\nu^m(V)$, then $d$ is det or per and the Laplace expansion theorem can be applied to each of the preceding summands to obtain (for $\alpha \in \Omega_{m,n}$)

$$
\left[ T(T/I) \right]_{E^*_{m,n}} = \sum_{\omega \in \Omega_{m,n}} d(A[\omega(l), \ldots, \omega(r)| \omega(l), \ldots, \omega(r)]). (41)
$$

(d) Let $1 \leq r \leq m \leq n$ and let $d_r(A[\alpha|\alpha])$ ($p_r(A[\alpha|\alpha])$) be the sum of all principal $r$-square subdeterminants (subpermanents) of $A[\alpha|\alpha]$, $\alpha \in \Omega_{m,n}$. If $A$ is a normal matrix, then $d_r(A[\alpha|\alpha])$ is in the convex hull of the numbers $E_r(\lambda_1, \ldots, \lambda_m)$, $\gamma \in \Omega_{m,n}$, and $p_r(A[\alpha|\alpha])$ is in the convex hull of the numbers $E_r(\lambda_1, \ldots, \lambda_m)$, $\gamma \in \Omega_{m,n}$. For,

$$
\sqrt{n} \sum_{\nu(\alpha)} e^*_\alpha, \quad \alpha \in \Lambda,
$$

is an o.r. basis of the symmetry class and with respect to this basis the main diagonal elements in positions $(\alpha, \alpha)$, $\alpha \in \Omega_{m,n}$ are given by (41). Now the main diagonal entries of a normal matrix lie in the convex hull of the eigenvalues of the matrix (see Exercise 12). Moreover $D(T, I)$ is normal when $T$ is normal because $\delta(T, I)$ is obviously normal (see Exercise 13). The assertion then follows from (b) above. This result should be compared with Example 4.4(c), Chapter 2.

(e) Another formulation of (d) can be made. Let $x_1, \ldots, x_m$ be any o.n. set of vectors in the unitary space $V$. Let $T \in \mathcal{L}(V, V)$ be normal and for $1 \leq r \leq m \leq n$ consider
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\[ f(x_1, \ldots, x_m) = \sum_{\omega \in Q_{x,m}} (c^\omega(T)x^\omega, x^\omega). \]

Then \( f(x_1, \ldots, x_m) \) is in the convex hull of the numbers
\[ \frac{1}{r!} E_{\gamma}(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(m)}), \quad \gamma \in Q_{m,n}. \]
Similarly, if
\[ g(x_1, \ldots, x_m) = \sum_{\omega \in Q_{x,m}} (p^\omega(T)x^\omega, x^\omega), \]
then \( g(x_1, \ldots, x_m) \) is in the convex hull of the numbers
\[ \frac{1}{r!} E_{\gamma}(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(m)}), \quad \gamma \in G_{m,n}. \]

The first result follows from (d) by recalling that
\[ (c^\omega(T)x^\omega, x^\omega) = \frac{1}{r!} \det([(T_{\omega(i)}, x_{\omega(j)})]); \]
and also if \( x_1, \ldots, x_m \) is completed to an o.n. basis, then
\[ [(T_{\omega(i)}, x_{\omega(j)}), \quad i, j = 1, \ldots, n, \] is a normal matrix with \( [(T_{\omega(i)}, x_{\omega(j)}), \]
\[ i, j = 1, \ldots, m, \] as a principal submatrix; similarly for the result about \( g(x_1, \ldots, x_m) \). It is worth noting here that many special cases of these results have appeared in the literature over the last 25 years.

(f) If \( T_1, \ldots, T_p \) are pairwise commuting normal transformations, then \( D(T_1, \ldots, T_p) \) is normal and for a suitable ordering of the e.v.'s \( \lambda_{i1}, \ldots, \lambda_{in} \) of \( T_i \), \( i = 1, \ldots, p \), the e.v.'s of
\( D(T_1, \ldots, T_p) \) are the numbers \( e^1_{q}(\lambda_{i\alpha(j)}), \quad q \in A \). For it is obvious that \( e^1(T_1, \ldots, T_p) \) is normal and hence the restriction
\( D(T_1, \ldots, T_p) \) is also. Moreover it is a standard result that there exists a common o.n. basis of e.v.'s of \( T_1, \ldots, T_p \) and then the result follows from the remarks following the proof of Theorem 2.4.

There is an important connection between symmetry classes over
spaces of linear transformations and partial derivations. We have encountered a result of this type in Theorem 2.7, Chapter 2.

**Theorem 2.5** Let $\mathcal{L} = L(V, V)$ and let $(\kappa, \chi)$ denote the symmetry class over $\mathcal{L}$ associated with the group $H \subseteq S_m$ and the character identically 1, i.e., $Q = \mathcal{L}^m_H$. Let $(P, \ast)$ denote the symmetry class over $V$ associated with the group $H$ and the character $\chi$, i.e., $P = V^m_\chi(H)$. Then there exists a unique linear transformation $f: Q \rightarrow L(P, P)$ such that

$$f(\kappa(T_1, \ldots, T_1, T_2, \ldots, T_{r_2}, \ldots, T_{r_p}, \ldots, T_{r_p})) = r_1! \cdots r_p! D(T_1, \ldots, T_p), \quad (42)$$

where $T_i \in L(V, V)$, $i = 1, \ldots, p$, are arbitrary and $D(T_1, \ldots, T_p)$ is the partial derivation corresponding to the partition $\tau: r_1 + \cdots + r_p = m$. Moreover, if $H = S_m$, then $f$ is surjective.

**Proof:** From Example 2.6 (j) it is clear that we need only prove (42) for the partition given by $p = m$ and $r_1 = \cdots = r_m = 1$. For, from (30) we have (since $r_i = 1$, $i = 1, \ldots, m$)

$$D(T_1, \ldots, T_m)^\chi = \sum_{\varphi \in S_m} T\varphi(1)^{\chi_1} \ast \cdots \ast T\varphi(m)^{\chi_m}, \quad (43)$$

and it is obvious from (43) that $D(T\sigma(1), \ldots, T\sigma(m)) = D(T_1, \ldots, T_m)$ for any $\sigma \in H$. Moreover $D(T_1, \ldots, T_m)$ is linear in each $T_i$ so that there exists a unique linear $f: Q \rightarrow L(P, P)$ such that

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\[ f(\kappa(T_1, \ldots, T_m)) = D(T_1, \ldots, T_n). \]  \( \text{(44)} \)

Then from Example 2.6 (j),

\[
\begin{align*}
& f(\kappa(T_1, \ldots, T_1, \ldots, T_p, \ldots, T_p)) = D(T_1, \ldots, T_1, \ldots, T_p, \ldots, T_p) \\
& \quad = r_1! \ldots r_p! D(T_1, \ldots, T_p).
\end{align*}
\]

It remains to prove that if \( H = S_m \) then \( f \) is surjective, and this is done by showing that the transformations \( D(T_1, \ldots, T_m) \) span \( L(P, P) \). Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and let \( T_{ij} \in L \) be given by

\[ T_{ij} e_k = \delta_{ik} e_j, \quad k = 1, \ldots, n. \]

For \( \alpha, \beta, \) and \( \gamma \) in \( \Delta \) we compute that

\[
D(T_{\alpha(1)}, T_{\beta(1)}, \ldots, T_{\alpha(m)}, T_{\beta(m)}) e^* \gamma
\]

\[ = \sum_{\varphi \in S_m} T_{\alpha \varphi(1)}, T_{\beta \varphi(1)} e_{\gamma(1)}^{*} \ldots \ldots \ldots T_{\alpha \varphi(m)}, T_{\beta \varphi(m)} e_{\gamma(m)}^{*} \]

\[ = \sum_{\varphi \in S_m} \delta_{\alpha \varphi(1)}, \gamma(1) e_{\beta \varphi(1)}^{*} \ldots \ldots \ldots \delta_{\alpha \varphi(m)}, \gamma(m) e_{\beta \varphi(m)}^{*} \]

\[ = \sum_{\varphi \in S_m} \delta_{\alpha \varphi}, \gamma e_{\beta \varphi}^{*}. \]

Now \( \alpha \varphi = \gamma \), iff \( \alpha \) and \( \gamma \) are in the same \( S_m \)-orbit, but since they are elements of a system of distinct representatives it follows \( \alpha \varphi = \gamma \), iff \( \alpha = \gamma \) and \( \varphi \) is in the stabilizer group of \( \alpha \), \((S_m)_{\alpha}\). Thus
Several remarks are in order concerning Theorem 2.5. First, it is definitely not the case that \( f \) is surjective in general.

For, if \( H = \{ e \} \) then \( P = \bigoplus V \) and

\[
D(T_1', \ldots, T_m') = \delta(T_1', \ldots, T_m').
\]

As we saw in (27), \( \delta(T_1', \ldots, T_m') \) commutes with every permutation operator. Hence every l.t. in \( Im \ f \) commutes with every permutation operator. However, not every l.t. in \( L(\bigotimes V, \otimes V) \) has this property, e.g., if \( \sigma = (1,2) \), \( (T \otimes I)P(\sigma)e_1 \otimes e_2 = T_{e_1} \otimes e_2 \) and \( P(\sigma)(T \otimes I)e_1 \otimes e_2 = e_2 \otimes T_{e_1} \). Hence \( f \) is not surjective. In case \( H = S_m \) there are of course only the two symmetry classes, \( \bigwedge V \) and \( V^{(m)} \), so that the computation showing that \( f \) is surjective is actually being done in the proof of Theorem 2.5 for these two symmetry classes only.

Example 2.9 (a) Theorem 2.5 can be used to show that the transformations \( C_m(T), \ T \in L(V,V), \ ) span \( L(\bigwedge V, \bigwedge V) \). For, in this case \( H = S_m \) and hence \( Q = \mathcal{L}^{(m)}, \) the \( m^{th} \) completely symmetric space. Now the elements \( \kappa(T, \ldots, T) \) span \( \mathcal{L}^{(m)} \) according to Theorem 1.7. The transformation \( f \) is surjective so the transformations \( f(\kappa(T, \ldots, T)) = m! \ D(T) = m! \ K(T) = m! \ C_m(T) \) [see
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Example 2.6 (a)] must span \( L(\land V, \land V) \).

(b) By precisely the same argument as in (a) we see that the transformations \( P_m(T), \ T \in L(V, V) \), span \( L(V^m), v^m) \).

(c) The linear closure of the set of transformations \( D(T_1, \ldots, T_m) \) is the same as the linear closure of the set of transformations \( K(T) \), i.e.,

\[
\langle D(T_1, \ldots, T_m), \ T_1 \in L(V, V) \rangle = \langle K(T), \ T \in L(V, V) \rangle.
\]

To see this consider (32) for \( p = m \);

\[
K(\sum_{j=1}^{m} \xi_j \cdot T_j) = \sum_{\tau: r_1 + \cdots + r_m = m} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_m^{r_m} D(T_1, \ldots, T_m), \quad (45)
\]

where we have temporarily subscripted the partial derivation corresponding to \( \tau \) for purposes of the present discussion. Let \( N \) be the number of partitions \( \tau: r_1 + \cdots + r_m = m \) (i.e., we allow \( r_i = 0 \) so there are plenty of them) and arrange the polynomials \( \xi_1^{r_1} \cdots \xi_m^{r_m} \) lexicographically in \( \tau \) in an \( N \)-tuple:

\[
v(\xi_1, \ldots, \xi_m) = (\xi_1^{m}, \xi_1^{m-1}\xi_2, \ldots, \xi_1^{r_1} \cdots \xi_m^{r_m}, \ldots, \xi_1^{r_1} \cdots \xi_m^{r_m}).
\]

By Exercise 10 the polynomials \( \xi_1^{r_1} \cdots \xi_m^{r_m} \) are l.i. over \( R \) and hence there exist \( N \) specializations of the \( \xi_1, \ldots, \xi_n \) to elements in \( R \) so that the resulting values of \( v(\xi_1, \ldots, \xi_m) \) are \( N \) l.i. \( N \)-tuples over \( R \). For, if not then all specializations \( v(a_1, \ldots, a_n) \) \( a_i \in R \) would lie in a proper subspace of \( V_N(R) \) and hence there would exist \( N \) elements of \( R \), \( c_{r_1} \cdots r_s \), not all 0 such that
\[ \sum c_{r_1 \ldots r_m} a_1 \ldots a_m = 0 \quad (46) \]

for all \( a_i \in R \). Consider the polynomial in \( R[\xi_1, \ldots, \xi_m] \),

\[ \sum c_{r_1 \ldots r_m} \xi_1^{r_1} \cdots \xi_m^{r_m} \quad (47) \]

From (46), the polynomial (47) vanishes for arbitrary specializations of the \( \xi_i \) and since \( R \) has characteristic 0 and hence contains an infinite number of elements, it follows that every coefficient \( c_{r_1 \ldots r_m} \) is 0, a contradiction. Thus let \( v(a_1(i), \ldots, a_m(i)) = (\ldots, a_{1\tau}, \ldots), i = 1, \ldots, N \), denote the \( N \) l.i. specializations. From (45) we have

\[ K \left( \sum_{j=1}^{m} a_j(i)T_j \right) = \sum_{\tau} a_{1\tau} D_\tau(T_1, \ldots, T_m), \quad i = 1, \ldots, N. \]

The \( N \)-square matrix of coefficients \([a_{1\tau}]\) is nonsingular so that each \( D_\tau(T_1, \ldots, T_m) \) can be expressed as a linear combination of induced transformations \( K(T) \), \( T \in L(V, V) \), with constant coefficients, which are independent of the particular choices of \( T_1, \ldots, T_m \).

Now according to Example 2.6 (j) any \( D_\tau(T_1, \ldots, T_m) \) is a multiple of \( D(T_1, \ldots, T_m) \), the partial derivation corresponding to \( r_1 = \cdots = r_m = 1 \). Hence it follows that

\[ D(T_1, \ldots, T_m) \in \langle K(T), T \in L(V, V) \rangle. \]

Conversely,

\[ m:K(T) = D(T, \ldots, T) \]

so that trivially
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\[ K(T) \in \langle D(T_1, \ldots, T_m), \quad T_i \in L(V, V) \rangle. \]

(d) By (a), (b), and (c) above the partial derivations

\[ D(T_1, \ldots, T_m) \text{ span } L(V_{\chi}^m(H), V_{\chi}^m(H)) \]

when the symmetry class is either \( V \) or \( V_m \). We show now that if \( \chi = 1 \) and if \( H \) is a proper subgroup of \( S_m \), then the transformations \( K(T) \) do not span \( L(V_{\chi}^m(H), V_{\chi}^m(H)) \). First observe that

\[ x \ast \cdots \ast x = \frac{1}{|H|} \sum_{\sigma \in H} P(\sigma) x \otimes \cdots \otimes x = x \otimes \cdots \otimes x = x \cdots x. \]

Let \( W = \langle x \ast \cdots \ast x, \ x \in V \rangle \) so that by Theorem 1.7, \( V_m = W \subset V_{1}^m(H) \).

The previous inclusion is strict. For, let \( \alpha = (1, \ldots, m) \) and suppose \( \sigma \in S_m, \ \sigma \notin H \). Then \( \alpha \sigma \neq \alpha \) and also \( \alpha \sigma \) is in the same \( H \)-orbit with some \( \beta \in \Delta = \overline{\Delta} \), i.e., \( \beta = \alpha \sigma \varphi, \ \varphi \in H \). Moreover, \( \beta \neq \alpha \) otherwise \( \alpha \varphi = \alpha \) and hence \( \varphi = e, \ \sigma = \varphi^{-1} \in H \), a contradiction. Thus the indexing set for a basis of \( V_{1}^m(H) \) contains at least one more sequence, namely, \( \beta \), than the indexing set for a basis of \( V_m \). But \( K(T)x \ast \cdots \ast x = T x \ast \cdots \ast T x \in W \) so that \( W \) is a proper invariant subspace of every \( K(T) \). It follows immediately that the \( K(T), \ T \in L(V, V), \) do not span \( L(V_{1}^m(H), V_{1}^m(H)) \).

Although the partial derivations \( D(T_1, \ldots, T_m) \) on the tensor space \( \otimes \ ) do not span \( L(\otimes V, \otimes V) \), i.e., they commute with all permutation operators (see the remarks following Theorem 2.5), it is important for applications to the representation theory of the group of all nonsingular l.t.'s in \( L(V, V) \) to know that the only l.t.'s commuting with every permutation operator are linear.
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combinations of the partial derivations \( D(T_1, \ldots, T_m) \). In view of Example 2.9(c) we then want to show that any l.t. in \( L(\otimes V, \otimes V) \) that commutes with every \( F(\sigma), \sigma \in S_m \), is in \( \langle \otimes T, T \in L(V, V) \rangle \). This leads us to define the following important class of l.t.'s.

**Definition 2.4 (Bisymmetric transformation)** If \( S \in L(\otimes V, \otimes V) \) and \( S \) commutes with every permutation operator \( P(\sigma), \sigma \in S_m \), then \( S \) is called a bisymmetric transformation. The totality of all bisymmetric transformations on \( \otimes \) is denoted by \( \mathcal{B}_m \).

**Theorem 2.6** Let \( E = \{e_1, \ldots, e_n\} \) be a basis of \( V \) and let \( E^\otimes \) be the lexicographically ordered basis \( \{e_\alpha^\otimes, \alpha \in \Gamma_n^m\} \) of \( \otimes V \). Let \( A = [S]_{E^\otimes} \). Then \( S \) is bisymmetric iff for every \( \sigma \in S_m \),

\[
a_{\alpha, \beta} = a_{\sigma \alpha, \sigma \beta}, \quad \alpha, \beta \in \Gamma_n^m.
\]

(48)

**Proof:** We have

\[
S e_\beta^\otimes = \sum_{\alpha \in \Gamma_n^m} a_{\alpha, \beta} e_\alpha^\otimes
\]

so that

\[
P(c)S e_\beta^\otimes = \sum_{\alpha \in \Gamma_n^m} s_{\alpha, \beta} e_{\alpha^{-1}}^\otimes.
\]

Since \( \alpha^{-1} \) runs over \( \Gamma_n^m \) as \( \alpha \) does, we have

\[
P(\sigma)S e_\beta^\otimes = \sum_{\alpha \in \Gamma_n^m} a_{\sigma \alpha, \beta} e_\alpha^\otimes.
\]

(49)

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Also

\[ SP(\sigma)e_\beta^\otimes = Se_\beta^{\otimes I_n} \]

\[ = \sum_{\alpha \in \Gamma_n} a_{\alpha, \beta} \varepsilon_\alpha^\otimes. \]  

(50)

Now \( P(\sigma)S = SP(\sigma) \) if and only if every \( \beta \in \Gamma_n \), \( S \alpha_\otimes \beta = P(\sigma)S_\beta^\otimes \).

Thus from (49) and (50), \( S \) is bisymmetric, if

\[ a_{\alpha, \beta}^{\otimes I_n} = a_{\alpha, \beta}, \quad \alpha, \beta \in \Gamma_n, \quad \sigma \in S_m. \]  

(51)

If we set \( \gamma = \beta^{\otimes I_n} \) in (51) and then replace \( \gamma \) by \( \beta \), we have

(48). \( \square \)

**Theorem 2.7** A linear transformation is bisymmetric, if and only if it is a linear combination of tensor product transformations \( m \) \( \Pi_l^T \), \( T \in L(V, V) \) [see Section 2.1, formula (4) for notation]. In other words,

\[ \mathcal{B}_m = \{ \Pi_l^T, T \in L(V, V) \}. \]

(52)

**Proof:** First observe that \( \mathcal{B}_m \) is a subalgebra of \( L^m \)(\( m \) \( \otimes V, \otimes V \)) \( l \) \( l \) as is the right side of (52) and that any \( \Pi_l^T \) commutes with every \( P(\sigma), \sigma \in S_m \). Thus the right side of (52) is obviously a subset of \( \mathcal{B}_m \). We apply Theorem 2.7 in Chapter 2 to obtain a unique surjective linear \( f: L^m \)(\( \otimes V, \otimes V \)) \( l \) \( l \) \( l \) \( l \) \( l \), \( \mathcal{L} = L(V, V) \), which satisfies

\[ f(\Pi_l^T) = \kappa(T_1, \ldots, T_m). \]

(53)

In (53) we have used \( \kappa \) for the canonical map \( \kappa: \mathcal{L} \rightarrow \otimes \mathcal{L} \) in
order to avoid possible confusion with the notation for the tensor product of l.t.'s as an element of \( I(\otimes^m V, \otimes^m V) \). Let \( \mathcal{L}(m) \) be the completely symmetric space in \( \otimes^m \mathcal{L} \), which according to Theorem 1.7 is spanned by elements \( T \cdots T = \kappa(T, \ldots, T) \), \( T \in \mathcal{L} \). Now from (53),

\[
f(\Pi^m T) = \kappa(T, \ldots, T)
\]

so that

\[
f(\langle \Pi^m T, T \in L(V, V) \rangle) = \mathcal{L}(m).
\]  

(54)

If we can prove that

\[
f(\Theta_m) = \mathcal{L}(m),
\]  

(55)

then since \( f \) is l-l, it will follow that

\[
\Theta_m = f^{-1}(\mathcal{L}(m)) = \langle \Pi^m T, T \in L(V, V) \rangle.
\]

Now, as we noted above

\[
\langle \Pi^m T, T \in L(V, V) \rangle \subset \Theta_m
\]

so that from (54),

\[
f(\Theta_m) \supset \mathcal{L}(m).
\]  

(56)

Now let \( S \in \Theta_m \) and the problem is to prove that \( f(S) \in \mathcal{L}(m) \).

Equivalently, if \( \sigma \in S_m \) we want to prove that

\[
P(\sigma)f(S) = f(S),
\]  

(57)

where \( P(\sigma) \) is the permutation operator on \( \otimes^m \mathcal{L} \), i.e.,
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\[ P(\sigma) x(T_1, \ldots, T_m) = x(T_{\sigma^{-1}(1)}, \ldots, T_{\sigma^{-1}(m)}) \]. Let \( E = \{e_1, \ldots, e_n\} \) be a basis of \( V \). Let \( T_{ij} \in L(V, V) \) be defined by \( T_{ij}(e_k) = \delta_{ik} e_j \), \( k = 1, \ldots, n \), \( i, j = 1, \ldots, n \). By the usual calculation, if

\[ T_{\alpha, \beta} = \prod_{i=1}^{m} T_{\alpha(i), \beta(i)} \]

then

\[ T_{\alpha, \beta}^\otimes = \xi_{\alpha, \gamma} \xi_{\gamma, \beta} \cdot \]

Thus the \( T_{\alpha, \beta}, \alpha, \beta \in \Gamma_n^m \), span \( L(\otimes V, \otimes V) \) so that we can write

\[ S = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta} T_{\alpha, \beta} \]

\[ = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta} \prod_{i=1}^{m} T_{\alpha(i), \beta(i)} \]

where of course \( ([S])_{\otimes}^{\otimes} \alpha, \beta = a_{\beta, \alpha} \). Then from (53),

\[ f(S) = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta} x(T_{\alpha(1), \beta(1)}, \ldots, T_{\alpha(m), \beta(m)}) \]

and hence by Theorem 2.6,

\[ P(\sigma)f(S) = \sum_{\alpha, \beta} a_{\alpha, \beta} P(\sigma)x(T_{\alpha(1), \beta(1)}, \ldots, T_{\alpha(m), \beta(m)}) \]

\[ = \sum_{\alpha, \beta} a_{\alpha, \beta} x(T_{\alpha^{-1}(1), \beta^{-1}(1)}, \ldots, T_{\alpha^{-1}(m), \beta^{-1}(m)}) \]

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\[ = \sum_{\alpha, \beta} a_{\alpha \beta} \kappa(T_\alpha(1), \beta(1), \ldots, T_\alpha(m), \beta(m)) \]

\[ = \sum_{\alpha, \beta} a_{\alpha \beta} \kappa(T_\alpha(1), \beta(1), \ldots, T_\alpha(m), \beta(m)) \]

\[ = f(S). \]

The general linear group on \( V, \ GL_n(V) \), is the group of all non-singular linear transformations \( T \in L(V, V), \dim V = n \). We can prove that \( GL_n(V) \) may be substituted for \( L(V, V) \) in (52). For, let \( x_{ij}, \ i, j = 1, \ldots, n, \) be \( n^2 \) independent indeterminates over \( R \) and let \( X = [x_{ij}] \). Let \( f \) be a linear functional on \( M_n(R), \ y = n^m \), for which \( f(\prod_{l=1}^m A) = 0 \) for any nonsingular \( A \in M_n(R). \) \( \prod_{l} A \) is the Kronecker product of \( A \) with itself \( m \) times.) Then obviously

\[ f(\prod_{l=1}^m A) \det A = 0 \]

for all \( A = [a_{ij}] \in M_n(R) \). Consider the polynomial

\[ g(X) \in R[x_{11}, \ldots, x_{nn}] \]

defined by

\[ g(X) = f(\prod_{l=1}^m X) \det X. \]

Then \( g(X) \) vanishes for all specializations of the \( x_{ij} \) to elements in \( R \) and hence by Exercise 14, \( g(X) \) is the 0 polynomial in

\( R[x_{11}, \ldots, x_{nn}] \). Since \( \det X \) is not the 0 polynomial and \( R[x_{11}, \ldots, x_{nn}] \) is an integral domain, it follows that \( f(\prod_{l=1}^m X) = 0 \)

is the 0 polynomial. But then \( f(\prod_{l=1}^m A) = 0 \) for all \( A \in M_n(R). \)

In other words, if \( f(\prod_{l=1}^m A) = 0 \) for all nonsingular \( A \), then

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\[ f(\Pi^m A) = 0 \text{ for all } A \in M_n(\mathbb{R}). \text{ It follows that} \]
\[ \langle \Pi^m A, \det A \neq 0, A \in M_n(\mathbb{R}) \rangle = \langle \Pi^m A, A \in M_n(\mathbb{R}) \rangle \]  
(58)

and hence by the usual procedure of taking matrix representations that
\[ \langle \Pi^m T, T \in GL_n(V) \rangle = \langle \Pi^m T, T \in L(V, V) \rangle. \]

Thus
\[ \rho_m = \langle \Pi^m T, T \in GL_n(V) \rangle. \]  
(59)

From time to time we denote the set of matrices in (58) by \( \rho_m \) and refer to any such matrix as being bisymmetric.

Exercises

1. Let \( A \in L(V, V) \). Show that there exists a unique linear map
\[ h_m : \bigotimes^m V \to \bigotimes^m V \text{ satisfying} \]
\[ h_m(\bigotimes x) = \sum_{i=1}^m x_1 \bigotimes \cdots \bigotimes x_{i-1} \bigotimes A x_i \bigotimes x_{i+1} \bigotimes \cdots \bigotimes x_m. \]

Hint: In fact,
\[ h_m = \sum_{i=1}^m I \bigotimes \cdots \bigotimes I \bigotimes A \bigotimes I \bigotimes \cdots \bigotimes I. \]

2. Show that formula (3) is valid if either \( \deg u = 0 \) or \( \deg v = 0 \).

Hint: If \( \deg u = 0 \), say \( u = r \in \mathbb{R} \) then \( u \bigotimes v = rv \), and \( J^p u = J^p r = r \) (i.e., from the definition of \( J \), \( Jr = r \)).
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Hence since \( h_0 = 0 \) we have \( h(u \otimes v) = h(rv) = rh(v) \) and

\[
h(u) \otimes v + J^p u \otimes h(v) = h_0(r) \otimes v + r \otimes h(v) = rh(v).
\]

If \( \deg v = 0 \), say \( v = r \), then the left side of (3) is \( rh(v) \) and the right side is \( h(v)r \). Note that this argument does not depend on the parity of \( p \).

3. Show that if \( J \) is the main involution in \( \mathbb{T}_0(v) \), then

\[
J^p x_1 \otimes \cdots \otimes x_m = \begin{cases} 
 x_1 \otimes \cdots \otimes x_m, & \text{if } p \text{ is even} \\
 (-1)^m x_1 \otimes \cdots \otimes x_m, & \text{if } p \text{ is odd.}
\end{cases}
\]

Hint: If \( m \) is even, then \( Jx^\otimes = x^\otimes \) so \( J^p x^\otimes = x^\otimes \). If \( m \) is odd \( Jx^\otimes = -x^\otimes \) so \( J^p x^\otimes = (-1)^p x^\otimes \). If \( p \) is even \( J^p x = x \). If \( p \) is odd \( J^p x = -x = (-1)^m x \).

4. Complete the proof of Theorem 2.1 in the case that \( p \) is even.

Hint: In the notation of Theorem 2.1

\[
h(u) \otimes v + J^p u \otimes h(v) \]

\[
= \sum_{k=1}^{m} x_1 \otimes \cdots \otimes D_{x^k} \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_{m+q} \\
+ \sum_{k=1}^{m+q} x_1 \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes D_{x^k} \otimes \cdots \otimes x_{m+q} \\
= \sum_{k=1}^{m+q} x_1 \otimes \cdots \otimes D_{x^k} \otimes \cdots \otimes x_{m+q} \\
= h(x_1 \otimes \cdots \otimes x_{m+q}) \\
= h(u \otimes v).
\]
5. Show that if $D$ is a derivation of the $\mathbb{Z}$-graded algebra $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} V(n)$ of degree $p$, then $D(r) = 0$ for any $r \in R$. Show also that if $\mathcal{A}$ is generated by $\{1_\mathcal{A}\} \cup V(1)$, then $D$ is completely determined by its values on $V(1)$. 

Hint: What is meant here is that $D(r1_\mathcal{A}) = 0$. Now since $D$ is linear we need only show $D(1_\mathcal{A}) = 0$. But since $\deg 1_\mathcal{A} = 0$ we have $J^p(1_\mathcal{A}) = 1_\mathcal{A}$ and hence

$$D(1_\mathcal{A}) = D(1_\mathcal{A}1_\mathcal{A}) = D(1_\mathcal{A})1_\mathcal{A} + J^p(1_\mathcal{A})D(1_\mathcal{A}) = 2D(1_\mathcal{A}).$$

Also since any element of $\mathcal{A}$ is a sum of finite products of elements in $\{1_\mathcal{A}\} \cup V(1)$, repeated application of the formula (1) defining a derivation shows that all values of $D$ are determined by the values on $V(1)$.

6. Let $v_i \in V$, $i = 1, \ldots, m$, be l.i. Show that if $g \in V^*$ and $g(v_i) \neq 0$ for some $i = 1, \ldots, m$, then there exists a basis $u_1, \ldots, u_m$ of $\langle v_1, \ldots, v_m \rangle$ such that $g(u_m) \neq 0$, $g(u_1) = \cdots = g(u_{m-1}) = 0$ and $u^A = v^A$.

Hint: We can assume $V = \langle v_1, \ldots, v_m \rangle$. Write $g = g_m \neq 0$ and augment to a basis $g_1, \ldots, g_{m-1}, g_m$ of $V^*$. Let $u_1, \ldots, u_m$ be a basis of $V$ to which $g_1, \ldots, g_m$ is dual. From Example 3.3 (c), Chapter 2 [see also Theorem 4.5 (b), Chapter 2] $u^A = sv^A$. Now replace $u_m$ by $a^{-1}u_m$.

7. Let $\mathcal{A}$ be an algebra over $R$ and let $I$ be a two-sided ideal in $\mathcal{A}$. Let $T: \mathcal{A} \to \mathcal{A}$ be an algebra homomorphism satisfying $I \subset \ker T$. If $q: \mathcal{A} \to \mathcal{A}/I$ is the quotient map, show that
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\( \overline{T}: \mathcal{U}/I \rightarrow \mathcal{U}/I, \quad \overline{T}q(w) = q(Tw) \), is a well-defined algebra homomorphism.

Hint: From Theorem 1.1 (a) \( \mathcal{U}/I \) is an algebra and \( q(x)q(y) = q(xy) \). Suppose \( q(w_1) = q(w_2) \) so that \( q(w_1 - w_2) = 0, w_1 - w_2 \in I \subset \ker T \). Then \( w_1 = w_2 + x, \ x \in \ker T \), and hence \( q(Tw_1) = q(T(w_2 + x)) = q(Tw_2) \). Thus \( \overline{T} \) is well defined. It is simple to verify that \( \overline{T} \) is an algebra homomorphism.

8. Verify the formula (29).

Hint: Let \( S \) denote the symmetrizer defined by \( H \) and \( \chi \).

From (25) we compute that

\[
(D(T_1^*, \ldots, T_p^*)u^*, v^*) = (u^*, D(T_1, \ldots, T_p)v^*) \\
= (S_u^\otimes, \delta(T_1, \ldots, T_p)S_v^\otimes) \\
= (S_u^\otimes, \delta(T_1^*, \ldots, T_p)v^\otimes) \\
= (S_u^\otimes, \sum_w \pi_w(T_1, \ldots, T_p)v^\otimes) \\
= \sum_w (S_u^\otimes, \pi_w(T_1, \ldots, T_p)v^\otimes) \\
= \sum_w (S\pi_w(T_1^*, \ldots, T_p^*)u^\otimes, v^\otimes) \\
= \sum_w (\pi_w(T_1^*, \ldots, T_p^*)S_u^\otimes, S_v^\otimes) \\
= (\sum_w \pi_w(T_1^*, \ldots, T_p^*)u^*, v^*) \\
= (\delta(T_1^*, \ldots, T_p^*)u^*, v^*) \\
= (D(T_1^*, \ldots, T_p^*)u^*, v^*).
\]
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9. Verify the formula (30).

Hint: We have

\[ D(T_1, \ldots, T_p)^* = D(T_1, \ldots, T_p) S v^\otimes \]
\[ = \delta(T_1, \ldots, T_p) S v^\otimes \]
\[ = S \delta(T_1, \ldots, T_p) v^\otimes \]
\[ = S \sum_w \pi_w(T_1, \ldots, T_p) v^\otimes \]
\[ \quad \downarrow \]
\[ w^i(1) \quad w^i(r_i) \]
\[ \downarrow \]
\[ S \sum_w \cdots \otimes T_i v^i(1) \otimes \cdots \otimes T_i v^i(r_i) \otimes \cdots \]
\[ \downarrow \]
\[ \sum_w \cdots \ast T_i v^i(1) \ast \cdots \ast T_i v^i(r_i) \ast \cdots . \]

10. Let \( \xi_1, \ldots, \xi_p \) be independent indeterminates over \( R \). Show that the polynomials \( \xi_1^{r_1} \cdots \xi_p^{r_p} \) are l.i. over \( R \) where \( r_1 + \cdots + r_p = m \) varies over all partitions of \( m \) into \( p \) nonnegative parts.

Hint:

\[ \sum_{r_1 + \cdots + r_p = m} c_{r_1} \cdots c_{r_p} \xi_1^{r_1} \cdots \xi_p^{r_p} = 0. \]

Then for a fixed partition \( k_1 + \cdots + k_p = m \) compute the partial derivative

\[ \frac{\partial^m}{\partial \xi_1^{k_1} \partial \xi_2^{k_2} \cdots \partial \xi_p^{k_p}} \]

to obtain \( k_1! \cdots k_p! c_{k_1} \cdots k_p = 0. \)
11. Let $E = \{e_1, \ldots, e_n\}$ be a basis of $V$ and $\{f_1, \ldots, f_n\}$ a
dual basis of $V^*$. Show that if $h_\beta \otimes = f_\beta(1) \cdots f_\beta(m)$, then
for $\alpha$ and $\beta$ in $\mathcal{A}$, $h_\beta(e_\alpha^*) = \delta_{\alpha\beta} \frac{\nu(\alpha)}{|H|}$.

Hint:

$h_\beta(e_\alpha^*) = h_\beta(S e_\alpha^*)$

$= h_\beta \left( \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) e_{\alpha \sigma^{-1}} \right)$

$= \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) h_\beta(e_{\alpha \sigma^{-1}})$

$= \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) f_\beta(1) \left(e_{\alpha \sigma^{-1}(1)}\right) \cdots f_\beta(m) \left(e_{\alpha \sigma^{-1}(m)}\right)$

$= \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) \delta_{\alpha \sigma^{-1}, \beta}$

$= \frac{\delta_{\alpha\beta}}{|H|} \sum_{\sigma \in H} \chi(\sigma)$

$= \delta_{\alpha\beta} \frac{\nu(\alpha)}{|H|}$.

12. Prove that if $A$ is an $n$-square complex normal matrix with
eigenvalues $\lambda_1, \ldots, \lambda_n$, then every main diagonal entry of $A$
is in the convex hull of the eigenvalues of $A$.

Hint: Let $A u_i = \lambda_i u_i$, where $\{u_1, \ldots, u_n\}$ is an o.n. basis
of eigenvectors. Let $e_i = (\delta_{i1}, \ldots, \delta_{in})$ and compute that

$a_{ii} = (A e_i, e_i) = \left( A \sum_{k=1}^{n} (e_i, u_k) u_k, \sum_{k=1}^{n} (e_i, u_k) u_k \right) = \sum_{k=1}^{n} |(e_i, u_k)|^2 \lambda_k$.
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Now
\[ \sum_{k=1}^{n} |(e_k, u_k)|^2 = \|e_1\|^2 = 1 \]
so that \( a_{ij} \) is a convex combination of \( \lambda_1, \ldots, \lambda_n \).

13. Prove that if \( T \) is normal, then \( \delta(T, I) \) is normal.

14. Prove Weyl's principle of the irrelevance of algebraic inequalities: Let \( I \) be an infinite integral domain and let \( \xi_1, \ldots, \xi_p \) be independent indeterminates over \( I \). Let
\( f(\xi_1, \ldots, \xi_p), g_t(\xi_1, \ldots, \xi_p) \) be polynomials in \( I[\xi_1, \ldots, \xi_p] \)
and suppose that \( f(a_1, \ldots, a_p) = 0 \) whenever \( g_t(a_1, \ldots, a_p) \neq 0 \),
t = 1, \ldots, r, for arbitrary \( a_i \in I \). Then \( f(\xi_1, \ldots, \xi_p) \) is
the 0 polynomial.

Hint: Discard any of the \( g_t(\xi_1, \ldots, \xi_p) \) that are the 0 polynomial and consider the product
\[ h(\xi_1, \ldots, \xi_p) = f(\xi_1, \ldots, \xi_p) \prod_{t=1}^{r} g_t(\xi_1, \ldots, \xi_p) \in I[\xi_1, \ldots, \xi_p]. \]

Then \( h(a_1, \ldots, a_p) = 0 \) for all \( a_i \in I \). Now it is a classic theorem in algebra that the only polynomial in \( I[\xi_1, \ldots, \xi_p] \),
which vanishes for all specializations of the \( \xi_i \) to the
infinite integral domain \( I \), is the 0 polynomial. (This is
proved by extending \( I \) to the quotient field, using the theorem
that a polynomial over a field only vanishes for finitely many
values of the indeterminate and then proceeding by induction on
\( p \).) But it is also classic that \( I[\xi_1, \ldots, \xi_p] \) is an integral
domain and hence has no zero divisors. Thus one of the factors of $h(\xi_1, \ldots, \xi_p)$ is 0 and none of the $g_l(\xi_1, \ldots, \xi_p)$ is 0. Thus $f(\xi_1, \ldots, \xi_p) = 0$.

15. Let $GL(n, R)$ be the group of all $n$-square nonsingular matrices in $M_n(R)$. Let $F: GL(n, R) \to GL(N, R)$, $N = n^m$, be a homogeneous polynomial representation of degree $m$, i.e., $F(XY) = F(X)F(Y)$ for all $X, Y \in M_n(R)$, and the entries of $F(X)$ are fixed homogeneous polynomials of degree $m$ in the entries of $X$. Show that there exists a unique algebra homomorphism $\Theta: \mathfrak{g}_m \to M_N(R)$ such that $F(X) = G(\Pi^l X)$ for all $X \in GL(n, R)$. [By $\mathfrak{g}_m$ we mean the set (58).]

Hint: Index the elements of $M_N(R)$ by $\Gamma_n^m$ so that the $\alpha, \beta$ entry of $F(X)$ is

$$F(X)_{\alpha, \beta} = \sum_{\gamma, \delta \in \Gamma_n^m} a_{\gamma, \delta}^\alpha \prod_{i=1}^m X_{\gamma(i), \delta(i)} = \sum_{\gamma, \delta} a_{\gamma, \delta}^\alpha \prod_{i=1}^m X_{\gamma(i), \delta(i)}.$$

Define $G: \mathfrak{g}_m \to M_N(R)$ by

$$G(Z)_{\alpha, \beta} = \sum_{\gamma, \delta} a_{\gamma, \delta}^\alpha Z_{\gamma, \delta}$$

for any $Z \in \mathfrak{g}_m$, so that $G$ is obviously linear. Then to show that $G$ is multiplicative, it clearly suffices to prove it for the generators of $\mathfrak{g}_m$. In view of (58) these are the matrices $\Pi^l X$, $X \in GL(n, R)$. But

$$G((\Pi^l X)(\Pi^l Y)) = G(\Pi^l XY) = F(XY) = F(X)F(Y) = G(\Pi^l X)G(\Pi^l Y)$$

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for all $X, Y \in GL(n, R)$. The uniqueness of $G$ is clear since

$H: R^m \to M_1^N(R)$ and $H(P \times Y) = F(X)$ for all $X \in GL(n, R)$,

$H(\Pi X) = G(\Pi X)$

for all $X \in GL(n, R)$. But the matrices $\Pi X, X \in GL(n, R)$,

span $R^m$.

16. Show that if $D_1$ and $D_2$ are derivations of $\mathcal{U}$ of degrees

$p_1$ and $p_2$, respectively, and $p_2 p_2$ is even, then $D_1 D_2 -

D_2 D_1$ is a derivation of $\mathcal{U}$.

Hint: Apply Theorem 2.2.

17. Show that if $D$ is a derivation of $\mathcal{U}$ of odd degree, then $D^2

is a derivation of $\mathcal{U}$.

Hint: Apply Theorem 2.2.

In Exercises 18-32 the special notations $\mu_X$, $L_X$, $D_X$, $M_X$, $\delta_X$,

$G(T_1, \ldots, T_m)$, and $G(T_1, \ldots, T_m)$ will be defined and used.

18. Let $x \in V$. Show that there exists a unique linear

$\mu_x: T_0(V) \to T_0(V)$ such that $\mu_x(r) = rx, r \in R$, and

$\mu_x(v^\otimes) = \sum_{k=0}^{m} (-1)^k v_1 \otimes \cdots \otimes v_k \otimes x \otimes v_{k+1} \otimes \cdots \otimes v_m$

for each $m \geq 1$, $v_i \in V$, $i = 1, \ldots, m$. The transformation

$\mu_x$ is called a multiplication operator. Show that $\mu_x(y) =

x \otimes y - y \otimes x$ and $\mu_x(x) = 0$, for any $x$ and $y$ in $V$.

Hint: The existence of $\mu_x$ is an easy consequence of the
universal factorization property of $\otimes V$ and the preceding
identities are trivial.
19. For any \( x \in V \) let \( I_x : T_0(V) \to T_0(V) \) denote left multiplication by \( x \), i.e., \( I_x(u) = x \otimes u \). Show that \( I_x \) is homogeneous of degree \( 1 \).

20. Let \( D_x = p_x - I_x \) as defined in the preceding two exercises. Prove that \( D_x \) is a derivation of degree \( 1 \) on \( T_0(V) \).

Hint: It suffices to prove that

\[
D_x(u \otimes v) = D_x(u) \otimes v + Ju \otimes D_x v,
\]

for \( u = x_1 \otimes \cdots \otimes x_m \), \( v = x_{m+1} \otimes \cdots \otimes x_n \). Now,

\[
D_x(u) \otimes v
= \left( (\mu_x - I_x)x_1 \otimes \cdots \otimes x_m \right) \otimes \left( x_{m+1} \otimes \cdots \otimes x_n \right)
= \sum_{k=0}^{m} (-1)^k x_1 \otimes \cdots \otimes x_k \otimes x \otimes x_{k+1} \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_n
- x \otimes x_1 \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_n
= \sum_{k=1}^{m} (-1)^k x_1 \otimes \cdots \otimes x_k \otimes x \otimes x_{k+1} \otimes \cdots \otimes x_n.
\]

Also,

\[
Ju \otimes D_x v
= (-1)^n x_1 \otimes \cdots \otimes x_m \otimes \left[ \sum_{k=0}^{n-m} (-1)^k x_{m+1} \otimes \cdots \otimes x_{m+k} \otimes x \otimes x_{m+k+1} \otimes \cdots \otimes x_{n} \right.
- x \otimes x_{m+1} \otimes \cdots \otimes x_{n} \right]
\approx \sum_{k=0}^{n-m} (-1)^{m+k} x_1 \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_{m+k} \otimes x \otimes x_{m+k+1} \otimes \cdots \otimes x_{n}
- (-1)^m x_1 \otimes \cdots \otimes x_m \otimes x \otimes x_{m+1} \otimes \cdots \otimes x_{n}.
\]
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\begin{align*}
&= \sum_{k=1}^{n-m} (-1)^{m+k} x_1 \otimes \cdots \otimes x_m \otimes x_{m+k} \otimes x_{m+k+1} \otimes \cdots \otimes x_n \\
&= \sum_{k=m+1}^{n} (-1)^k x_1 \otimes \cdots \otimes x_k \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_n.
\end{align*}

Finally,

\begin{align*}
D_x(u \otimes v)
&= \sum_{k=0}^{n} (-1)^k x_1 \otimes \cdots \otimes x_k \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_n - x \otimes x_1 \otimes \cdots \otimes x_n \\
&= \sum_{k=1}^{n} (-1)^k x_1 \otimes \cdots \otimes x_k \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_n.
\end{align*}

The result is now clear.

21. Show that for any \(u\) and \(v\) in \(T_0(V)\),

\[\mu_x(u \otimes v) = \mu_x(u) \otimes v + Ju \otimes D_x(v).\]

Hint: From Exercise 20,

\[D_x(u \otimes v) = D_x(u) \otimes v + Ju \otimes D_x(v).\]

But \(D_x = \mu_x - I_x\), so

\[\mu_x(u \otimes v) - I_x(u \otimes v) = (\mu_x(u) - I_x(u)) \otimes v + Ju \otimes D_x(v)\]

or

\[\mu_x(u \otimes v) - x \otimes u \otimes v = \mu_x(u) \otimes v - x \otimes u \otimes v + Ju \otimes D_x(v),\]

i.e.,

\[\mu_x(u \otimes v) = \mu_x(u) \otimes v + Ju \otimes D_x(v).\]
22. Show that $D_x(1) = 0$ and $D_x(y) = -y \otimes x$, $x, y \in V$.

Hint: Since $D_x$ is a derivation by Exercise 20, $D_x(1) = 0$ (see Exercise 18). From the definition of $\mu_x$ in Exercise 18, $\mu_x(y) = x \otimes y - y \otimes x$ so that $\mu_x(x) = 0$. Also

\[ D_x(y) = \mu_x(y) - L_x(y) = x \otimes y - y \otimes x - x \otimes y = -y \otimes x. \]

23. Show that for any $x \in V$, $\mu_x^2 = 0$, i.e., $\mu_x^2$ is the zero transformation on $T_0(V)$.

Hint: From Exercise 21, if $u \in T_0(V)$, then

\[ \mu_x(x \otimes u) = \mu_x(x) \otimes u + Jx \otimes D_x(u) \]
\[ = Jx \otimes (\mu_x(u) - L_x(u)). \]

This second equality comes from the fact that $\mu_x(x) = 0$ (see Exercise 22), and $D_x = \mu_x - L_x$. Continuing,

\[ \mu_x(x \otimes u) = -x \otimes (\mu_x(u) - L_x(u)) \]
\[ = -L_x \mu_x(u) + L_x^2(u). \]

Since $x \otimes u = L_x(u)$, we have

\[ \mu_x L_x + L_x \mu_x = L_x^2. \]

Thus

\[ D_x^2 = (\mu_x - L_x)^2 \]
\[ = \mu_x^2 - (\mu_x L_x + L_x \mu_x) + L_x^2 = \mu_x^2. \]

By Exercise 20, $D_x$ is a derivation of degree 1 so that by Exercise 17, $D_x^2$ is a derivation. But by Exercise 22, if
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\[ y \in V, \text{ then} \]
\[
D_x^2(y) = D_x D_x(y)
= D_x(-y \otimes x)
= -D_x(y) \otimes x - Jy \otimes D_x(x)
= y \otimes x \otimes x + y \otimes (-x \otimes x)
= y \otimes x \otimes x - y \otimes x \otimes x
= 0.
\]

From Exercise 5, \( D_x^2 \) must be 0 on all of \( T_0(V) \) because it is 0 on \( V \).

24. Let \( x_1, \ldots, x_m, x_{m+1} \) be in \( V \). Then show that

\[
\mu_{x_{m+1}}(x_1 \wedge \cdots \wedge x_m) = (m+1)(-1)^{m+1} x_1 \wedge \cdots \wedge x_m \wedge x_{m+1}.
\]

Hint: Let \( h_1, \ldots, h_{m+1} \) be arbitrary in \( V^* \), and define

\[
f \in \left( \bigotimes_{1}^{m+1} V \right)^*
\]

by

\[
f \otimes = \prod_{t=1}^{m+1} h_t.
\]

Then

\[
f(\mu_{x_{m+1}}(x^\otimes)) = f \mu_{x_{m+1}} \left( \frac{1}{m!} \sum_{\sigma \in S_m} e(\sigma)x_\sigma ^\otimes \right) = \frac{1}{m!} \sum_{\sigma \in S_m} e(\sigma)f \mu_{x_{m+1}}(x_\sigma ^\otimes).
\]

If we use the definition of \( \mu_{x_{m+1}} \) in Exercise 13 and interchange the order of summation, we obtain from the Laplace expansion,
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\[ f(\mu_{x_{m+1}}(x^\lambda)) \]

\[ = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k h_{k+1}(x_{m+1}) \sum_{\sigma \in S_m} \epsilon(\sigma) h_1(x_{\sigma(1)}) \cdots h_k(x_{\sigma(k)}) \cdot h_{k+2}(x_{\sigma(k+1)}) \cdots h_{m+1}(x_{\sigma(m)}) \]

\[ = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k h_{k+1}(x_{m+1}) \det \begin{bmatrix}
    h_1(x_1) & \cdots & h_k(x_1) & h_{k+2}(x_1) & \cdots & h_{m+1}(x_1) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    h_1(x_m) & \cdots & h_k(x_m) & h_{k+2}(x_m) & \cdots & h_{m+1}(x_m)
\end{bmatrix} \]

\[ = (-1)^m \frac{1}{m!} \det [h_j(x_1)], \quad i, j = 1, \ldots, m+1 \]

\[ = (m+1)(-1)^m f(x_1 \wedge \cdots \wedge x_{m+1}). \]

(See formula (45), Example 4.5 (d), Chapter 2.) We normalize \( \mu_x \) as follows: \( M_x(u) = \frac{1}{m+1} \mu_x(u) \), if \( \deg u = m \), so that the preceding formula immediately implies that

\[ M_x(y_1 \wedge \cdots \wedge y_m) = x \wedge y_1 \wedge \cdots \wedge y_m. \]

25. Let \( \varphi \in \mathcal{M}_2(V, R, S_2, \mathcal{S}) \), i.e., \( \varphi: V \times V \rightarrow R \) is bilinear and \( \varphi(x, y) = \varphi(y, x) \) for all \( x \) and \( y \). For a fixed \( x \in V \), let \( g \in V^* \) be defined by \( g(y) = \varphi(x, y) \). Instead of writing \( \delta_g \) (see Example 2.2) we write \( \delta_x \), so that from formula (6)

\[ \delta_x(x_1 \otimes \cdots \otimes x_m) = \sum_{k=1}^{m} (-1)^{k-1} \varphi(x_1, x_k^\otimes) x_k^\otimes. \]

Show that \( \delta_x \) is linear in \( x \), i.e., \( \delta_{ax + bz} = a \delta_x + b \delta_z \).
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\(a, b \in \mathbb{R}, \ x, z \in V.\)

Hint: This is obvious from the bilinearity of \(\varphi.\)

26. For the fixed symmetric bilinear \(\varphi\) in Exercise 25 define
\[P_x : T_0(V) \to T_0(V)\] by \(P_x = M_x + \delta_x\) (\(M_x\) is defined in Exercise 24). Show that \(P_x\) is linear in \(x.\) Also show that
\[P_x(x_1 \wedge \cdots \wedge x_m) = x \wedge x_1 \wedge \cdots \wedge x_m + \sum_{k=1}^{m} (-1)^{k-1} \varphi(x_k, x) x_k.\]

Hint: This follows from Exercises 24 and 25 and Theorem 2.3 [i.e., formula (11)].

27. Prove that the mapping \(x \to P_x\) is injective.

Hint: If \(P_x = P_z,\) then \(P_x(1) = P_z(1)\) and \((M_x + \delta_x)(1) = (M_z + \delta_z)(1).\) Now \(\delta_x\) is a derivation, and hence from Exercise 5, \(\delta_x(1) = 0.\) Also from Exercise 18, \(M_x(1) = x\) so \(M_x(1) = M_z(1)\) implies \(x = z.\)

28. Prove that as an operator on the Grassmann algebra, \(P^2_x = \varphi(x, x) I_{\wedge V}.\) In other words, \(P^2_x\) holds the \(m^{th}\) Grassmann space invariant for each \(m = 0, 1, \ldots\) and in fact \(P^2_x\) is a multiple of the identity on \(\wedge V.\)

Hint:
\[P^2_x = (M_x + \delta_x)^2 = M_x^2 + M_x \delta_x + \delta_x M_x + \delta_x^2\]

and from Example 2.3 (c), \(\delta_x^2 = 0.\) If \(m = 0,\) then \(M^2_x(1) = \mathcal{L}_x(M_x(1)) = M_x(x) = 0\) (see Exercises 18 and 22). Also
Theorem 2.3, 
\[ \delta_x M_x(y^\wedge) = \delta_x (x \wedge y^\wedge) = \varphi(x, x)y^\wedge + \sum_{k=1}^{m} (-1)^k \varphi(x, y_k)x \wedge y_k^\wedge \]
and
\[ M_x \delta_x (y^\wedge) = M_x \sum_{k=1}^{m} (-1)^{k-1} \varphi(x, y_k)y_k^\wedge = \sum_{k=1}^{m} (-1)^{k-1} \varphi(x, y_k)x \wedge y_k^\wedge . \]
Thus \( M_x \delta_x (y^\wedge) + \delta_x M_x (y^\wedge) = \varphi(x, x)y^\wedge . \)

29. Show that if \( x \) and \( y \) are in \( V \), then \( P_x P_y + P_y P_x = 2\varphi(x, y)I_{\wedge V} \). In other words, \( P_x P_y + P_y P_x - 2\varphi(x, y)I_{\wedge V} \) is the zero transformation on \( \wedge V \).

Hint: From Exercises 26 and 28 and the symmetry of \( \varphi \), we compute that
\[ P_{x+y}^2 = \varphi(x+y, x+y)I_{\wedge V} = \varphi(x, x) + 2\varphi(x, y) + \varphi(y, y)I_{\wedge V} \]
and
\[ P_{x+y}^2 = (P_x + P_y)^2 = P_x^2 + P_y^2 + P_x P_y + P_y P_x \]
\[ = \varphi(x, x)I_{\wedge V} + \varphi(y, y)I_{\wedge V} + P_x P_y + P_y P_x . \]
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30. Let \( T_i \in \mathcal{L}(V, V) \), \( i = 1, \ldots, m \), \( H \) a subgroup of \( S_m \), and \( \chi \) a character of degree 1. Define the \textit{symmetrized product} of \( T_1, \ldots, T_m \) by the formula

\[
\mathcal{S}(T_1, \ldots, T_m) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma^{-1}) T_{\sigma(1)} \cdots T_{\sigma(m)}.
\]

When \( H = S_m \) and \( \chi = \varepsilon \), then \( \mathcal{S}(T_1, \ldots, T_m) \) is written with a subscript: \( \mathcal{S}_m(T_1, \ldots, T_m) \). (If \( m = 0 \), \( \mathcal{S}_0 = \mathcal{L}_V \) by convention.) Show that \( \mathcal{S}(T_1, \ldots, T_m) \) is linear in each \( T_i \) and symmetric with respect to \( H \) and \( \chi \). Let \( \mathcal{L} = \mathcal{L}(V, V) \) and prove that there exists a unique linear \( f: \mathcal{L}_m^m(H) \to \mathcal{L} \) such that \( f(T_1 \ast \cdots \ast T_m) = \mathcal{S}(T_1, \ldots, T_m) \).

Hint: The first assertion is routine. The second follows in the usual way from the universal properties of \( T_1 \ast \cdots \ast T_m \).

31. Show that the following recursion formula holds for \( \mathcal{S}_m \):

\[
\mathcal{S}_m(T_1, \ldots, T_m) = \frac{1}{m} \sum_{k=1}^{m} (-1)^{k-1} T_k \mathcal{S}_{m-1}(T_1, \ldots, T_k, \ldots, T_m).
\]

Hint: Let \( L_j = T_j \), \( j = 1, \ldots, k-1 \), and \( L_k = T_{k+1}, \ldots, L_{m-1} = T_m \). Let \( e^k \) be the bijection on \( \{1, \ldots, m-1\} \) onto \( \{1, \ldots, k-1, k+1, \ldots, m\} \) defined by \( e^k(j) = j \), \( j = 1, \ldots, k-1 \), and \( e^k(j) = j+1 \), \( j = k, \ldots, m-1 \). Then \( L_j = T_{e^k(j)} \), \( j = 1, \ldots, m-1 \), so that
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\[ \Sigma_{m-1}(T_1, \ldots, T_k, \ldots, T_m) = \Sigma_{m-1}(L_1, \ldots, L_{m-1}) \]
\[ = \frac{1}{(m-1)!} \sum_{\varphi \in S_{m-1}} \varepsilon(\varphi) \prod_{l=1}^{m-1} T_{\varphi(l)} \]
\[ = \frac{1}{(m-1)!} \sum_{\varphi \in S_{m-1}} \varepsilon(\varphi) \prod_{l=1}^{m-1} T_{e_k \varphi(l)}. \]

Thus the required identity is equivalent to
\[ \sum_{\sigma \in S_m} \varepsilon(\sigma) \prod_{i=1}^{m} T_{\sigma(i)} = \sum_{k=1}^{m} (-1)^{k-1} T_k \sum_{\varphi \in S_{m-1}} \varepsilon(\varphi) \prod_{i=1}^{m-1} T_{e_k \varphi(i)}. \]

The terms on either side of this statement are formally identical if the sign is disregarded. Thus consider a typical term on the right: \((-1)^{k-1} \varepsilon(\varphi) T_k \prod_{i=1}^{m-1} T_{e_k \varphi(i)}\). The term on the left in which the factors in the product of the \(T_j\)'s occur in the same order corresponds to the permutation
\[ \sigma = \begin{pmatrix} 1 & 2 & \cdots & m \\ k & e_k \varphi(1) & \cdots & e_k \varphi(m-1) \end{pmatrix}. \]

Define \(e_k \in S_m\) by \(e_k(m) = k\) and \(e_k(j) = e_k(j), j = 1, \ldots, m-1\) and define \(\varphi \in S_m\) by \(\varphi(m) = m\) and \(\varphi(j) = \varphi(j), j = 1, \ldots, m-1\). Then \(e_k \varphi(m) = k\) and \(e_k \varphi(j) = e_k \varphi(j), j \neq m,\) and \(\varepsilon(\varphi) = \varepsilon(\varphi).\) Moreover,
\[ \sigma = \begin{pmatrix} 1 & 2 & \cdots & m \\ e_k \varphi(m) & e_k \varphi(1) & \cdots & e_k \varphi(m-1) \end{pmatrix} = \begin{pmatrix} 1 & \cdots & m \\ e_k \varphi(\theta(1)) & \cdots & e_k \varphi(\theta(m)) \end{pmatrix}, \]

where \(\theta\) is the \(m\)-cycle \(\theta = (m \ m-1 \ m-2 \ \cdots \ 2 \ 1).\) Thus
\[ \varepsilon(\phi) = \varepsilon(\varepsilon^k) \varepsilon(\phi) \varepsilon(\theta) = \varepsilon(\varepsilon^k) \varepsilon(\phi)(-1)^{m-1}. \]

Now
\[ \varepsilon^k = \left( \begin{array}{cccccccc} 1 & \cdots & k-1 & k & k+1 & \cdots & m-1 & m \\ l & \cdots & k-1 & k+1 & k+2 & \cdots & m & k \end{array} \right) = \left( \begin{array}{cccccccc} k & k+1 & \cdots & m-1 & m \end{array} \right) \]

so that \( \varepsilon(\varepsilon^k) = (-1)^{m-k} \). Hence
\[ \varepsilon(\phi) = (-1)^{2m-k-1} \varepsilon(\phi) = (-1)^{k-1} \varepsilon(\phi). \]

32. Let \( x_1, \ldots, x_m \) be vectors in \( V \). Show that
\[ \mathfrak{S}_m(x_1, \ldots, x_m)(1) = x_1 \wedge \cdots \wedge x_m. \]

Hint: From Exercise 26 we know that each \( P_x \) maps the Grassmann algebra (as a vector space) into itself so that \( \mathfrak{S}_m(x_1, \ldots, x_m) \) can be regarded as a mapping on \( \wedge V \). The argument is by induction on \( m \). For \( m = 1 \),
\[ P_{x_1}(1) = M_{x_1}(1) + \delta_{x_1}(1) = M_{x_1}(1) = x_1. \]

Suppose the result is valid for \( m-1 \) or fewer vectors. Then from Exercise 31, the induction hypothesis, and Exercise 26
\[ \mathfrak{S}_m(x_1, \ldots, x_m)(1) \]
\[ = \frac{1}{m} \sum_{k=1}^m (-1)^{k-1} P_{x_k} x_k^\wedge \]
\[ = \frac{1}{m} \sum_{k=1}^m (-1)^{k-1} \left[ x_k^\wedge x_k^\wedge + \sum_{j=1}^{k-1} (-1)^{j-1} \phi(x_k^\wedge, x_j) z_{jk} \right. \]
\[ + \left. \sum_{j=k+1}^m (-1)^j \phi(x_k^\wedge, x_j) z_{kj} \right], \]
where $z_{kj}$ is the exterior product (in order) of the vectors $x_1, \ldots, x_j, \ldots, x_k, \ldots, x_m$. Obviously $z_{kj} = z_{jk}$ and moreover we extend the definition of $z_{jk}$ to $z_{jj} = 0$, $j = 1, \ldots, m$.

Since $x_k \wedge x_k = (-1)^{k-1} x_k$, we have

$$\Theta_m(P_{x_1}, \ldots, P_{x_m})(1) = x_1 \wedge \cdots \wedge x_m +$$

$$\sum_{k=1}^{m} (-1)^{k-1} \left[ \sum_{j<k} (-1)^{j-1} \varphi(x_k, x_j) z_{jk} + \sum_{j>k} (-1)^{j} \varphi(x_j, x_k) z_{kj} \right].$$

Write $(-1)^j = (-1)^{j-1}$ so that the second term in the preceding sum becomes

$$\sum_{k,j=1}^{m} (-1)^{k-1} (-1)^{j-1} [\varphi(x_k, x_j) - \varphi(x_j, x_k)] z_{jk},$$

which in view of the symmetry of $\varphi$ is 0.

In Exercises 33–37 the notations $\Omega(T_1, \ldots, T_m)$, $\varphi, \psi, \chi, \kappa$ are used.

33. Let $T_i \in L(V, V)$, $i = 1, \ldots, m$, and define $\theta : x \in V \to V$ by

$$\theta(v_1, \ldots, v_m) = \frac{1}{|H|} \sum_{\sigma \in H} T_{\sigma(1)} v_1 \otimes \cdots \otimes T_{\sigma(m)} v_m.$$ 

Show that $\theta$ is symmetric with respect to $H$ and $\chi$ and hence that there exists a unique $\Omega(T_1, \ldots, T_m) \in L(V^H, V^H)$, which satisfies

$$\Omega(T_1, \ldots, T_m) v^* = \theta(v_1, \ldots, v_m).$$

The transformation $\Omega(T_1, \ldots, T_m)$ is called the star product of
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\[ T_1, \ldots, T_m. \]

Hint: Let \( \varphi \in H \) so that

\[ \theta(\varphi(1), \ldots, \varphi(m)) = \frac{1}{|H|} \sum_{\sigma \in H} T_{\sigma(1)}^{\varphi(1)} \ast \cdots \ast T_{\sigma(m)}^{\varphi(m)}. \]

Set \( u_t = T_{\sigma(t)}^{\varphi(t)} \) so that \( u_{\varphi^{-1}(t)} = T_{\sigma \varphi^{-1}(t)}^{\varphi(t)} \) and

\[ \chi(\varphi^{-1})u^* = u_{\varphi^{-1}}^* = T_{\sigma \varphi^{-1}(1)}^{\varphi(1)} \ast \cdots \ast T_{\sigma \varphi^{-1}(m)}^{\varphi(m)}. \]

Thus

\[ \theta(\varphi(1), \ldots, \varphi(m)) = \chi(\varphi) \frac{1}{|H|} \sum_{\sigma \in H} T_{\sigma \varphi^{-1}(1)}^{\varphi(1)} \ast \cdots \ast T_{\sigma \varphi^{-1}(m)}^{\varphi(m)} \]

and \( \sigma \varphi^{-1} \) runs over \( H \) as \( \sigma \) does.

34. Show that \( \Omega(T_1, \ldots, T_m) \) is linear in each \( T_i \) and \( \Omega(T, \ldots, T) = K(T) \).

35. Let \( \mathcal{L} = L(V, V) \) and let \( (Q, \kappa) \) denote the symmetry class over \( \mathcal{L} \) associated with the group \( H \) and character identically \( 1 \), i.e., \( Q = \mathcal{L}_1^H \). Let \( (P, \ast) \) denote the symmetry class over \( V \) associated with the group \( H \) and character \( \chi \), i.e., \( P = V_\chi^H \). Show that there exists a unique surjective linear transformation \( f: Q \to L(P, P) \) such that

\[ f(\kappa(T_1, \ldots, T_m)) = \Omega(T_1, \ldots, T_m). \]

Hint: This result should be compared with Theorem 2.5. The only difference is that \( f \) is always surjective in this case, whereas in Theorem 2.5, \( f \) is not general surjective (see the
36. Let \( \tau: r_1 + \cdots + r_p = m \) be a partition of \( m \) into \( p \) non-negative integers. Let \( \alpha^t \in Q_{r_t,m}^t, \ t = 1, \ldots, p \), be fixed nonoverlapping sequences (i.e., \( \Im \alpha^s \cap \Im \alpha^t = \emptyset, \ s \neq t \)).

For each \( p \)-tuple of sequences \( w_1, \ldots, w_p, \ w^t \in Q_{r_t,m}^t, \ t = 1, \ldots, p \), let
\[
g_{\alpha}(w_1, \ldots, w_p) = |\{ \sigma \in H | \Im \sigma \alpha^t = \Im w^t \}|,
\]
i.e., \( g_{\alpha}(w_1, \ldots, w_p) \) is the number of permutations \( \sigma \in H \) for which the two unordered sets \( \{ \sigma \alpha^t(1), \ldots, \sigma \alpha^t(r_t) \} = \{ w^t(1), \ldots, w^t(r_t) \} \) are the same. Show that the nonzero values of \( g_{\alpha} \) are all equal. We thus let \( Q_{\alpha}(\tau, H) \) denote the set of all \( p \)-tuples of sequences \( (w_1, \ldots, w_p), \ w^t \in Q_{r_t,m}^t, \ t = 1, \ldots, p \), for which there exists \( \sigma \in H \) such that \( \Im \sigma \alpha^t = \Im w^t \), i.e., \( (w_1, \ldots, w_p) \in Q_{\alpha}(\tau, H) \), iff \( g_{\alpha}(w_1, \ldots, w_p) > 0 \).

Hint: Suppose that \( g_{\alpha}(w_1, \ldots, w_p) \) and \( g_{\alpha}(\gamma_1, \ldots, \gamma_p) \) are positive. Then there exist \( \sigma_0 \) and \( \sigma_1 \) in \( H \) for which
\[
g_{\sigma_0}(\alpha^t(1), \ldots, \alpha^t(r_t)) = [w^t(1), \ldots, w^t(r_t)], \ t = 1, \ldots, p
\]
and
\[
g_{\sigma_1}(\alpha^t(1), \ldots, \alpha^t(r_t)) = [w^t(1), \ldots, w^t(r_t)], \ t = 1, \ldots, p.
\]
Let \( A_w = \{ \sigma \in H | \Im \sigma \alpha^t = \Im w^t, \ t = 1, \ldots, p \} \). Then
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\[ g_\alpha(\omega^1, \ldots, \omega^p) = |A_w|, \text{ and if } \sigma \in A_w, \text{ then} \]
\[ \sigma^{-1}_o \sigma(\omega^1, \ldots, \omega^p) = (\alpha^1(1), \ldots, \alpha^r(r_t)), \quad t = 1, \ldots, p. \]

If \( \varphi \in A_\gamma \), then
\[ \varphi(\alpha^1(1), \ldots, \alpha^r(r_t)) = (\gamma^1(1), \ldots, \gamma^r(r_t)), \quad t = 1, \ldots, p \]
and hence
\[ \varphi \sigma^{-1}_o \sigma(\omega^1, \ldots, \omega^p) = (\gamma^1, \ldots, \gamma^p), \quad t = 1, \ldots, p. \]
Hence \( \varphi \sigma^{-1}_o A_w \subseteq A_\gamma \). But \( |\varphi \sigma^{-1}_o A_w| = |A_w| \) so that \( |A_w| \leq |A_\gamma| \) and similarly \( |A_\gamma| \leq |A_w| \).

37. Let \( \tau \) and \( \alpha \) be as in Exercise 36 and let \( T^1, \ldots, T^p \) be l.t.'s in \( L(V, V) \). Define
\[ \Omega_\alpha(T^1, \ldots, T^p) = \Omega(\ldots, T^1, \ldots, T^1, \ldots), \]
i.e., \( T^1 \) appears in positions \( \alpha^1(1), \ldots, \alpha^r(r_t), \quad t = 1, \ldots, p \).
Show that
\[ \omega^1(1) \quad \omega^p(r_t) \]
\[ \Omega_\alpha(T^1, \ldots, T^p)^* = \frac{g}{|H|} \sum_{w \in C_\alpha} \cdots \cdots * T^1 v^* \quad * \cdots * T^p w^* \quad * \cdots * T^1 v^* \quad * \cdots \]
where \( g \) is the constant positive value of \( g_\alpha \).

Hint: According to Exercise 33, \( \Omega(X^1 \ldots X^m)^* \) involves terms in which a typical summand is \( X_\sigma(1) v^1 * \cdots * X_\sigma(m) v^m \). According to the definition of \( \Omega_\alpha(T^1, \ldots, T^p) \),
\[ X_\alpha^1(1) = \cdots = X_\alpha^r(r_t) = T^1, \quad t = 1, \ldots, p, \]
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so that \( T_t \) appears in positions \( w^t(l), \ldots, w^t(r_t) \),
\( t = 1, \ldots, p \), iff there is \( \sigma \in \mathcal{H} \) for which

\[
(\alpha^t(1), \ldots, \alpha^t(r_t)) = (\sigma w^t(1), \ldots, \sigma w^t(r_t)),
\]
\( t = 1, \ldots, p \), i.e. (since \( \sigma^{-1} \) and \( \sigma \) both run over \( \mathcal{H} \)), \( T_t \)
appears in the positions \( w^t(1), \ldots, w^t(r_t) \), \( t = 1, \ldots, p \),
exactly \( g_\alpha(w^1, \ldots, w^p) \) times and since the positive values of
\( g_\alpha(w^1, \ldots, w^p) \) are constant we obtain the required formula.

38. Let \( z \in \wedge^V \), \( m \geq 1 \), \( z \neq 0 \). Show that there exist \( m \) deri-
vations \( \delta_1, \ldots, \delta_m \) such that \( \delta_1 \cdots \delta_m(z) \neq 0 \).

Hint: Let \( e_1, \ldots, e_n \) be a basis of \( V \) and write

\[
z = \sum_{w \in Q_{m,n}} a_w e_w
\]
with \( a_\gamma \neq 0 \). Let \( f_1, \ldots, f_n \) be a basis of \( V^* \) dual to
\( e_1, \ldots, e_n \) and set \( g_1 = f_\gamma(m)^* \), \( \ldots, g_m = f_\gamma(1)^* \). Then from
Theorem 2.3

\[
\delta_{f_j} (e_\omega^*) = \sum_{k=1}^m (-1)^k f_j (e_{w(k)}) e_{w(1)} \wedge \cdots \wedge e_{w(k-1)} \wedge e_{w(m)}
\]

Obviously \( \delta_{f_j} (e_\omega^*) = 0 \) unless \( j \in \text{Im } w \). It follows that
\( \delta_1 \cdots \delta_m (e_\omega^*) = 0 \) unless \( \text{Im } \gamma \subset \text{Im } w \), i.e., \( w = \gamma \) since
\( w, \gamma \in Q_{m,n} \). Also

\[
\delta_{g_1} \cdots \delta_{g_m} (e_\omega^*) = \delta_{g_1} \cdots \delta_{g_{m-1}} e_\gamma(2) \wedge \cdots \wedge e_\gamma(m) = \cdots = 1.
\]
Thus \( \delta_{g_1} \cdots \delta_{g_m} (z) = a_\gamma \neq 0 \).
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39. Prove that if \( z \in \wedge V \), then \( D(z) = 0 \) for every derivation of \( \wedge V \) of degree \(-1\) iff \( z \in R \).

Hint: If \( z \in R \), then \( D(z) = 0 \) (see Exercise 5). Conversely, let \( z = \sum_{k=0}^{n} z_k \), \( z_k \in \wedge V \), and suppose \( z \notin R \). Then for some \( m > 1 \), \( z_m \neq 0 \) and \( z_1 = \cdots = z_{m-1} = 0 \). Then by Exercise 38 there exists \( \delta_g \) such that \( \delta_g(z_m) \neq 0 \) and \( \delta_g(z) = \delta_g(z_m) + \delta_g(z_{m+1}) + \cdots + \delta_g(z_n) \) with deg \( \delta_g(z_k) = k-1 \). Hence \( \delta_g(z) \neq 0 \). Since \( \delta_g \) is a derivation of degree \(-1\) it follows that if \( z \notin R \) there exists a derivation \( D \) of degree \(-1\) for which \( D(z) \neq 0 \).

40. Prove that if \( D \) is a derivation of \( \wedge V \) of degree \(-1\) and \( m \geq 2 \), then \( (x_1 \land \cdots \land x_m) \land D(x_1 \land \cdots \land x_m) = 0 \).

Hint:
\[
D(x_1 \land \cdots \land x_m) = D(x_1 \land (x_2 \land \cdots \land x_m)) = D(x_1) \land (x_2 \land \cdots \land x_m) + (x_1) \land D(x_2 \land \cdots \land x_m).
\]

Now \( J^2 = I \) so \( J^{-1} = J \) and \( Jx_1 = -x_1 \). Thus \( x_1 \land \cdots \land x_m \land D(x_1 \land \cdots \land x_m) = 0 \)
since the product is a sum of exterior products in which at least two factors are the same element of \( V \).

41. Let \( z_1, \ldots, z_p \) be decomposable elements in \( \wedge V \) of positive even degree and suppose that \( z_1 + \cdots + z_p = 0 \). Then for each \( m, \ 1 \leq m \leq p, \)
\[
\sum_{\omega \in Q_{m,p}} z_{\omega(1)} \land \cdots \land z_{\omega(m)} = 0.
\]

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Hint: From Example 1.5(e), we know that \( z_s \wedge z_t = z_t \wedge z_s \)
and \( z_s \wedge z_s = 0 \) (because \( z_s \) is decomposable). Then by the
multinomial theorem

\[
0 = (z_1 + \cdots + z_p)^m = \sum_{k_1 + \cdots + k_p = m} \binom{m}{k_1, \ldots, k_p} z_1^{k_1} \wedge z_2^{k_2} \wedge \cdots \wedge z_p^{k_p}.
\]

Now any term for which \( k_i > 1 \) must be 0. Thus \( k_i = 0 \) or
\( 1, \ i = 1, \ldots, p \). Hence \( m \) of the \( k_i \) are 1, the rest 0
and for such a choice \( \binom{m}{k_1, \ldots, k_p} = m! \). Thus the indicated
expansion becomes

\[
m! \sum_{w \in Q_{m,p}} z_{w(1)} \wedge \cdots \wedge z_{w(m)}.
\]

42. Let \( z_1, \ldots, z_p \) and \( w_1, \ldots, w_q \) be decomposable elements in
\( \wedge V \) each of even degree. Show that if \( z_1 + \cdots + z_p = w_1 + \cdots + w_q \),
then

\[
(1+z_1) \wedge \cdots \wedge (1+z_p) = (1+w_1) \wedge \cdots \wedge (1+w_q).
\]

Hint: Let \( z_{p+1} = -w_1 \) and \( z_{p+2} = -w_2, \ldots, z_{p+q} = -w_q \) so that
\( z_1 + \cdots + z_p + z_{p+1} + \cdots + z_{p+q} = 0 \). Hence by Exercise 41

\[
\sum_{w \in Q_{m,p+q}} z_{w(1)} \wedge \cdots \wedge z_{w(m)} = 0,
\]

\( m = 1, \ldots, p+q \). Since the \( z_1, \ldots, z_{p+q} \) pairwise commute, we
conclude after expanding that \( (1+z_1) \wedge \cdots \wedge (1+z_{p+q}) = 1 \).
But then \( (1+z_1) \wedge \cdots \wedge (1-z_p) \wedge (1-w_1) \wedge \cdots \wedge (1-w_q) = 1 \).
Also \((1 - w_t) \wedge (1 + w_t) = 1\) so that
\[
(1 + z_1) \wedge \cdots \wedge (1 + z_p) = (1 + w_1) \wedge \cdots \wedge (1 + w_q).
\]

43. Let \(\dim V = n\). Let \((Q, \chi)\) be the symmetry class over \(L = L(V, V)\) associated with the group \(S_n\) and the character \(1\). Prove that there exists a unique linear functional \(\text{ald}_r : Q \to R\), which satisfies
\[
\text{ald}_r(\chi(T, \ldots, T, I, \ldots, I)) = E_r(T)
\]
where \(E_r(T)\) is the \(r\)th elementary symmetric function of the eigenvalues of \(T\). (In the case \(r = n\), \(\text{ald}_n\) is called Iversen's linear determinant.)

Hint: According to Example 2.8(a), if \(T\) is the partition given by \(p = 2, r_1 = r,\) and \(r_2 = n - r\) and we take the symmetry class to be the 1-dimensional \(n\)th Grassmann space over \(V\), then \(D(T, I)\) possesses the single e.v. \(E_r(T)\) (i.e., it is an l.t. on a 1-dimensional space \(\bigwedge^1 V\)). According to Theorem 2.5 there exists a unique \(f : Q \to L(\bigwedge^n V, \bigwedge^n V)\) such that
\[
f(\chi(T, \ldots, T, I, \ldots, I)) = r! (n-r)! D(T, I).
\]

Clearly \(\text{ald}_r = \frac{1}{r!(n-r)!} \text{tr} f\).

44. Let \(A_1, \ldots, A_p\) be p.s.d. \(n\)-square hermitian matrices and let \(\mu_1, \ldots, \mu_p\) be arbitrary complex numbers. If \(H \subset S_n\) and \(\chi\) is a character of degree 1, show that
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\[ \left| d^H_X \left( \sum_{k=1}^{p} \mu_k A_k \right) \right| \leq d^H_X \left( \sum_{k=1}^{p} |\mu_k| A_k \right). \]

Hint: Let \( E = \{ e_1, \ldots, e_n \} \) be an o.n. basis of a unitary space \( V \) for which \( A_1^T = [T_1]^E \), where \( T_1 \) are p.s.d. hermitian transformations in \( L(V, V) \). We compute that

\[ \left| d^H_X \left( \sum_{k=1}^{p} \mu_k A_k \right) \right| = \left| d^H_X \left( \left( \sum_{k=1}^{p} \mu_k e_k \right) e_j \right) \right| = |H| (K \left( \sum_{k=1}^{p} \mu_k \right) e_1 * \ldots * e_n, e_1 * \ldots * e_n) = |H| \left( \sum_{r_1 + \ldots + r_p = m} \mu_1 ^{r_1} \ldots \mu_p ^{r_p} D(T_1, \ldots, T_p) e_1 *, e_1 * \right) \]  

\[ = |H| \sum \mu_1 ^{r_1} \ldots \mu_p ^{r_p} D(T_1, \ldots, T_p) e_1 *, e_1 * \]

\[ \leq |H| \sum |\mu_1| ^{r_1} \ldots |\mu_p| ^{r_p} D(T_1, \ldots, T_p) e_1 *, e_1 * \]

\[ = |H| \left( \sum |\mu_1| ^{r_1} \ldots |\mu_p| ^{r_p} D(T_1, \ldots, T_p) e_1 *, e_1 * \right) = d^H_X \left( \sum_{k=1}^{p} |\mu_k| A_k \right). \]

45. Let \( V = V_n(\mathbb{R}) \). Take \( \wedge V \) to be the space of n-square skew-symmetric matrices over \( \mathbb{R} \) and \( x \wedge y = \frac{1}{2} ([x^T y] - [y^T x]) \).
Show that if \( P \in M_{n,n}(\mathbb{R}) \) [regarded as a linear transformation on \( V_n(\mathbb{R}) \)] and \( A \in \wedge^2 V \) (i.e., \( A \) is \( n \)-square skew-symmetric), then \( C_2(P)A = PAP^t \).

Hint: Let \( e_1, \ldots, e_n \) be the unit vectors, \( e_i = (\delta_{i1}, \ldots, \delta_{in}) \).

Then

\[
C_2(P)(E_{ij} - E_{ji}) = 2C_2(P)e_i \wedge e_j
= 2Pe_i \wedge Pe_j
= 2P(1) \wedge P(j)
\]

\[
= 2 \sum_k P_{ki}^1 e_k \wedge \sum_m P_{mj}^m e_m
= 2 \sum_{k \neq m} P_{ki} P_{mj} e_k \wedge e_m
= 2 \left[ \sum_{k < m} P_{ki} P_{mj} e_k \wedge e_m + \sum_{s > t} P_{si} P_{tj} e_s \wedge e_t \right]
= 2 \left[ \sum_{k < m} P_{ki} P_{mj} e_k \wedge e_m - \sum_{t < s} P_{tj} P_{si} e_t \wedge e_s \right]
= 2 \sum_{k < m} (P_{ki} P_{mj} - P_{kj} P_{mi}) e_k \wedge e_m
= \sum_{k < m} (P_{ki} P_{mj} - P_{kj} P_{mi})(E_{km} - E_{mk})
\]

Now

\[
P_{E_{ij}} = \sum_{k,t,r,m} P_{kt} P_{mr} e_k E_{rt} E_{ij} E_{xm} = \sum_{k,m} P_{ki} P_{mj} E_{km}
= \sum_{k < m} P_{ki} P_{mj} E_{km} + \sum_{k > m} P_{ki} P_{mj} E_{km} + \sum_{m} P_{mi} P_{mj} E_{km}
\]

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and subtracting we have

\[ P(E_{ij} - E_{ji}) = \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} + \sum_{k > m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} + \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} \]

\[ = \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} + \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} + \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})E_{km} \]

\[ = \sum_{k < m} (P_{ki}P_{mj} - P_{kj}P_{mi})(E_{km} - E_{mk}) \]

4.6. Let \( V = V_n(R) \) where \( n = 2m \) and let \( z = \sum_{k=1}^{m} \alpha_k x_{2(k-1)} \wedge x_{2k} \).

Show that \( z^m = m! \alpha_1 \cdots \alpha_m x_1 \wedge x_2 \wedge \cdots \wedge x_n \), where the \( m \)th power is in the Grassmann algebra \( \wedge V \).

Hint: Since \( z \) is homogeneous of degree 2, \( z^m \) is homogeneous of degree \( 2m = n \). Moreover, since all terms in \( z \)
are of degree 2, it follows that they are pairwise commutative [see Example 1.5(e)] and hence the expansion of \( z^m \) can be done with the usual multinomial expansion theorem. But any product involving some \( x_{2k-1} \wedge x_{2k} \) more than once is 0 so

\[ z^m = \left( \sum_{k=1}^{m} \alpha_k x_{2(k-1)} \wedge x_{2k} \right)^m \]

\[ = m! \alpha_1 \cdots \alpha_m x_1 \wedge \cdots \wedge x_n. \]
47. Let \( A = \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij} - E_{ji}) \) be an \( n \)-square skew-symmetric matrix, \( n = 2m \). Prove that \( \det A \) is the square of a polynomial in the \( a_{ij} \), \( Pf(A) \), called the pfaffian of \( A \), and that

\[
Pf(A) = \sum_{\sigma(\varphi)} \prod_{k=1}^{m} a_{\varphi(2k-1), \varphi(2k)}.
\]

\( \varphi \in S_n, \varphi(2k) < \varphi(2k-1) \)

\( \varphi(1) < \varphi(3) < \cdots < \varphi(2m-1) \)

Hint: It is a standard theorem in matrix theory that there exists an \( n \)-square matrix \( P \) such that

\[
A = P \sum_{k=1}^{m} \alpha_k (E_{2k-1,2k} - E_{2k,2k-1}) P',
\]

the entries of \( P \) are polynomials with integral coefficients in the entries of \( A \) and \( \det P = \pm 1 \). Thus from Exercise 45

\[
C_2(P) \sum_{k=1}^{m} \alpha_k (E_{2k-1,2k} - E_{2k,2k-1}) = A.
\]

Now \( E_{2k-1,2k} - E_{2k,2k-1} = 2e_{2k-1} \wedge e_{2k} \) so that

\[
A = C_2(P) \sum_{k=1}^{m} 2\alpha_k e_{2k-1} \wedge e_{2k} = 2 \sum_{k=1}^{m} \alpha_k p_{2k-1} \wedge p_{2k}
\]

\[
= 2 \sum_{k=1}^{m} \alpha_k x_{2k-1} \wedge x_{2k}, \quad x_j = p(j).
\]

By the preceding exercise,
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\[ A \wedge \cdots \wedge A = n! \; z_1^{m} \alpha_1 \cdots \alpha_n x_1 \wedge \cdots \wedge x_n \]
\[ = m! \; z_1^{m} \alpha_1 \cdots \alpha_m x_1^{(1)} \wedge \cdots \wedge x_1^{(n)} \]
\[ = m! \; z_1^{m} \alpha_1 \cdots \alpha_m \det P_{1} \wedge \cdots \wedge P_{n}. \]

But

\[ A = \sum_{1 \leq i < j \leq n} a_{ij} (E_{i} - E_{j}) = 2 \sum_{i < j} a_{ij} e_i \wedge e_j. \]

Consider the \( m^\text{th} \) power in \( \wedge V, \) \( (\sum_{i < j} a_{ij} e_i \wedge e_j)^m. \) The elements are all of degree 2 and hence pairwise commute so we can use the ordinary multinomial theorem to obtain

\[ \left( \sum_{i < j} a_{ij} e_i \wedge e_j \right)^m = \sum_{k_1 + \cdots + k_r = m} \binom{m}{k_1 \cdots k_r} a_{1}^{k_1} \cdots a_{r}^{k_r} \cdot (e_1 \wedge e_2 \wedge \cdots \wedge (e_{n-1} \wedge e_n)^{k_r}, \]

\[ r = \frac{n(n-1)}{2} = \frac{2m(2m-1)}{2} = m(2m-1). \]

Now clearly any term in which \( k_i > 1 \) is 0 so that the sum extends only over all choices for which precisely \( m \) of the \( k_i \) are 1 and the rest 0. The value of the multinomial coefficient for such a choice of the \( k_i \) is \( m! \) Thus the preceding expression is equal to

\[ m! \sum_{i_1 < \cdots < i_m, i_1 < j_1, i_2 < j_2, \ldots, i_m < j_m} a_{i_1,j_1} a_{i_2,j_2} \cdots a_{i_m,j_m} (e_{i_1} \wedge e_{j_1}) \wedge (e_{i_2} \wedge e_{j_2}) \wedge \cdots \wedge (e_{i_m} \wedge e_{j_m}). \]
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Now if any of the integers \(i_1, j_1, i_2, j_2, \ldots, i_m, j_m\) are the same the summand is 0 so that these integers must be \(1, \ldots, n\) in some order. Set

\[
\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_m & j_m \end{pmatrix}
\]

so that the last expression reads

\[
m! \sum_{\varphi \in S_n} \prod_{k=1}^{m} \epsilon_{\varphi(2k-1), \varphi(2k)} e_\varphi
\]

\[
\varphi(2k-1) < \varphi(2k), \varphi(1) < \varphi(3) < \cdots < \varphi(2m-1)
\]

\[
= m! \sum_{\varphi \in S_n} \epsilon(\varphi) \prod_{k=1}^{m} a_{\varphi(2k-1), \varphi(2k)} e_1 \wedge \cdots \wedge e_n.
\]

\[
\varphi(2k-1) < \varphi(2k), \varphi(1) < \varphi(3) < \cdots < \varphi(2m-1)
\]

Thus

\[
\underbrace{A \wedge \cdots \wedge A}_{m} = 2^m m! \sum_{\varphi \in S_n} \epsilon(\varphi) \prod_{k=1}^{m} a_{\varphi(2k-1), \varphi(2k)} e_1 \wedge \cdots \wedge e_n.
\]

\[
\varphi(2k-1) < \varphi(2k), \varphi(1) < \varphi(3) < \cdots < \varphi(2m-1)
\]

Comparing this with the other value of \(\underbrace{A \wedge \cdots \wedge A}_{m}\) we have

\[
\alpha_1 \cdots \alpha_m \det P = Pf(A).
\]

But \((\alpha_1 \cdots \alpha_m \det P)^2 = \det A\) since

\[
A = P \sum_{k=1}^{m} \alpha_k (E_{2k-1, 2k-1} - E_{2k, 2k-1}) P^T.
\]
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