Proof of Large-cube Theorems for Multiverse PL

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We need a preliminary definition:

Definition (order-type equivalence) Let \( N \) be the nonnegative integers and let \( k>0 \) be a fixed integer. Two vectors \( x=(x_1,x_2,...,x_k) \) and \( y=(y_1,y_2,...,y_k) \) in \( N^k \) are order-type equivalent if the set \( \{(i,j)\mid i\leq j\}\) equals \( \{(i,j)\mid i\leq j\} \) and \( \{(i,j)\mid i\leq j\} \) equals \( \{(i,j)\mid i\leq j\} \).

We also need a version of Ramsey’s Theorem related to order-type equivalence (see Graham, R. L., Rothschild, B. L., and Spencer, J. H. Ramsey Theory, 2nd edition, John Wiley, New York, 1990, page 23).

**Ramsey’s Theorem (version)** If \( f \) is a function on \( N \) such that \( \text{Image}(f)=\{f(z)\mid z \text{ an element of } N\} \) is finite then there is an infinite set \( H \subseteq N \) such that \( f \) is constant on the order-type equivalence classes of \( H \).

Lemma (Multiverse PL) Let \( G=(N^2,\Theta) \) be any directed graph on the lattice \( N^2 \). There is an infinite subset \( H=\{h_0,h_1,h_2,...\} \) of \( N \) such that for \( L=H \cap H \) the labeling function \( f_L \) has at most one significant label on \( L \). In fact, the set \( \{f_L(z)\mid z \in L, f_L(z)<\min(z)\} \subseteq \{h_0\} \).

**Proof:** We define a function \( g \) on \( N^2 \times N^2 \) (which we identify with \( N \)). For each \( (x,y) \) in \( N^2 \times N^2 \), define \( g(x,y)=0 \) if \( (x,y) \) is not an edge of \( G \) and \( g(x,y)=1 \) otherwise. By Ramsey’s theorem, there is an infinite subset \( H \) of \( N \) where \( g \) is constant on order-type equivalence classes of \( H \) (which we identify with \( H \times H \)). Consider the infinite subset \( L=H \cap H \) of \( N \). We are going to show that \( G \) has at most one significant label on \( L \). Let \( z \) be a vertex of \( G \) with significant label \( f_L(z) \). Let \( z=x_1,x_2,...,x_t \) be a path in \( G \) from \( z \) to a vertex \( x_t \) with \( \min(x_t)=f_L(z) \). Assume that this path is minimal in the sense that \( \min(x_j)>f_L(z) \) for \( 1\leq j\leq t \). Note that \( g(z,x_1,x_2,...,x_t)=1 \) because \( (x_0, x_1, x_2,...,x_t) \) is an edge of \( G \) (and hence an edge of \( H \)).

Replace any coordinate of \( x_t \) that equals \( \min(x_t) \) by \( h_0=\min(H) \) (the smallest integer in the set \( H \)) and denote the resulting vertex of \( G \) by \( w \). Note that \( (x_0, x_1, x_2,...,x_t) \) and \( (x_0, x_1, w) \) are order-type equivalent in \( N \) and thus \( g(x_0, x_1, w)=1 \). Thus, \( (x_0, x_1, w) \) is an edge of \( G \). Hence, \( z=x_1,x_2,...,w \) is a path in \( G \) from \( z \) to a vertex \( w \) with \( \min(w)=h_0 \). By the definition of \( f_L(z) \), we must have \( f_L(z)=h_0 \) and thus \( h_0 \) is the only possible significant label for a vertex in \( G \). We have shown that \( G \) has at most one significant label. In fact, \( \{f_L(z)\mid z \in L, f_L(z)<\min(z)\} \subseteq \{h_0\} \).

**Theorem (Multiverse PL)** Let \( G=(N^2,\Theta) \) be any directed graph on the lattice \( N^2 \). Let \( p>0 \) be any integer. There are finite subsets \( E \) and \( F \) such that \( E=F \subseteq N, |E|=p \), and for \( D=F \times F \) and \( S=E=F \times E \), \( f_D \) has at most one significant label on \( S \). In fact, \( \{f_D(z)\mid z \in S, f_D(z)<\min(z)\} \subseteq \{\min(E)\} \).

**Proof:** There is an infinite subset \( H=\{h_0,h_1,h_2,...\} \) of \( N \) such that for \( L=H \cap H \), \( \{f_D(z)\mid z \in L, f_D(z)<\min(z)\} \subseteq \{h_0\} \). This assertion follows from the previous lemma. Let \( E=\{h_0,h_1,h_2,...,h_p\} \). Define \( S=E=F \). The infinite graph \( G \) has at most one significant label on \( S \), but we want to find a finite set \( F \), \( E=F \subseteq H \), such that \( G_D \), where \( D=F \times F \), has at most one significant label on \( S \). For each \( z \) in \( S \), let \( z=x(z),x_2(z),...,x_t(z) \) be a path in \( G \) from \( z \) to a vertex \( x(z) \) with \( \min(x(z))=f(z) \). For vertices \( z \) with \( f(z)=\min(z) \), this path consists of a single vertex, \( z \) itself. If \( f(z) \) is significant, so \( h_0=f(z)<\min(z) \), this path may have arbitrary but finite length and involve vertices in \( L \) but not in \( S \). Since \( S \) is finite, we can choose a set \( F \), \( E=F \subseteq H \), such that \( D=F \times F \) contains the vertices of all such paths. For this choice of \( F \), \( f_D \) has at most one significant label on \( S \). In fact, \( \{f_D(z)\mid z \in S, f_D(z)<\min(z)\} \subseteq \{h_0\} \subseteq \{\min(E)\} \).

**NOTE:** In the examples TL and SL, we shall assume the graph \( G \) satisfies the condition that if \( (x,y) \) is an edge of \( G \) then \( \max(x) < \min(x) \). With this “downward” condition on edges of \( G \), we can take \( D=S \) in the above theorem. In fact, we could take \( D=S \) if we assumed \( \max(y) \leq \max(x) \).

**HIGHER DIMENSIONS:** The above results and proofs extend in a straightforward way to graphs \( G \) on \( N^n \).