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CHAPTER 6
REPRESENTATION THEORY

6.1 Rational Representations

A groupoid is simply a set $S$ together with a binary operation on $S \times S$ with values in $S$ that we shall usually denote by juxtaposition. If the associative law $x(yz) = (xy)z$ holds, then $S$ is called a semi-group. If $S$ and $G$ are groupoids and $L : S \rightarrow G$ is a function satisfying

$$L(s_1s_2) = L(s_1)L(s_2) \quad s_1,s_2 \in S,$$  \hspace{1cm} (1)

then $\text{Im} L$ is called a representation of $S$ in $G$. Such a function $L$ is called a representation function, and if $L$ is injective then the representation is called faithful. The general linear group over a vector space $V$, $\dim V = n$, is the multiplicative group of non-singular linear transformations $T \in L(V,V)$. As in Section 3.2 this multiplicative group is denoted by $\text{GL}_n(V)$. Of course, $L(V,V)$ is a semi-group and if

$$L : S \rightarrow L(V,V)$$

is a representation, then $V$ is called a representation module for $S$ or simply an $S$-module. If

$$\text{Im} L \subseteq \text{GL}_n(V)$$
then \( L \) is called a **proper** representation, and \( V \) is a **proper** representation module. The dimension of \( V \) is called the **degree** of the representation. If

\[
L: S \rightarrow \mathcal{M}_n(R)
\]

and \( L \) satisfies (1), then \( L \) is called a **matrix representation** of \( S \). If \( \text{GL}(n,R) \) is the multiplicative group of nonsingular \( n \)-square matrices over \( R \) and \( \text{Im} \, L \subseteq \text{GL}(n,R) \), then \( L \) is called a **proper** matrix representation.

An important construction is embodied in the following definition.

**Definition 1.1 (Tensor product representation)** Let \( L_i: S \rightarrow L(V_i, V_i), \ i = 1, \ldots, m, \) be representations of a groupoid \( S \). Then the tensor product of these representations is the function

\[
L: S \rightarrow L(\bigotimes_{i=1}^m V_i, \bigotimes_{i=1}^m V_i)
\]

defined by

\[
L(s) = L_1(s) \otimes \cdots \otimes L_m(s) \quad s \in S.
\]

We write

\[
L = L_1 \otimes \cdots \otimes L_m
\]

\[
= \otimes_{i=1}^m L_i.
\]

Observe that \( L \) is indeed a representation:
\[ L(s_1)L(s_2) = (L_1(s_1) \otimes \cdots \otimes L_m(s_1))(L_1(s_2) \otimes \cdots \otimes L_m(s_2)) \]

\[ = L_1(s_1) L_1(s_2) \otimes \cdots \otimes L_m(s_1) L_m(s_2) \]

\[ = L_1(s_1 s_2) \otimes \cdots \otimes L_m(s_1 s_2) \]

\[ = L(s_1 s_2). \]

If \( V_1 = \cdots = V_m = V \), \( \dim V = n \), \( L_1 = \cdots = L_m = L \), then \( L_1 \otimes \cdots \otimes L_m \) is written

\[ L^\otimes \] (4)

or sometimes

\[ \bigotimes_{\ell=1}^{m} L_{\ell} \]

and is called the \( m^{th} \) tensor power of the representation \( L \). If \( S \subseteq L(V, V) \), \( \dim V = n \), then \( S \) can be regarded as a representation of itself, i.e., \( L(T) = T \) for each \( T \in S \), and the \( m^{th} \) tensor power representation \( L^\otimes \) is then just the mapping

\[ T \rightarrow T \otimes \cdots \otimes T. \] (5)

Recall the notation \( \prod^m \Pi^\ell T \) introduced in Section 2.1, formula (4), for the transformation on the right in (5). We can think of \( \prod^m \Pi^\ell \) as the \( m^{th} \) tensor power of the identity representation of \( S \), i.e.,

\[ \prod^m \Pi^\ell (T_1 T_2) = T_1^T_2 \otimes \cdots \otimes T_1^T_2 \]
\[
= (T_1 \otimes \cdots \otimes T_1)(T_2 \otimes \cdots \otimes T_2)
\]
\[
= \prod_{\lambda}^m T_{\lambda 1} \prod_{\lambda}^m T_{\lambda 2}.
\]

(6)

Of course, if \( S \subset GL_n(V) \), then \( \prod_{\lambda}^m S \to L(\otimes V, \otimes V) \) is a proper representation [see Section 2.1, formula (23)]. We shall usually write \( \prod_{\lambda}^m T \) as \( \Pi_{\lambda}^m T \), i.e., we omit the superfluous 1 in the pi notation.

We recall (Section 2.2, Definition 2.1) that a permutation operator \( P(\sigma) \in L(\otimes V, \otimes V) \) is defined by

\[
P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \quad \sigma \in S_m.
\]

As we verified in Section 2.2, Example 2.3(b), the function

\[
P : S_m \to L(\otimes V, \otimes V)
\]

whose value at any \( \sigma \in S_m \) is \( P(\sigma) \), is a proper faithful (if \( \dim V > 1 \)) representation of the group \( S_m \).

Our central goal in this chapter is to analyze the structure of the two representations

\[
\prod_{\lambda}^m : GL_n(V) \to L(\otimes V, \otimes V)
\]

and

\[
P : S_m \to L(\otimes V, \otimes V).
\]
It turns out that these two representations are intimately connected. This might be surmised from Theorem 2.7 in Section 3.2 in which it is proved that a linear transformation on $\otimes V$ commutes with every $P(\sigma)$ iff it is in the linear closure of the space spanned by the $m$ $n^T, T \in GL_n(V)$. The representation (8), although hardly simple, at least has the virtue of being a representation of a finite group. Obviously the representation (7) does not share this property.

If $L:S \to L(V,V)$ is a representation and $E$ is a basis of $V$, then clearly the mapping

$$s \to M(s) = \left[ L(s) \right]_E^E$$

(9)

is a matrix representation of $S$. If $E'$ is another basis of $V$ and

$$M'(s) = \left[ L(s) \right]_{E'}^{E'}$$

then

$$M'(s) = AM(s)A^{-1}, \quad s \in S,$$

(10)

where $A$ is a fixed matrix in $GL(n,R)$. Suppose now that $S$ is a multiplicative semi-group in $L(U,U)$ where $U$ is some vector space over $R$.

**Definition 1.2 (Rational and integral representations)** The above representation $L$ is said to be rational (integral) if there exists a basis $F$ of $U$ such that each entry of $M(s)$ is a fixed rational function (polynomial) (i.e., independent of the
particular $s$) in the entries of $[s]_F^F$. Integral representations are also called polynomial representations.

To clarify this definition, first notice that from (10) the question of whether $L$ is a rational or integral representation is independent of the basis $E$. We also easily check that the definition is independent of the basis $F$ of $U$. For, if $F'$ is another basis of $U$ then

$$ [s]_F^{F'} = B[s]_F^F B^{-1} \tag{11} $$

where $B$ is a fixed matrix over $R$. Obviously, from (11), any polynomial over $R$ in the entries of $[s]_F^{F'}$ is a polynomial over $R$ in the entries of $[s]_F^F$ and conversely.

Recall that a character of degree 1 of a groupoid $S$ is simply a representation

$$ \chi : S \to R' \tag{12} $$

where $R'$ is the multiplicative group of nonzero elements of $R$ (see Definition 2.3, Section 2.2). Such representations are also called abelian characters of $S$. Our first result shows that the only polynomial abelian characters of $L(U,U)$ are powers of the determinant. To be precise we have the following result.

**Theorem 1.1** If $\chi : L(U,U) \to R$ is a polynomial representation, not identically 0, then there exists an integer $k \geq 0$ such that

$$ \chi(T) = (\det T)^k \tag{13} $$
for all \( T \in L(U,U) \).

**Proof**: From the above remarks we may replace \( L(U,U) \) by \( M_n(R) \) so that the problem is to show that if \( \chi(X) \) is not identically 0,

\[
\chi(XY) = \chi(X)\chi(Y),
\]

and \( \chi(X) \) is a fixed polynomial in the entries \( x_{ij} \) of \( X \), then

\[
\chi(X) = (\det X)^k.
\]

First note that \( \chi(I_n) = \chi(I_n^2) = \chi(I_n)^2 \) so \( \chi(I_n) \) is 0 or 1. But if \( \chi(I_n) \) were 0, it would follow that \( \chi(A) = \chi(AI_n) = \chi(A)\chi(I_n) = 0 \) for any \( A \in M_n(R) \), a contradiction. Thus \( \chi(I_n) = 1 \). Observe that for any nonsingular \( A \), \( A^{-1} = (\det A)^{-1}\text{adj} A \), and the entries of \( \text{adj} A \) are fixed polynomials in the entries of \( A \). Thus for any \( A \in GL(n,R) \)

\[
1 = \chi(A A^{-1}) = \chi(A)\chi\left(\frac{\text{adj} A}{\det A}\right).
\]

Since \( \chi(A) \) is a polynomial in the entries of \( A \) we see that by multiplying through (16) by a sufficiently high power of \( \det A \) we can conclude that

\[
\chi(A)g(A) = (\det A)^p, \quad p > 0,
\]

where \( g(A) \) is a polynomial in the entries of \( A \). Now, let \( x_{ij} \) be \( n^2 \) independent indeterminates over \( R \) and observe that the polynomial
\[ \chi(X)g(X) = (\det X)^p \]  

is 0 for arbitrary specializations of \( X = [x_{ij}] \) to \( X = A \in M_n(\mathbb{R}) \) subject only to the polynomial inequality \( \det A \neq 0 \). But then (18) must be the 0 polynomial by Weyl's irrelevancy principle (Exercise 14, Section 3.2). Hence

\[ \chi(X)g(X) = (\det X)^p. \]  

The polynomial \( \det X \) is irreducible in the unique factorization domain \( R[x_{11}, \ldots, x_{nn}] \) (see Section 2.4, Exercise 11) and hence (19) implies that \( \chi(X) = r(\det X)^k \), \( r \in \mathbb{R}, \ k \) a nonnegative integer. Since \( \chi(I_n) = \det I_n = 1 \) we conclude that \( r = 1 \) and (15) follows.

Our next result allows us to replace any rational representation of \( GL_n(U) \) by a polynomial representation.

**Theorem 1.2** Let \( \varphi : GL_n(U) \to GL_N(V) \) be a rational representation with \( \varphi(I_U) = I_V \). Then there exists an integer \( k \) such that

\[ \varphi(T) = (\det T)^k \theta(T), \quad T \in GL_n(U), \]  

where \( \theta : GL_n(U) \to GL_N(V) \) is a polynomial representation.

**Proof:** We can assume bases of \( U \) and \( V \) have been chosen so that \( \varphi \) can be regarded as a rational representation of \( GL(n,\mathbb{R}) \):

\[ \varphi : GL(n,\mathbb{R}) \to GL(N,\mathbb{R}). \]
Let \( x_{ij} \) be \( n^2 \) independent indeterminates over \( \mathbb{R} \) and let \( \varphi_{st}(X) \), \( X = [x_{ij}] \), \( s, t = 1, \ldots, N \), be the rational functions that appear in \( \varphi(X) \). Let \( q(X) \) be the monic l.c.m. of the denominators of the \( \varphi_{st}(X) \) and \( p(X) \) be the monic g.c.d. of the polynomials \( q(X) \varphi_{st}(X) \). We can then write

\[
\varphi(X) = \frac{p(X)}{q(X)} \theta(X)
\]  

(21)

where \( \theta_{st}(X) \in \mathbb{R}[x_{11}, \ldots, x_{nn}] \) and

\[
g.c.d.\{\theta_{st}(X), \ s, t = 1, \ldots, N\} = 1.
\]  

(22)

In case \( N = 1 \) take \( \theta(X) = 1 \) in (21) and then \( \varphi(X) = \frac{p(X)}{q(X)} \).

We can also normalize the equation (21) as follows: cancel out any common factors that \( p(X) \) and \( q(X) \) have so that they can be assumed to be relatively prime. Since \( p(I_n) = a \neq 0 \), \( q(I_n) = b \neq 0 \) we can replace \( p(X) \) by \( \frac{1}{a} p(X) \), \( q(X) \) by \( \frac{1}{b} q(X) \) and \( \theta(X) \) by \( \frac{a}{b} \theta(X) \) in (21), so that we can assume that

\[
p(I_n) = q(I_n) = 1
\]  

(23)

and, since \( \varphi(I_n) = I_N \),

\[
\theta(I_n) = I_N.
\]  

(24)

For any \( A \) and \( B \) in \( GL(n, \mathbb{R}) \) we have

\[
\frac{p(AB)}{q(AB)} \theta(AB) = \varphi(AB)
\]

\[
= \varphi(A) \varphi(B)
\]
\[ \frac{p(A)}{q(A)} \theta(A) \frac{p(B)}{q(B)} \theta(B), \]

or

\[ q(AB)p(A)p(B)\theta(A)\theta(B) - q(A)q(B)p(AB)\theta(AB) = 0, \quad (26) \]

Let \( y_{ij} \) be \( n^2 \) indeterminates so that \( x_{ij}, y_{ij} \) are \( 2n^2 \) independent indeterminates over \( R \). We see from (25) that every entry of the \( N \)-square matrix

\[ F(X,Y) = q(XY)p(X)p(Y)\theta(X)\theta(Y) - q(X)q(Y)p(XY)\theta(XY) \quad (26) \]

is 0 for arbitrary specializations of \( X \) and \( Y \) to matrices \( A \) and \( B \) in \( \text{M}_n(R) \) subject only to the inequalities \( \det A \neq 0 \), \( \det B \neq 0 \). Applying Weyl's irrelevancy principle (Exercise 14, Section 3.2) we conclude that

\[ F(X,Y) = 0. \quad (27) \]

Thus if \( B \in \text{GL}(n,R) \)

\[ q(XB)p(X)p(B)\theta(X)\theta(B) = q(X)q(B)p(XB)\theta(XB) \]

and since \( \theta(B) \in \text{GL}(N,R) \),

\[ p(X)p(B)q(XB)\theta(X) = q(X)q(B)p(XB)\theta(XB)\theta(B)^{-1}. \quad (28) \]

The polynomial \( q(X) \) divides each entry on the right in (28) and hence must divide each entry on the left. Now \( p(X) \) and \( q(X) \) are relatively prime as are the \( N^2 \) entries \( \theta_{ij}(X) \). If \( N = 1 \) the \( \theta \) terms do not appear in (28). In other words,
q(X) \mid p(X)q(XB)\theta_{ij}(X), \quad i,j = 1,\ldots,N

and since q(X) and p(X) are relative prime,

q(X) \mid q(XB)\theta_{ij}(X), \quad i,j = 1,\ldots,N,

in which \( \theta_{ij}(X) \) is replaced by 1 if \( N = 1 \). Any prime factor \( q'(X) \) of \( q(X) \) must divide either \( q(XB) \) or \( \theta_{ij}(X) \) for each \( i,j = 1,\ldots,N \). Since the \( \theta_{ij}(X) \) are relatively prime they have no common prime divisor and hence for at least one pair \( i,j \), \( q'(X) \) is not a divisor of \( \theta_{ij}(X) \). Thus any prime factor of \( q(X) \) must divide \( q(XB) \) so that

\[
q(X) \mid q(XB) .
\] (29)

Now

\[
\deg q(XB) \leq \deg q(X)
= \deg q(XB^{-1})
\leq \deg q(XB)
\]

so that \( \deg q(X) = \deg q(XB) \). Hence from (29) there is an element \( r_B \in R \) such that

\[
q(XB) = r_B q(X) .
\] (30)

But then

\[
x_B = r_B \cdot 1
\]
\[ = r_b \cdot q(I_n) \]
\[ = q(I_n)B \quad \text{[from (30)]} \]
\[ = q(B) , \]

so that (30) becomes
\[ q(X)q(B) = q(XB) , \quad (31) \]

\( B \in \text{GL}(n, \mathbb{R}) \). By the irrelevancy principle (31) implies that
\[ q(A)q(B) = q(AB) \]

for all \( A, B \) in \( M_n(\mathbb{R}) \) and hence by Theorem 1.1
\[ q(A) = (\det A)^{m} , \quad A \in M_n(\mathbb{R}) , \]

for some nonnegative integer \( m \). Thus
\[ q(X) = (\det X)^{m} . \quad (32) \]

We must next show that a formula similar to (32) holds for \( p(X) \).

The formula (32) implies that \( q(X)q(X^{-1}) = 1 \) so that
\[ p(X)p(X^{-1})\Theta(X)\Theta(X^{-1}) = \frac{p(X)}{q(X)} \frac{p(X^{-1})}{q(X^{-1})} \Theta(X)\Theta(X^{-1}) \]
\[ = \phi(X)\phi(X^{-1}) \]
\[ = \phi(XX^{-1}) \]
\[ = \phi(I_n) \]
\[ = I_n . \quad (33) \]
Now \( X^{-1} = \frac{\text{adj } X}{\text{det } X} \) so that there is a nonnegative integer \( d \) such that

\[
H(X) = (\text{det } X)^d p(X^{-1}) \theta(X) \theta(X^{-1})
\]

has polynomial entries and from (33) we compute that

\[
p(X)H(X) = (\text{det } X)^d I_N. \quad (34)
\]

The irreducibility of \( \text{det } X \) then implies that

\[
p(X) = (\text{det } X)^e \quad (35)
\]

for some nonnegative integer \( e \). If we combine (21), (32) and (35) we conclude that

\[
\varphi(X) = (\text{det } X)^k \theta(X) \quad (36)
\]

for some integer \( k \). But then

\[
\theta(AB) = \frac{\varphi(AB)}{(\text{det } AB)^k} = \frac{\varphi(A)}{(\text{det } A)^k} - \frac{\varphi(B)}{(\text{det } B)^k}
\]

\[
= \theta(A)\theta(B)
\]

and \( \theta: \text{GL}(n, \mathbb{R}) \to \text{GL}(N, \mathbb{R}) \) is a polynomial representation. \( \square \)

**Definition 1.3 (Equivalent representations)** Let \( L: S \to \text{L}(V, V) \) and \( M: S \to \text{L}(U, U) \) be two representations of a groupoid \( S \). If
T : U → V is a linear bijection that satisfies

\[ L(s)T = TM(s), \ s \in S, \]  \hspace{1cm} (37)

then \( L \) and \( M \) are said to be equivalent. We write \( L \sim M \).

Even if \( T \) is not necessarily bijective in (37) we say that \( L \) and \( M \) are linked by \( T \).

There is a similar notion of linking sets of matrices. Thus let \( \Omega \subseteq M_n(R), \Gamma \subseteq M_m(R) \) and \( M \in M_{m,n}(R) \). Assume that for each \( A \in \Omega \) there is a \( B \in \Gamma \) such that

\[ BM = MA \]  \hspace{1cm} (38)

and moreover for each \( B \in \Gamma \) there is an \( A \in \Omega \) such that (38) holds. Then \( M \) is said to link \( \Omega \) and \( \Gamma \). Observe that by choosing bases \( E \) and \( F \) for \( U \) and \( V \) we see that if \( L \) and \( M \) are linked as in (37) then the two sets of matrices \( [L(s)]_F^E \) and \( [M(s)]_E^E \), \( s \in S \), are linked. In fact

\[ [L(s)]_F^E [T]_E^F = [T]_E^F [M(s)]_E^E \]  \hspace{1cm} (39)

so that if \( L \sim M \) then \( \dim U = \dim V \), \( A = [T]_E^F \) is nonsingular, and (39) is equivalent to

\[ A^{-1}[L(s)]_F^E = [M(s)]_E^E. \]  \hspace{1cm} (40)

In general, if \( \Delta : S \rightarrow M_n(R) \), \( \varphi : S \rightarrow M_n(R) \) are matrix representations of \( S \) then \( \Delta \) and \( \varphi \) are said to be equivalent if there is a nonsingular \( A \in M_n(R) \) such that
\[ A^{-1} \Delta(s) A = \varphi(s), \ s \in S. \]

Let \( S \subseteq L(U, U) \) be a multiplicative semi-group and let \( L : S \to L(V, V) \) be a polynomial representation of \( S \). We say that \( L \) is homogeneous of degree \( m \) if each entry of

\[
\begin{bmatrix}
L(s)_{ij}^E
\end{bmatrix}_E
\]

is a homogeneous polynomial of degree \( m \) in the entries of \( [s]_F \).

It is easy to see that this property is independent of the bases \( E \) and \( F \). For, suppose \( x_{ij} \) are \( n^2 \) indeterminates over \( R \), \( X = \begin{bmatrix} x_{ij} \end{bmatrix} \) and

\[ \Theta(X) = \begin{bmatrix} \theta_{ij}(X) \end{bmatrix}, \]

where \( \theta_{ij}([s]_F^E) = (L(s)_{ij}^E)_E \), \( s \in S \). If the bases \( E \) and \( F \) are replaced by bases \( E' \) and \( F' \) respectively, then clearly \( \Theta(X) \) will be replaced by

\[ \varphi(X) = A \begin{bmatrix} \theta_{ij}(B X B^{-1}) \end{bmatrix} A^{-1} = A \theta(B X B^{-1}) A^{-1} \]

where \( A \) and \( B \) are fixed nonsingular matrices over \( R \) [see (10) and (11)]. Obviously

\[ \varphi(tX) = A \theta(B(tX)B^{-1}) A^{-1} \]

\[ = t^m A \theta(BX B^{-1}) A^{-1} \]

\[ = t^m \varphi(X). \]
Hence \( \wp(X) \) is also homogeneous of degree \( m \). Our next result shows that any polynomial representation of \( \text{GL}_n(U) \) is equivalent to a direct sum of homogeneous polynomial representations. In general, we say that the representation \( L : S \to L(V,V) \) is a direct sum of the representations \( L_1, \ldots, L_k \) if there is a direct sum decomposition of the space, \( V = \sum_{i=1}^{k} V_i \), such that each \( V \) is an invariant subspace of every \( L(s), \ s \in S \). If we let \( L_i(s) = L(s)|_{V_i} \), then clearly \( L_i : S \to L(V_i,V_i) \) is a representation as well. We write

\[
L = \sum_{i=1}^{k} L_i .
\]

If \( L : S \to L(V,V) \), then \( L = \sum_{i=1}^{k} L_i \) iff there is a basis \( E \) of \( V \) such that

\[
\Delta(s) = \left[ L(s) \right]_E = \sum_{i=1}^{k} \Delta_i(s) \tag{42}
\]

where \( \Delta_i(s) \) is a matrix representation of \( L_i(s), \ i = 1, \ldots, k \). Similarly, if \( \Delta : S \to M_n(R) \), then to say that \( \Delta \) is equivalent to a direct sum of representations \( \Delta_i \) of \( S \) means that there is a nonsingular matrix \( A \in M_n(R) \) such that

\[
A^{-1} \Delta(s) A = \sum_{i=1}^{k} \Delta_i(s), \ s \in S . \tag{43}
\]

**Theorem 1.3** Let \( \wp : \text{GL}_n(U) \to \text{GL}_n(V) \) be a polynomial repre-
sentation. Then $\varphi$ is equivalent to a direct sum of proper homogeneous polynomial representations.

**Proof:** We can assume that bases of $U$ and $V$ have been chosen so that $\varphi$ can be regarded as a polynomial representation of $GL(n, R)$,

$$\varphi : GL(n, R) \to GL(N, R).$$

The above discussion shows that the choice of bases is immaterial.

If $x_{ij}$ are $n^2$ independent indeterminates over $R$, then to say that $\varphi$ is a polynomial representation means that there are $n^2$ polynomials $\varphi_{st}(X) \in R[x_{11}, x_{12}, \ldots, x_{nn}], X = [x_{ij}]$, such that $\varphi(A) = [\varphi_{ij}(A)]$ for each $A \in GL(n, R)$. Now let $\lambda$ and $\mu$ be two more independent indeterminates and write

$$\varphi(\lambda I_n) = A_0 + \lambda A_1 + \cdots + \lambda^p A_p,$$  \hspace{1cm} (44)

$A_i \in M_n(R), \; i = 1, \ldots, p$. Similarly, $\varphi(\mu I_n) = \sum_{i=0}^p \mu^i A_i$ so that

$$\sum_{i=0}^p (\lambda \mu)^i A_i = \varphi(\lambda \mu I_n)$$

$$= \varphi(\lambda I_n) \varphi(\mu I_n)$$

$$= \sum_{i=0}^p \lambda^i A_i \sum_{j=0}^p \mu^j A_j$$

$$= \sum_{i=0}^p \lambda^i \mu^i A_i^2 + \sum_{i \neq j} \lambda^i \mu^j A_i A_j.$$  \hspace{1cm} (45)
Hence matching coefficients, we conclude that the \( p+1 \) \( N \)-square matrices \( A_0, \ldots, A_p \) satisfy
\[
A_i^2 = A_i, \quad i = 0, \ldots, p, \quad (46)
\]
and
\[
A_i A_j = 0, \quad i \neq j, \quad i, j = 0, \ldots, p. \quad (47)
\]
From Exercise 1 there is a fixed nonsingular matrix \( B \) in \( GL(N, \mathbb{R}) \) such that
\[
BA_i B^{-1} = \begin{bmatrix}
0 & \cdots & 1 & \cdots & 0 \\
-1 & \cdots & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
-1 & \cdots & -1 & \cdots & -1 \\
0 & \cdots & 1 & \cdots & 1 \\
\end{bmatrix}, \quad i = 0, \ldots, p, (48)
\]
where the identity blocks appear down the main diagonal in immediate succession and are totally disjoint, \( i = 0, \ldots, p \). Of course some of the \( A_i \) may be 0 so that \( B^{-1}A_i B \) is 0 also and in actual fact does not appear in (48). Thus assume \( A_{k_1}, \ldots, A_{k_r} \) are precisely the nonzero \( A_{k_i} \) and set \( D_{k_i} = A_{k_i}, \quad i = 1, \ldots, r \).

Then we have
\[
R_F(\lambda I_n) B^{-1} = \sum_{i=1}^{r} \lambda^{k_i} B D_{k_i} B^{-1}
\]
\[
= \sum_{i=1}^{r} \lambda^{k_i} N_{k_i}. \quad (49)
\]
Notice that \( \sum_{i=1}^{r} N_{k_i} = N \). Partition \( B \phi(X)B^{-1} \) conformally with (49), i.e., write

\[
B \phi(X)B^{-1} = \begin{bmatrix}
\theta_{11}(X) & \cdots & \theta_{1r}(X) \\
\vdots & \ddots & \vdots \\
\theta_{r1}(X) & \cdots & \theta_{rr}(X)
\end{bmatrix}
\]

where \( \theta_{ij}(X) \) is \( N_{k_i} \times N_{k_j} \), \( i,j = 1, \ldots, r \). Then since \( \lambda I_n \) and \( X \) commute, we can conclude that \( B \phi(\lambda I)B^{-1} \) and \( B \phi(X)B^{-1} \) must commute. By the usual block multiplication argument

\[
\lambda^{k_i} \theta_{ij}(X) = \lambda^{k_j} \theta_{ij}(X), \quad i,j = 1, \ldots, r,
\]

and hence for \( i \neq j \), \( \theta_{ij}(X) = 0 \) because \( k_i \neq k_j \). Also it is clear that each \( \theta_{ii} \) defines a polynomial representation of \( GL(n, \mathbb{R}) \) and hence

\[
\theta_{ii}(\lambda I_n)X = \theta_{ii}(\lambda I_n)\theta_{ii}(X).
\]

But \( \theta_{ii}(\lambda I_n) \) is the \( i^{th} \) block down the main diagonal in \( B \phi(\lambda I_n)B^{-1} \) which by (49) is precisely \( \lambda^{k_i} N_{k_i} \). Hence (50) implies that

\[
\theta_{ii}(\lambda X) = \lambda^{k_i} \theta_{ii}(X).
\]

We have proved that
\[ B_{\mathfrak{f}}(X) R^{-1} = \sum_{i=1}^{r} \theta_{ii}(X) \]

and hence that \( \theta_{ii}(X) \) is nonsingular whenever \( X \) is nonsingular. Therefore \( \theta_{ii} \) is a proper homogeneous polynomial representation of degree \( k_i \), \( i = 1, \ldots, r \).

If we combine Theorems 1.2 and 1.3 we have

**Theorem 1.4** Let \( \varphi: GL_n(U) \to GL_n(V) \) be a rational representation. Then there exists an integer \( k \) and proper homogeneous polynomial representations \( \theta_1, \ldots, \theta_r \) such that \( \varphi \) is equivalent to the representation

\[ T \to (\det T)^k \sum_{i=1}^{r} \theta_i(T). \]

One of the important problems in representation theory is to determine whether a representation is equivalent to a direct sum of representations acting on proper subspaces. This leads us to make the following definition.

**Definition 1.4** (Reducibility) Let \( S \) be a groupoid and \( L: S \to L(V, V) \) a representation. If \( W \) is a proper subspace of \( V \) such that \( L(s)W \subseteq W, \ s \in S \), then \( W \) is called an invariant subspace of \( L \) and \( L \) is said to be reducible. If for any proper invariant subspace \( W \) there is an invariant subspace \( U \) such that \( V = W + U \), then \( L \) is said to be completely or fully reducible. "Irreducible" means not reducible.
We sometimes refer to $V$ as being reducible or irreducible, e.g., $V$ is a reducible $S$-module.

**Example 1.1** Let $S = \{a^k\}$ be an infinite cyclic group and let

$$L(a^k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$  

Clearly

$$L(a^k a^m) = L(a^{k+m}) = \begin{bmatrix} 1 & k+m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = L(a^k) L(a^m).$$

We can regard $L(a^k)$ as acting on $V_2(R)$ with the obvious invariant subspace $\langle (1,0) \rangle$. This $L$ is reducible. But $L$ is not fully reducible, otherwise there would exist a fixed matrix $B$ such that $B^{-1} L(a^k) B$ is diagonal for every $k$, which is clearly impossible.

A general concept of reducibility is available for an arbitrary set of matrices $\Omega \subseteq M_n(R)$. Thus $\Omega$ is reducible if there is a fixed nonsingular $B \in M_n(R)$ and fixed integers $p$ and $q$ such that

$$B^{-1} AB = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A \in \Omega, \quad (51)$$

where $A_{11}$ is $p$-square, and $A_{22}$ is $q$-square. If whenever (51)
holds there exists a $C$ such that

$$C^{-1}AC = D_{11} + D_{22}, \ A \in \Omega,$$

where $D_{11}$ is $p$-square and $D_{22}$ is $q$-square, then $\Omega$ is said to be fully reducible.

**Example 1.2** Let $G'$ denote the multiplicative group of nonzero complex numbers and define the representation (verify it)

$$L(x+iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = xI_2 + yQ, \ x, y \in \mathbb{R},$$

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Note that $Q$ is the companion matrix of the polynomial $\lambda^2 + 1$, irreducible over $\mathbb{R}$. Thus $L$ is not reducible over $\mathbb{R}$. However, if $P = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$, then $P^{-1}QP = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$ and $L$ is fully reducible over $\mathbb{C}$ (why?). Thus we see that the question of reducibility sometimes depends on the underlying field.

**Definition 1.5 (Absolute irreducibility)** Let $V$ be a vector space over the field $\mathbb{R}$. The representation $L: S \to \text{GL}_n(V)$ is **absolutely irreducible** over $\mathbb{R}$ if the following condition holds. Let $F$ be any extension field of $\mathbb{R}$ and regard $V$ as a vector space over $F$ (see Theorem 4.3, Section 1.4). Then there are no proper invariant subspaces of $L$ over $F$. 

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We sometimes say that $V$ is an absolutely irreducible space if $L$ is understood.

A similar concept for matrices is the following: if $\Omega \subset M_n(R)$, then $\Omega$ is absolutely irreducible if there is no extension field $F$ of $R$ for which there exists $B \in GL(n,F)$ such that (51) holds.

**Example 1.3** (a) Consider the tensor representation

$$\Pi' : GL_n(V) \rightarrow GL_n^2(V \otimes V). \quad (52)$$

Notice that $\Pi'$ is in fact a homogeneous polynomial representation of $GL_n(V)$. For, let $E$ be a basis of $V$ and $E^{\otimes}$ the usual induced basis for $V \otimes V$. If $[T]_E^E = A$ then

$$\begin{bmatrix} \Pi'(T) \end{bmatrix}^{\otimes}_{E^{\otimes}} = A \otimes A,$$

so let

$$\varphi(A) = A \otimes A.$$

Then $\varphi(tA) = t^2 \varphi(A)$ and $\varphi$ is homogeneous of degree 2. In fact

$$\varphi(A)_{\alpha, \beta} = a_{\alpha(1), \beta(1)} A_{\alpha(2), \beta(2)}, \quad \alpha, \beta \in \Gamma_n^2.$$ 

Now

$$V \otimes V = \bigwedge V + V^{(2)} \quad (53)$$
and $V^2$ and $V^{(2)}$ are invariant subspaces of $V^2$.

\[ V^2 \ (T) \mid \wedge V = C_2(T), \]
\[ V^{(2)} \ (T) \mid V^{(2)} = P_2(T). \]

We assert that both $C_2 : GL_n(V) \rightarrow GL(V^{2})$ and $P_2 : GL_n(V) \rightarrow GL(V^{(2)})$ are absolutely irreducible polynomial representations of degree 2 whatever the field $R$ is. To see this we refer to Examples 2.9 (a), (b), Section 3.2, in which it is proved that

\[ \langle C_2(T), T \in L(V,V) \rangle = L(V^2, \wedge V^2) \]

and

\[ \langle P_2(T), T \in L(V,V) \rangle = L(V^{(2)}, V^{(2)}). \]

In fact, in both (54) and (55) $GL_n(V)$ may be substituted for $L(V,V)$ (see Exercise 3). If $C_2$ were reducible over some extension field $F$ of $R$, there would exist a nonsingular matrix $H \in GL(n, F)$ and fixed positive integers $p$ and $q$, $p+q = \binom{n}{2}$, such that

\[ H^{-1}C_2(A)H = \begin{bmatrix} D_{11}(A) & 0 \\ D_{21}(A) & D_{22}(A) \end{bmatrix} \]
for all \( A \in \text{GL}(n, \mathbb{R}) \). This clearly contradicts (54) because every matrix in \( \mathbb{M}_{n}^{(2)}(\mathbb{R}) \) is a linear combination of matrices \( C_2(A) \), \( A \in \text{GL}(n, \mathbb{R}) \). A similar argument shows that \( P_2 \) is absolutely irreducible.

(b) Let \( S_\varepsilon \) and \( S_1 \) be the two symmetrizers that define \( \wedge^2 V \) and \( V^{(2)} \) respectively, i.e.,

\[
S_\varepsilon = \frac{1}{2}(P(e) - P((12)))
\]

\[
S_1 = \frac{1}{2}(P(e) + P((12))).
\]

Notice that

\[
I_V \otimes V = S_\varepsilon + S_1,
\]

\[
S_\varepsilon^2 = S_\varepsilon, \quad S_1^2 = S_1, \quad S_1 S_\varepsilon = S_\varepsilon S_1 = 0, \quad (56)
\]

\[
\wedge^2 V = S_\varepsilon (V \otimes V),
\]

\[
V^{(2)} = S_1 (V \otimes V),
\]

and (53) holds. Thus \( S_\varepsilon \) and \( S_1 \) form a so-called Pierce decomposition [i.e., the equations (56) hold] of the identity into orthogonal idempotents and the absolutely irreducible invariant subspaces \( \wedge^2 V \) and \( V^{(2)} \) are images of \( V \otimes V \) under these idempotents.

One of the goals of this chapter is to generalize Example 1.3(a), (b) according to the following program which we now outline.

I. Any degreee \( m \) homogeneous polynomial representation \( \theta : \text{GL}_n(U) \rightarrow \)
GLₙ(V) has the form

\[ \beta = \sum_{i}^{m} h \Pi_i' \]

where \( h: \mathfrak{B}_m \to L(V, V) \) is a unique algebra homomorphism. Here \( \mathfrak{B}_m \) is the algebra of bisymmetric linear transformations on \( \bigotimes_{1}^{m} U \) which by Theorem 2.7, Section 3.2, is the algebra generated by \( \Pi'(T), T \in GL_n(U) \). This result was essentially done in Exercise 15, Section 3.2, but we shall recapitulate it here in somewhat more detail.

II. The algebra \( \mathfrak{B}_m \) is the commutator algebra in \( L(\bigotimes_{1}^{m} U, \bigotimes_{1}^{m} U) \), i.e., centralizer, of the algebra generated by all symmetry operators on \( \bigotimes_{1}^{m} U \). This latter algebra is generated by the permutation transformations \( P(\sigma) \), \( \sigma \in S_m \), and we shall prove that \( \mathfrak{B}_m \) is in fact equivalent to a direct sum of complete matrix algebras.

III. Every representation of a direct sum of complete matrix algebras \( \mathcal{M} \) is fully reducible, and the different summands constitute the complete list of absolutely irreducible representations of \( \mathcal{M} \) (to within equivalence, of course).

IV. There exists a set of mutually orthogonal idempotents

\[ Y_1, \ldots, Y_p \]

in the algebra generated by the permutation transformations:

\[ Y_1 + \cdots + Y_p = I_{\bigotimes_{1}^{m} V} \]
\( Y_i^2 = Y_i, \ Y_i \ Y_j = 0, \ i \neq j \). The minimal invariant subspaces of \( m \) \( \Pi' \) are precisely the nonzero symmetry classes of tensors.

\[ \text{Im} \ Y_k, \ \ k = 1, \ldots, p. \]

To carry out this program we must do a sizeable piece of general group representation theory. Some of the results, however, are immediately accessible.

**Theorem 1.5** Let \( \theta : \text{GL}_n(U) \to \text{GL}_N(V) \) be a degree \( m \) homogeneous polynomial representation. Then there exists a unique algebra homomorphism \( h : \beta_m \to \text{L}(V, V) \) such that

\[ \theta = h \Pi'. \quad (58) \]

**Proof:** Choose bases of \( U \) and \( V \) so that we can assume \( \theta \) is a matrix representation

\[ \theta : \text{GL}(n, \mathbb{R}) \to \text{GL}(N, \mathbb{R}). \quad (59) \]

Let \( X = [x_{ij}] \) be a matrix of \( n^2 \) independent indeterminates so that the entries of \( \Pi'(X) \) are the polynomials

\[ \prod_{i=1}^{m} x_{\alpha(i), \beta(i)}, \ \alpha, \beta \in \Gamma_n^m. \quad (60) \]

The polynomials (60) constitute all homogeneous degree \( m \) monic monomials in the indeterminates \( x_{ij} \). Thus any homogeneous
degree \( m \) polynomial in the \( x_{ij} \) is a scalar linear combination of the polynomials (60). In particular we can write

\[
\vartheta_{st}(X) = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta}^{s, t} \prod_{i=1}^{m} x^{\alpha(i), \beta(i)}
\]

\[
= \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta}^{s, t} \left( \prod_{i=1}^{m} x_i \right)^{\alpha, \beta}
\]  \hspace{1cm} (61)

Now define \( h \) by the formula

\[
h(Z)_{s, t} = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta}^{s, t} z^{\alpha, \beta}
\]

for any \( Z \in \mathfrak{g}_m \). Note that from (61) \( h = \left( \prod_{i=1}^{m} x_i \right)_{s, t} = \sum_{\alpha, \beta \in \Gamma_n^m} a_{\alpha, \beta}^{s, t} \left( \prod_{i=1}^{m} x_i \right)^{\alpha, \beta} = \vartheta_{st}(X) = \vartheta(X)_{s, t} \) so that

\[
h \left( \prod_{i=1}^{m} x_i \right) = \vartheta(X).
\]  \hspace{1cm} (62)

Obviously \( h \) is linear, so to show that it is multiplicative we need only verify the property for the generators of \( \mathfrak{g}_m \), \( \prod_{i=1}^{m} (A)_i \), \( A \in \text{GL}(n, \mathbb{R}) \). But

\[
h \left( \prod_{i=1}^{m} (A) \right) \cdot \left( \prod_{i=1}^{m} (B) \right) = h \left( \prod_{i=1}^{m} (AB) \right)
\]

\[
= \vartheta(AB) \hspace{1cm} \text{[by (62)]}
\]

\[
= \vartheta(A) \vartheta(B)
\]

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\begin{equation}
= h(\| A \|) h(\| B \|) \quad [\text{by (62)}.]
\end{equation}

The uniqueness of \( h \) is clear because if \( h(\| X \|) = h_1(\| X \|) \), then the algebra homomorphisms \( h \) and \( h_1 \) agree on a generating set of \( G_m \).

The first assertion in II to the effect that \( G_m \) is the centralizer of

\begin{equation}
\langle P(\sigma), \sigma \in S_m \rangle
\end{equation}

in \( L_1(\otimes U, \otimes U) \) is precisely the content of Theorem 2.7 in Section 3.2. To prove the second statement - that the centralizer of (63) is isomorphic to a direct sum of complete matrix algebras requires the basic result that any representation of a finite group is fully reducible and hence can be expressed as a direct sum of irreducible representations. This is Maschke's theorem, one of the central results in the representation theory for finite groups. The next sequence of theorems is devoted to proving the above assertion concerning (63).

The following is a basic result in the representation theory for finite groups.

**Theorem 1.6** (Maschke's theorem) Let \( S \) be a finite group and let \( L: S \rightarrow \text{GL}_n(V) \) be a representation. Then \( L \) is fully reducible.

**Proof:** We remark that the following proof will work even over fields of finite characteristic as long as the order \( h \) of \( S \) is
not a multiple of the characteristic of the field.

If \( V \) is irreducible there is no problem. Thus assume \( V \) is reducible. This means that there exists a basis \( E \) of \( V \) such that

\[
\Delta(s) = \begin{bmatrix} L(s) \end{bmatrix}_E = \begin{bmatrix} \Delta_{11}(s) & 0 \\ \Delta_{21}(s) & \Delta_{22}(s) \end{bmatrix}
\]

where \( \Delta_{11}(s) \) is \( p \)-square, \( \Delta_{22}(s) \) is \( q \)-square, \( p \) and \( q \) fixed, \( p + q = n \). The theorem will be proved if we exhibit a fixed \( P \in \text{GL}(n, \mathbb{R}) \) such that

\[
P^{-1}\Delta(s)P = \Delta_{11}(s) \oplus \Delta_{22}(s), \quad s \in S.
\]

Let

\[
C = \sum_{s \in S} \Delta_{21}(s)\Delta_{11}(s)^{-1} \in M_{q, p}(\mathbb{R})
\]

and set

\[
P = \begin{bmatrix} I_p & 0 \\ \frac{1}{h}c & I_q \end{bmatrix}.
\]

Now

\[
\Delta_{11}(st) = \Delta_{11}(s)\Delta_{11}(t), \quad i = 1, 2,
\]

and
\[ \Delta_{21}(st) = \Delta_{21}(s)\Delta_{11}(t) + \Delta_{22}(s)\Delta_{21}(t) \]  

(64)

follow immediately from \( \Delta(st) = \Delta(s)\Delta(t) \) and block multiplication. Multiply both sides of (64) on the right by \( \Delta_{11}(t)^{-1} \) and transpose terms to obtain

\[ \Delta_{21}(s) = \Delta_{21}(st)\Delta_{11}(t)^{-1} - \Delta_{22}(s)\Delta_{21}(t)\Delta_{11}(t)^{-1} \]

\[ = \Delta_{21}(st)\Delta_{11}(st)^{-1}\Delta_{11}(s) - \Delta_{22}(s)\Delta_{21}(t)\Delta_{11}(t)^{-1}, \]  

(65)

and summing both sides of (65) over all \( t \in S \) we obtain

\[ h\Delta_{21}(s) = \left( \sum_{t \in S} \Delta_{21}(st)\Delta_{11}(st)^{-1}\Delta_{11}(s) - \Delta_{22}(s)\sum_{t \in S} \Delta_{21}(t)\Delta_{11}(t)^{-1} \right). \]  

(66)

As \( t \) runs over \( S \), \( st \) runs over \( S \) so that (66) simplifies to

\[ h\Delta_{21}(s) = C\Delta_{11}(s) - \Delta_{22}(s)C. \]  

(67)

Notice that

\[ p^{-1} = \begin{bmatrix} I_p & 0 \\ \frac{1}{h}C & I_q \end{bmatrix} \]

so that

\[ p^{-1}\Delta(s)p = \begin{bmatrix} I_p & 0 \\ \frac{1}{h}C & I_q \end{bmatrix} \begin{bmatrix} \Delta_{11}(s) & 0 \\ \frac{1}{h}C & I_q \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ \frac{1}{h}C & I_q \end{bmatrix} \]

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\[ \begin{bmatrix} \Delta_{11}(s) & 0 \\ \frac{1}{h} \Delta_{11}(s) + \Delta_{21}(s) + \frac{1}{h} \Delta_{22}(s)C & \Delta_{22}(s) \end{bmatrix} \]

\[ = \Delta_{11}(s) + \Delta_{22}(s). \quad \text{[by (67)]} \]

By repeated application of Maschke's theorem we can obtain a breakdown of any representation of a finite group into a direct sum of irreducible components. The following result shows that by using nothing more than a finite algebraic extension of \( \mathbb{R} \) the representation can be broken down into a direct sum of absolutely irreducible components.

**Theorem 1.7** Let \( S \) be a finite group and let \( L: S \to \text{GL}_n(V) \) be a representation of \( S \). Then there exists a finite algebraic extension field \( F \) of \( \mathbb{R} \) such that \( L \) is equivalent over \( F \) to a direct sum of absolutely irreducible representations of \( S \).

**Proof:** Choose a basis \( E \) of \( V \) and consider the matrix representation \( \Delta : s \mapsto [L(s)]_E^E \in \text{GL}(n, \mathbb{R}) \). If \( \Delta \) is not absolutely irreducible, then by definition there exists some extension field \( \mathbb{R}_1 \) of \( \mathbb{R} \) and \( P \in \text{GL}(n, \mathbb{R}_1) \) such that

\[ P^{-1} \Delta(s) P = \begin{bmatrix} \Delta_{11}(s) & 0 \\ \Delta_{21}(s) & \Delta_{22}(s) \end{bmatrix}, \quad s \in S, (68) \]

where \( \Delta_{11}(s) \) is \( p \times p \) and \( \Delta_{22}(s) \) is \( q \times q \). By Maschke's theorem (Theorem 1.6) we can assume \( \Delta_{21}(s) = 0, s \in S \), in
(68). It follows that

\[ M_{\alpha,\beta} = P(\alpha I_p + \beta I_q)P^{-1} \]

commutes with every \( \Delta(s) \), \( s \in S \), for any \( \alpha, \beta \) in \( R_1 \). Thus as a system of homogeneous linear equations over \( R_1 \) (since \( S \) is finite)

\[ X_\Delta(s) - \Delta(s)X = 0 , \ s \in S , \]  

(69)

has a null space of dimension at least 2. Hence, over \( R \) the system (69) has a null space of dimension at least 2 also. It follows that there exists \( M \in M_n(R) \), \( M \neq \gamma I_n \) for any \( \gamma \in R \), and \( M_\Delta(s) = \Delta(s)M \) for all \( s \in S \). Let \( \mu \) be an eigenvalue of \( M \), set \( W = M - \mu I_n \) and note that \( W \neq 0 \) and \( W \) also satisfies

\[ W_\Delta(s) = \Delta(s)W , \ s \in S . \]  

(70)

Moreover \( R(\mu) \) is a finite algebraic extension of \( R \). Now \( W \) is singular so that \( 0 < k = \rho(W) < n \) and we obtain matrices \( A \) and \( B \) in \( GL(n, R(\mu)) \) such that

\[ ABW = I_k + 0 . \]  

(71)

From (70) we have

\[ (ABW)(B^{-1}_\Delta(s)B) = (A_\Delta(s)A^{-1})(ABW) . \]  

(72)

Set

\[ B^{-1}_\Delta(s)B = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix} \]
\[ A \Delta(s) A^{-1} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} , \]

in which the partitionings are conformal with (71). Then

\[
\begin{bmatrix} K_{11}(s) & K_{12}(s) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}
\]

\[
= \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} G_{11}(s) & 0 \\ G_{21}(s) & 0 \end{bmatrix}
\]

so that \( K_{12}(s) = 0 \). Thus over \( R(\mu) \), \( \Delta(s) \) is equivalent to the representation

\[
s = \begin{bmatrix} K_{11}(s) & 0 \\ K_{21}(s) & K_{22}(s) \end{bmatrix}
\]

and by Maschke's theorem \( \Delta(s) \) is equivalent over \( R(\mu) \) to the
representation

\[ s \sim K_{11}(s) \circ K_{22}(s). \]

We have proved that if \( \Delta : S \to GL(n, R) \) is not absolutely irreducible, then there exists a finite algebraic extension \( R(\mu) \) of \( R \) such that

\[ \Delta \sim K_{11} \circ K_{22} \]

over \( R(\mu) \). By repeating this argument we can reduce each of \( K_{11} \) and \( K_{22} \), if necessary, by making a finite algebraic extension of \( R(\mu) \). Obviously the process terminates after a finite number of steps with

\[ \Delta \sim \Delta_1 \circ \cdots \circ \Delta_p \]

over \( F \), a field obtained from \( R \) by a finite sequence of finite algebraic extensions. By a standard elementary theorem \( F \) is, in fact, a simple algebraic extension of \( R \).

There are several ways of reformulating Theorem 1.7. Thus, in the equivalence class of representations equivalent to \( \Delta \) over some field there is a direct sum of absolutely irreducible representations such that the entries of the representing matrices lie in a finite algebraic extension \( F \) of \( R \). Or, there exists a finite algebraic extension \( F \) of \( R \) such that the vector space \( V \) over \( R \) is a direct sum of absolutely irreducible subspaces. We shall temporarily defer the question of the uniqueness of the components.

Another cornerstone in the theory of group representations is
Theorem 1.8 (Schur's Lemma) (a) Let $\Omega \subset M_n(R)$, $\Gamma \subset M_m(R)$ be irreducible sets of matrices over $R$ and assume $M \in M_{m,n}(R)$ links $\Omega$ and $\Gamma$. Then either $M = 0$ or $m = n$, and $M$ is nonsingular.

(b) Let $L: S \to L(V,V)$, $K: S \to L(U,U)$ be two irreducible representations of the groupoid $S$ and assume $T$ links $L$ and $K$. Then either $T = 0$ or $\dim U = \dim V$, $T$ is a bijection, and $L \sim K$.

Proof: (a) Let $C(M)$ be the column space of $M$ and assume $M \neq 0$. If $A \in \Omega$ and $B \in \Gamma$ satisfy

$$MA = BM,$$  \hspace{1cm} (73)

then

$$BM(t) = MA(t)$$

$$= \sum_{j=1}^{n} M(j) a_{jt}$$

$$\in C(M)$$

so that

$$BC(M) \subset C(M).$$

Since for any $B \in \Gamma$ there is an $A \in \Omega$ such that (73) holds, it follows that

$$\Gamma C(M) \subset C(M).$$  \hspace{1cm} (74)
Suppose that $\dim C(M) = s < m$. Then it is easy to see that (74) implies that there exists $P \in GL(m, \mathbb{R})$ such that

$$P^{-1}BP = \begin{bmatrix}
B_{11} & 0 \\
B_{21} & B_{22}
\end{bmatrix}, \quad B \in \Gamma,$$

(75)

where $B_{11}$ is $(m-s)$-square and $B_{22}$ is $s$-square. But (75) contradicts the irreducibility of $\Gamma$. Thus $\dim C(M) = m$ (i.e., the rank cannot exceed the number of rows, $m$) so that $m \leq n$.

By taking transposes in (73) we have the following situation. For each $A^T \in \Omega^T$ there is a $B^T \in \Gamma^T$ such that

$$M^TB^T = A^TM^T.$$

(The set $\Omega^T$ of all transposes of matrices in $\Omega$ is also irreducible; see Exercise 2.) Repeating the above argument using the preceding equation, we conclude that

$$\dim C(M^T) = n$$

or equivalently

$$\dim R(M) = n$$

where $R(M)$ is the row space of $M$. But $M$ has $m$ rows so $m \geq n$. Thus $m = n$ and $M$ is nonsingular.

(b) We have

$$L(s)T = TK(s), \quad s \in S$$

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and choosing bases $E$ and $F$ of $V$ and $U$ respectively

$$MA(s) = B(s)M$$

where $M = [T]_F^E$, $A(s) = [K(s)]_F^E$, $B(s) = [L(s)]_E^E$, $s \in S$. The irreducibility of $K$ and $L$ implies that both \{$(A(s)| s \in S)$ and $(B(s) | s \in S)$\} are irreducible sets of matrices. If we apply (a) we conclude that $M = 0$ or $M$ is square, i.e., $\dim U = \dim V$, and $M$ is nonsingular, i.e., $T$ is a bijection.

As a consequence of Schur's Lemma we have

**Theorem 1.9 (a)** Let $\Omega \subseteq M_n(R)$ be an irreducible set of matrices over $R$ and assume that $M \in M_n(R)$ commutes with each $A \in \Omega$:

$$MA = AM, \ A \in \Omega.$$ 

Then exactly one of the following alternatives holds:

(i) $M = 0$;

(ii) $M \neq 0$, $R$ contains an eigenvalue of $M$ and $M = \alpha I_n$;

(iii) $M \neq 0$, $R$ contains no eigenvalue of $M$ and if $\alpha$ is any eigenvalue of $M$ then $\Omega$ is reducible over the algebraic extension field $R(\alpha)$.

(b) Let $L : S \rightarrow L(V, V)$ be an irreducible representation of a groupoid $S$ and assume that $T \in L(V, V)$ commutes with each $L(s) \in \text{Im} L$:

$$TL(s) = L(s)T.$$

Then exactly one of the following alternatives holds:
(i) $T = 0$;

(ii) $T \neq 0$, $R$ contains an eigenvalue of $T$ and $T = \alpha I_V$;

(iii) $T \neq 0$, $R$ contains no eigenvalue of $T$ and if $\alpha$ is any eigenvalue of $T$ then $L$ is reducible if $V$ is regarded as a vector space over the extension field $R(\alpha)$.

Proof: (a) Suppose $M \neq 0$. Let $\alpha$ be any eigenvalue of $M$ so that $M - \alpha I_n$ is singular and clearly commutes with every $A \in \Omega$. If $\alpha \in R$ then by Theorem 1.8(a) $M - \alpha I_n = 0$, so that (ii) holds. If $\alpha \notin R$ then $M - \alpha I_n$ commutes with every $A \in \Omega$ when the matrices are regarded as being over $R(\alpha)$. If $\Omega$ were irreducible over $R(\alpha)$, it would follow again from Theorem 1.8(a) that $M = \alpha I_n$ and since $M$ is over $R$ that $\alpha \in R$, contrary to assumption. Thus $\Omega$ is reducible over the extension field $R(\alpha)$.

The proof of (b) follows immediately from (a) by taking matrix representations.

If $\Delta : S \to GL(m, R)$ is a matrix representation of the group $S$, then each entry $\Delta_{ij}$ can be regarded as a function on $S$ taking on values in $R$:

$$\Delta_{ij} : S \to R.$$ 

In general suppose $S$ is a finite group of order $h$ and $f \in R^S$, i.e., $f$ is a function on $S$ taking on values in $R$. If $g \in R^S$ is another such function, then the scalar product of $f$ and $g$ is defined by
\[ (f, g) = \frac{1}{h} \sum_{s \in S} f(s)g(s^{-1}). \] (76)

One of the interesting consequences of Schur's Lemma is the following orthogonality statement for the \( \Delta_{ij} \) functions.

**Theorem 1.10** Let \( \Delta = [\Delta_{ij}] \), \( \kappa = [\kappa_{ij}] \) be proper irreducible matrix representations of the finite group \( S \) of degrees \( m \) and \( n \) respectively. If \( \Delta \) and \( \kappa \) are inequivalent then

\[ (\Delta_{ip}, \kappa_{jq}) = 0, \quad i, p = 1, \ldots, m, \quad j, q = 1, \ldots, n. \] (77)

If \( \Delta \) is absolutely irreducible then

\[ (\Delta_{ip}, \Delta_{jq}) = \frac{1}{m} \delta_{ij} \delta_{pq}. \] (78)

**Proof:** Let \( U \in M_{m,n}(\mathbb{R}) \) be any matrix and set

\[ V = \sum_{s \in S} \Delta(s)UK(s^{-1}) \in M_{m,n}(\mathbb{R}). \] (79)

Then for any \( t \in S \)

\[ \Delta(t)V\kappa(t^{-1}) = V \]

so that

\[ \Delta(t)V = V\kappa(t) \]

and \( V \) links \( \Delta \) and \( \kappa \). Since \( \Delta \) and \( \kappa \) are inequivalent, it follows from Theorem 1.8(a) that \( V = 0 \). Thus from (79),
\[ 0 = v_{ij} \]

\[ = \sum_{s \in S} \sum_{p, q} \Delta_{ip}(s) u_{pq} \kappa_{qj}(s^{-1}) \]

\[ = \sum_{p, q} u_{pq} h(\Delta_{ip}, \kappa_{qj}) , \]

\( h = |S| \). Since \( U \) is arbitrary (77) follows.

If \( \Delta = \kappa \) in (79) and \( \Delta \) is absolutely irreducible, then, by Theorem 1.9(a), \( m = n, V = \alpha I_m, \alpha \in \mathbb{R} \),

\[ m\alpha = \text{tr} \ V \]

\[ = \sum_{s \in S} \text{tr} \Delta(s) U \Delta(s^{-1}) \]

\[ = h \text{tr} \ U , \]

and

\[ \alpha = \frac{h \text{tr} U}{m} . \]

Thus

\[ \frac{h}{m} \text{tr} U \delta_{ij} = \alpha \delta_{ij} \]

\[ = v_{ij} \]

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\[
= \sum_{p,q} u_{pq} h(\Delta_{pq}, \Delta_{pq})
\]

or

\[
\frac{h}{m} \sum_{p=1}^{m} u_{pp} \delta_{ij} = \sum_{p,q=1}^{m} u_{pq} h(\Delta_{pq}, \Delta_{pq}). \tag{80}
\]

The matrix \( U \) is arbitrary so matching coefficients in (80) produces (78).

Schur's Lemma can be used to show that for an absolutely irreducible matrix representation \( \Delta \) of a finite group, the entry functions are l.i.. To be precise we have

**Theorem 1.11 (Burnside-Frobenius-Schur)** Let \( L_k : S \rightarrow GL_{n_k} (V_k) \), \( k = 1, \ldots, m \), be pairwise inequivalent absolutely irreducible representations of the finite group \( S \) and set \( L = \sum_{k=1}^{m} L_k \in L(\Sigma V_k, \Sigma V_k) \). Then \( L \) is a representation of \( S \) and

\[
\text{dim}(\text{Im} L) = n_1^2 + \cdots + n_m^2. \tag{81}
\]

**Proof:** Each \( L(s) \) operates on \( V = \sum_{k=1}^{m} V_k \), \( \text{dim} V = n_1 + \cdots + n_m \). Now choose bases of \( V_1, \ldots, V_m \) and switch over to a matrix representation of the form

\[
\Delta(s) = \sum_{k=1}^{m} \Delta^k(s). \tag{82}
\]
If (81) does not hold, then there exist matrices $K^k \in M_{n_k}(R)$ not all $0$, $k = 1, \ldots, m$, such that
\[ \sum_{k=1}^{m} \text{tr}(K^k \Delta^k(s)) = 0, \quad s \in S \]
or equivalently,
\[ \sum_{k=1}^{m} \sum_{p,i=1}^{n_k} K^k_{pi} \Delta^k_{ip}(s) = 0, \quad s \in S. \quad (83) \]

For a fixed $r$, multiply through (83) by $\Delta^r_{ij}(s^{-1})$ and sum on $s$ to obtain
\[ 0 = \sum_{k=1}^{m} \sum_{p,i=1}^{n_k} K^k_{pi} \Delta^k_{ip}(\Delta^r_{ij}, \Delta^r_{qj}) \]
\[ = \sum_{k=1}^{m} \sum_{p,i=1}^{n_k} K^k_{pi} \delta^r_{ij} \delta^r_{qj} \quad \text{[by (77)]} \]
\[ = \frac{1}{n_r} \sum_{p,i}^{n_r} K^r_{pi} \delta^r_{ij} \delta_{pq} \quad \text{[by (78)]} \]
\[ = \frac{1}{n_r} K^r_{qj}. \]

Thus every $K^k = 0$. 

We shall modify this proof of Theorem 1.11 in Exercises 4 and 5 in such a way that the assumption that $S$ is finite can be dropped. In any event, Theorem 1.11 states that if $\Delta^k : S \to GL(n_k, R)$,
k = 1, ..., m, are absolutely irreducible pairwise inequivalent matrix representations of the group S, then any matrix of the form

\[ \sum_{k=1}^{m} A_k, \quad A_k \in M_{n_k}(\mathbb{R}) \]

is a linear combination of matrices of the form (82), s ∈ S. Of course each set of matrices

\[ \Gamma_k = \{A_k(s), \quad s \in S\} \]

is absolutely irreducible. Now the representation (8) of \( S_m \) can be expressed by Theorem 1.6 as a direct sum of absolutely irreducible representations (over an algebraic field extension of \( \mathbb{R} \) if necessary). According to Theorem 1.11 each of these absolutely irreducible components of \( P \) spans an isomorphic copy of a complete matrix algebra over \( \mathbb{R} \), i.e., the linear closure of the range of each of these absolutely irreducible components contains all linear transformations of the space on which it operates. Now the linear closure of the range of the representation \( \hat{\Pi}' \) of \( GL_n(V) \) in (7) is precisely the algebra \( \mathfrak{g}_m \) of bisymmetric transformations as we noted in the remark I. Hence

\[ \langle \text{Im } \Pi' \rangle \]

is the set of all linear transformations on \( V \otimes V \) commuting with every element of

\[ \langle \text{Im } P \rangle \]
and \((\text{Im } P)\) is a direct sum of complete matrix algebras. Thus we are motivated to investigate the structure of the centralizer of a direct sum of complete matrix algebras. To state our result concerning this situation, let

\[
\Gamma_k \subset M_{n_k}(R), \quad k = 1, \ldots, m,
\]

be an absolutely irreducible set of matrices and assume that there exists a common indexing set \(\mathcal{J}\) for the \(\Gamma_k\), i.e.,

\[
\Gamma_k = \{A_k(\alpha) : \alpha \in \mathcal{J}\}. \quad (84)
\]

Let \(\Gamma\) consist of all matrices in \(M_n(R), \quad n = n_1 + \cdots + n_m\), of the form

\[
\sum_{k=1}^{m} A_k(\alpha), \quad \alpha \in \mathcal{J}.
\]

Also let \(C(\Gamma)\) denote the centralizer of \(\Gamma\), that is, the totality of matrices in \(M_n(R)\) commuting with every matrix in \(\Gamma\). If \(Q \in \text{GL}(n, R)\) then \(Q^{-1}\Gamma Q\) is the totality of matrices \(Q^{-1}A Q\), \(A \in \Gamma\). Obviously \(Q^{-1}\Gamma Q\) is equivalent to \(\Gamma\) and

\[
C(Q^{-1}\Gamma Q) = Q^{-1}C(\Gamma)Q.
\]

We will assume that \(\Gamma_1, \ldots, \Gamma_r\) are all the inequivalent sets among

\[
\Gamma_1, \ldots, \Gamma_m \quad (85)
\]

and that precisely \(\epsilon_t\) of the sets \(85)\) are equivalent to \(\Gamma_t\),
\( t = 1, \ldots, r \). That is, the sets \( \Gamma_1, \ldots, \Gamma_r \) are inequivalent in the sense that for \( i \neq j \), there is no fixed matrix \( Q \) such that \( Q^{-1} A_i(\alpha)Q = A_j(\alpha), \ \alpha \in \mathcal{G} \). Also, for each \( t = 1, \ldots, r \), there are precisely \( e_t \) values of \( k \) for which there exists a nonsingular matrix \( Q_t \) such that \( Q_t^{-1} A_k(\alpha)Q_t = A_t(\alpha), \ \alpha \in \mathcal{G} \).

Thus by replacing \( \Gamma \) by an equivalent set we can assume that the matrices in \( \Gamma \) have the form of a direct sum of Kronecker products:

\[
\sum_{k=1}^{r} \mathbf{I}_{e_k} \otimes A_k(\alpha), \ \alpha \in \mathcal{G}.
\]  

(86)

After this normalization we can assert the following important result.

**Theorem 1.12** If \( \Gamma \) consists of all matrices of the form (86) in which \( \Gamma_1, \ldots, \Gamma_r \) are pairwise inequivalent absolutely irreducible sets of matrices then \( C(\Gamma) \) is equivalent to the totality of matrices of the form

\[
\sum_{k=1}^{r} \mathbf{I}_{n_k} \otimes B_k, \ B_k \in \mathcal{M}_{e_k}(\mathbb{R}), \ k = 1, \ldots, r.
\]  

(87)

**Proof:** Assume that \( B \) commutes with every matrix (86) and partition \( B \) conformally with (86):

\[
B = [B_{st}], \ s, t = 1, \ldots, r
\]

where \( B_{st} \) is an \( e_s n_s \times e_t n_t \) block in \( B \). Then

\[
(I_{e_s} \otimes A_s)B_{st} = B_{st}(I_{e_t} \otimes A_t).
\]  

(88)
Now temporarily denote \( B^{st} \) by \( D \) to reduce the indices so that (88) becomes

\[
\begin{array}{c}
\varepsilon_s \\
(A_s + \ldots + A_s)D = D(A_t + \ldots + A_t).
\end{array}
\]  

(89)

Next, partition \( D \) conformally with the block multiplications indicated in (89): \( D = [D^{pq}] \) where every \( D^{pq} \) is \( n_s \times n_t \), \( p = 1, \ldots, e_s \), \( q = 1, \ldots, e_t \). Then (89) states that for any pair \( s, t \in \{1, \ldots, r\} \) and any \( \alpha \in \mathcal{F} \)

\[
A_s(\alpha)D^{pq} = D^{pq}A_t(\alpha),
\]  

(90)

so that \( D^{pq} \) links \( \Gamma_t \) and \( \Gamma_s \). If \( s \neq t \) then by Theorem 1.8(a) every \( D^{pq} = 0 \) so that \( D = B^{st} = 0 \). If \( s = t \), \( D^{pq} \) commutes with every \( A_t \in \Gamma_t \) and since \( \Gamma_t \) is absolutely irreducible it follows by Theorem 1.9(a)(i), (ii), that

\[
D^{pq} = \beta_{pq} I_{n_t}, \quad \beta_{pq} \in \mathbb{R}.
\]  

(91)

Thus we have proved

\[ B^{st} = 0, \quad s \neq t, \]

and

\[ B^{kk} = B_k \otimes I_{n_k}, \quad B_k \in \mathcal{M}_{n_k}(\mathbb{R}). \]

Hence
\[ B = \sum_{k=1}^{r} B_k \otimes I_{n_k}. \]

But we know that there exists a nonsingular \( Q_k \) such that
\[ Q_k^{-1}(B_k \otimes I_{n_k})Q_k = I_{n_k} \otimes B_k \]
and the result follows.

In view of Theorem 1.12, we can codify some of our remarks concerning the relationship between the representations (7) and (8).

**Theorem 1.13** Let
\[ \Pi^{'m} : GL_n(V) \to GL_N(\otimes^m V), \quad N = n^m, \]
denote the \( m^{\text{th}} \) tensor power representation. Then there is a simple algebraic extension field \( F \) of \( R \) such that over \( F \)
\[ \otimes^m V \]
is a direct sum of absolutely irreducible invariant subspaces.
That is, the representation \( \Pi^{'m} \) is equivalent over \( F \) to a direct sum of absolutely irreducible component representations. If
\[ P : S_m \to GL_N(\otimes^m V) \]
is the permutation operator representation of \( S_m \), then \( F \) is the field extension required to decompose \( P \) into a direct sum of absolutely irreducible representations according to Theorem 1.7.
Let
\[ p_1, \ldots, p_r \] (93)
be such a list of absolutely irreducible pairwise inequivalent components of \( P \) in which \( p_t \) occurs with multiplicity \( e_t \), and acts on a subspace of \( \otimes V \) of dimension \( n_t \), \( t = 1, \ldots, r \).

Then, corresponding to (93), there is a list of \( r \) absolutely irreducible pairwise inequivalent components of \( \Pi' \),
\[ \Pi^1, \ldots, \Pi^r \] (94)
in which \( \Pi^t \) occurs with multiplicity \( n_t \) and acts on a subspace of \( \otimes V \) of dimension \( e_t \), \( t = 1, \ldots, r \).

Proof: According to Theorem 1.7, the representation \( P \) of the finite group \( S_m \) is a direct sum of absolutely irreducible representations over some simple algebraic extension \( F \) of \( \mathbb{R} \). This means that \( \otimes V \) can be decomposed into invariant subspaces such that each restriction of \( P \) to one of these subspaces is absolutely irreducible. Some of these restrictions may themselves be equivalent, and we can denote the subset of pairwise inequivalent absolutely irreducible such representations by (93) in which \( p_t \) occurs with multiplicity \( e_t \) and acts on a subspace of \( \otimes V \) of dimension \( n_t \), \( t = 1, \ldots, r \). By choosing an appropriate basis of \( \otimes V \) over \( F \) we can then obtain a matrix representation \( \Delta : S_m \to GL(N, F) \) corresponding to \( P \) of the form
\[ \Delta(\sigma) = \sum_{k=1}^{r} I_{e_k} \otimes \Delta_k(\sigma), \quad \sigma \in \mathbb{S}_m, \]  

(95)

in which the \( \Delta_k(\sigma) \) are the absolutely irreducible pairwise inequivalent matrix representations of \( \mathbb{S}_m \) corresponding respectively to the \( P^k, \quad k = 1, \ldots, r \). According to Theorem 1.11, any matrix of the form

\[ \sum_{k=1}^{r} I_{e_k} \otimes A_k, \quad A_k \in M_{n_k}(F) \]  

(96)

is a linear combination of matrices \( \Delta(\sigma) \) as \( \sigma \) runs over \( \mathbb{S}_m \) and obviously any linear combination of the matrices (95) has the form (96). The algebra \( B_m = \langle \Pi'(T), \quad T \in \text{GL}_n(V) \rangle \) of bisymmetric transformations is precisely the centralizer of the algebra \( \langle P(\sigma), \quad \sigma \in \mathbb{S}_m \rangle \) of all symmetry operators (see Theorem 2.7, Section 3.2). Now represent \( \Pi' \) in the same basis of \( \otimes V \) used to obtain (95) and call the corresponding matrix representation \( \Omega \). Then \( \langle \text{Im} \Omega \rangle \) consists of the totality of matrices in \( M_n(F) \) commuting with every matrix (95). Thus by Theorem 1.12 \( \Omega \) is equivalent to a representation (that we continue to call \( \Omega \)) of the form

\[ \Omega(T) = \sum_{k=1}^{r} I_{n_k} \otimes \Omega_k(T), \quad T \in \text{GL}_n(V). \]

Moreover, any matrix of the form
\[ \sum_{k=1}^{r} I_{n_k} \otimes B_k, \quad B_k \in M(e_k)(F) \tag{97} \]

is a linear combination of matrices \( \Omega(T) \) as \( T \) runs over \( GL_n(V) \).

Corresponding to each \( \Omega_k \) is an absolutely irreducible component \( \Pi^k \) of \( \Pi' \) occurring with multiplicity \( n_k \) and acting on a subspace of \( \otimes V \) of dimension \( e_k \), \( k = 1, \ldots, r \).

The representations \( \Omega_k \), \( k = 1, \ldots, r \), are also pairwise inequivalent. For, suppose e.g., that \( \Omega_1 \sim \Omega_2 \). Then there exists a fixed \( e_1 \)-square matrix \( Q \) such that \( Q \Omega_1(T)Q^{-1} = \Omega_2(T) \). But then we easily conclude that \( \Omega \) itself is equivalent to the representation

\[ \Omega'(T) = I_{n_1+n_2} \otimes \Omega_2(T) + \sum_{k=3}^{r} I_{n_k} \otimes \Omega_k(T). \]

However (97) tells us that

\[ \dim(\Omega(T), \ T \in GL_n(V)) = \sum_{k=1}^{r} e_k^2 \]

so that the same equality must hold for \( \Omega' \) replacing \( \Omega \). It is clear from the form of \( \Omega' \) that \( \dim(\Omega'(T), \ T \in GL_n(V)) \) is at most \( e_2^2 + \sum_{k=3}^{r} e_k^2 \).

We remark that we have not proved that the representations (94) constitute a complete list (to within equivalence) of absolutely irreducible pairwise inequivalent components of \( \Pi' \). To
do this we will show if an algebra \( \mathfrak{U} \) of matrices is equivalent to a direct sum of complete matrix algebras, i.e., \( \mathfrak{U} \cong \bigoplus_{k=1}^{m} \mathfrak{U}_k \), as is \( \mathfrak{B}_m = \langle \Pi(T), T \in \text{GL}_n(V) \rangle \), then every matrix representation of \( \mathfrak{U} \) is completely reducible and any absolutely irreducible representation of \( \mathfrak{U} \) must be equivalent to one of the \( \mathfrak{U}_k \). This remark will be clarified immediately before Theorem 2.1 in the next section.

It is interesting to note that Theorem 1.12 also implies that the relationship between the bisymmetric transformations and the symmetry operators in Section 3.2, Theorem 2.7, is perfectly reciprocal. To be precise we have

**Theorem 1.14** The centralizer in \( L(\otimes V \otimes V) \) of the algebra \( \mathfrak{B}_m \) of bisymmetric transformations is the algebra of all symmetry operators. That is, a linear transformation on \( \otimes V \) commutes with every tensor power transformation \( \Pi(T), T \in \text{GL}_n(V) \), if and only if it is a linear combination of permutation operators. Thus

\[
\text{C}(\mathfrak{B}_m) = \langle P(\sigma), \sigma \in S_m \rangle.
\]

**Proof:** Obviously

\[
\langle P(\sigma), \sigma \in S_m \rangle \subseteq \text{C}(\mathfrak{B}_m),
\]

i.e., any \( P(\sigma) \) commutes with every \( \Pi(T) \). Now as we saw in the proof of Theorem 1.13, we can find a basis of \( \otimes V \) such that the set of all matrix representations of transformations in \( \mathfrak{B}_m \) is precisely the set of all matrices of the form (97). The central-
izer of all matrices of the form (97) is by Theorem 1.12 equivalent to precisely the set of all matrices of the form (96). We thus can write down the dimension of this set, i.e., the dimension of $C(\mathfrak{g}_m)$ as a subspace of $L(\otimes V, \otimes V)$:

$$\dim C(\mathfrak{g}_m) = \sum_{k=1}^{r} n_k^2.$$ 

But as we saw in (95) and (96) the matrices corresponding to $P(\sigma), \sigma \in S_m$, with an appropriate basis choice for $\otimes V$, span precisely the totality of matrices of the form (96). Thus, we also have

$$\dim \langle P(\sigma), \sigma \in S_m \rangle = \sum_{k=1}^{r} n_k^2.$$ 

Since both sides of the above inclusion have the same dimension, their equality follows.

We shall denote the algebra of all symmetry operators in $L(\otimes V, \otimes V)$ by $\varphi_m$:

$$\varphi_m = \langle P(\sigma), \sigma \in S_m \rangle.$$ 

Thus Theorem 1.14 and Theorem 2.7 in Section 3.2 can be very simply expressed using the centralizer notation:

$$C(\varphi_m) = \mathfrak{g}_m,$$  \hspace{1cm} (98)

(Theorem 2.7, Section 3.2) and
\[ C(\mathbb{R}_m) = \varnothing_m, \] (99)

(Theorem 1.14).

In the next section, after we develop some of the general properties of the group ring, we shall prove, using Theorem 1.13 [in particular, the formula (96)], that the representation \( \mathbb{N}' \) is fully reducible. If we assume this we can use (99) to prove:

**Theorem 1.15** Assume that the representation

\[ \mathbb{N}' : \text{GL}_m(\mathbb{V}) \rightarrow \text{GL}_m(\otimes V), \quad N = n_m, \]

is fully reducible, let \( W_1 \subset \otimes V \) be an invariant subspace of \( \mathbb{N}' \), and let \( W_2 \) be a complementary invariant subspace,

\[ \otimes V = W_1 + W_2. \]

Let \( Y \) be the projection along \( W_1 \) parallel to \( W_2 \) so that \( Y(\otimes V) = W_1 \). Then \( Y \) is a symmetry operator, i.e.,

\[ Y \in \varnothing_m. \]

**Proof:** Any \( w \in \otimes V \) can be uniquely expressed as \( w = w_1 + w_2 \), \( w_i \in W_i \), \( i = 1, 2 \), and

\[ Yw = w_1. \]

Then

\[ \mathbb{N}'(T)Yw = \mathbb{N}'(T)w_1. \]

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and

\[ Y_{\Pi'}(T)w = Y_{\Pi'}(T)(w_1 + w_2) \]
\[ = Y_{\Pi'}(T)w_1 + Y_{\Pi'}(T)w_2 \]
\[ = Y_{\Pi'}(T)w_1 \]
\[ = \Pi'(T)w_1 \]
\[ = \Pi'(T)Yw, \]

because \( \Pi'(T)w_i \in W_i, \ i = 1,2 \). Thus for any \( T \in \text{GL}_n(V) \),

\[ Y_{\Pi'}(T) = \Pi'(T)Y \]

so that

\[ Y \in C(\Theta_m). \]

But then by (99), \( Y \in \Theta_m \).

We will refine the preceding result in the next section. To do this, as well as to prove that \( \Pi' \) is fully reducible, we will have to study the structure of the group algebra of a finite group as well as the structure of the representations of a direct sum of complete matrix algebras.

The final sequence of results in this section is devoted to showing that the list of absolutely irreducible representations (93) and the multiplicities with which they occur are in fact uniquely determined. This is a consequence of some general facts about representations of finite groups.
Definition 1.6 (Character) Let \( L : S \to L(V, V) \) be a representation of the groupoid \( S \). The character of \( L \), denoted by \( \chi_L \), is the trace function of \( L \). That is, \( \chi_L : S \to \mathbb{R} \) and

\[
\chi_L(s) = \text{tr} L(s), \quad s \in S.
\]

If \( \Delta : S \to M_n(\mathbb{R}) \) is a matrix representation of \( S \) then

\[
\chi_\Delta(s) = \text{tr} \Delta(s), \quad s \in S.
\]

An irreducible character is just a character of an irreducible representation. An absolutely irreducible or simple character of a group \( S \) is a character of an absolutely irreducible representation of \( S \). The degree of a character is the degree of the corresponding representation. The principal character is the character of degree 1 that is identically equal to 1.

There are a number of obvious facts about characters. Thus, if \( \lambda_i(s), \quad i = 1, \ldots, n \), are the eigenvalues of \( L(s) \), then

\[
\chi_L(s) = \sum_{i=1}^{n} \lambda_i(s), \quad s \in S.
\]

If \( L \sim M \) are equivalent representations of \( S \), then

\[
\chi_L = \chi_M.
\]

If \( S \) is a group, \( R = \mathbb{C} \), \( L(e) = I_V \), and \( s^k = e \) for some positive integer \( k \), then

\[
\chi_L(s^{-1}) = \overline{\chi_L(s)}.
\]
For, \( L(s)^k = I \) so the eigenvalues of \( L(s) \) are of modulus 1 and \( \lambda_i(s)^{−1} = \lambda_i(s) \). Also \( L(s)L(s)^{−1} = L(e) = I \) so that \( L(s)^{−1} = L(s)^{−1} \). Hence

\[
\chi_L(s^{−1}) = \frac{\sum_{i=1}^{n} \lambda_i(s)}{n} = \frac{\sum_{i=1}^{n} \lambda_i(s)}{n} = \chi_L(s).
\]

**Theorem 1.16** Let \( L_k : S \to GL(n_k) \), \( k = 1, \ldots, m \), be pairwise inequivalent absolutely irreducible representations of the group \( S \). Then the characters \( \chi_k = \chi_{L_k} \), \( k = 1, \ldots, m \), are I.I. functions on \( S \) to \( R \).

**Proof:** Suppose that

\[
\sum_{k=1}^{m} c_k \chi_k = 0,
\]

choose bases for each \( V_k \), let \( \Delta_k : S \to GL(n_k, R) \) be the matrix representation corresponding to \( L_k \), and set

\[
\Delta_k(s) = [a_{ij}^k(s)].
\]

Then

\[
0 = \sum_{k=1}^{m} c_k \chi_k(s)
\]
\[
= \sum_{k=1}^{m} c_k \text{tr} \Delta_k(s) \\
= \sum_{k=1}^{m} c_k \sum_{i=1}^{n_k} a_{ii}^k(s).
\]

From Theorem 1.11, the functions \( a_{ij}^k : S \rightarrow \mathbb{R} \) are l.i., and thus \( c_1 = \cdots = c_m = 0 \).

If \( L : S \rightarrow L(V, V) \) is a representation of the group \( S \) and \( W_1 \) is an invariant subspace under \( L \), then two representations of \( S \) can be defined in terms of \( L \): namely

\[
L_1(s) = L(s)|_{W_1}
\]

and

\[
L_2(s)q(v) = q(L(s)v)
\]

where \( q : V \rightarrow V/W_1 \) is the quotient map (verify that \( L_2 \) is indeed a representation). If \( \{e_1, \ldots, e_{r+1}, \ldots, e_n\} \) is a basis of \( W_1 \) and \( \{e_1, \ldots, e_r\} \) is a completion of this basis to a basis of \( V \), then the matrix representation corresponding to \( L \) is

\[
\Delta(s) = \begin{bmatrix}
A(s) & 0 \\
\vdots & \ddots \\
0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
* & \ddots & \ddots & \ddots & 0 \\
* & \ddots & \ddots & \ddots & \ddots \\
B(s)
\end{bmatrix}.
\] (100)
Clearly $B(s)$ is a matrix representation for $L_1$ and $A(s)$ is a matrix representation for $L_2$. Now if $L_1$ and $L_2$ are themselves reducible, it means that there are nonsingular matrices $P$ and $Q$ of appropriate size such that

$$P^{-1}A(s)P = \begin{bmatrix}
C(s) & 0 \\
- & - & - & - \\
* & D(s)
\end{bmatrix}$$

and

$$Q^{-1}B(s)Q = \begin{bmatrix}
E(s) & 0 \\
- & - & - & - \\
* & H(s)
\end{bmatrix}.$$  

Then

$$(P+Q)^{-1}A(s)(P+Q) = \begin{bmatrix}
C(s) & 0 & 0 & 0 \\
- & - & - & - & - & - \\
* & D(s) & 0 & 0 \\
* & * & * & * & * & * \\
E(s) & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * \\
H(s)
\end{bmatrix}.$$  

We can proceed in this fashion until we finally obtain a matrix $U \in \text{GL}(n,R)$ such that
and each $\Delta_{ii}(s)$ is an irreducible matrix representation. In terms of L this means that there exists a basis $E$ of $V$ such that $[L(s)]_E$ is the partitioned matrix on the right in (101).

Next we define a chain of invariant subspaces of $V$ as follows:

$V_p$ is the space spanned by the basis vectors in $E$ that correspond to the box $\Delta_{pp}$; $V_{p-1}$ is the space spanned by the basis vectors in $E$ that correspond to the two boxes $\Delta_{pp}$ and $\Delta_{p-1,p-1}$; $V_{p-2}$ is the space spanned by the basis vectors in $V$ that correspond to the three boxes $\Delta_{pp}$, $\Delta_{p-1,p-1}$, and $\Delta_{p-2,p-2}$, etc.

Clearly

$$V_p \subseteq V_{p-1} \subseteq V_{p-2} \subseteq \cdots \subseteq V_1 = V.$$ (102)

The space $V_p$ is an irreducible invariant subspace because $\Delta_{pp}$ is irreducible. Moreover we know from our discussion that $\Delta_{p-1,p-1}$ is the matrix representation corresponding to the quotient representation on $V_{p-1}/V_p$. Since $\Delta_{p-1,p-1}$ is irreducible it follows that this quotient representation is irreducible and hence $V_p$ must be maximal in $V_{p-1}$ in the sense that the only invariant subspace of $V_{p-1}$ containing $V_p$ is $V_{p-1}$ itself.
(confirm this). Similarly \( V_{p-1} \) is maximal in \( V_{p-2} \) with corresponding matrix representation \( \Delta_{p-2,p-2} \), \( V_2 \) is maximal in \( V_1 \) with corresponding matrix representation \( \Delta_{11} \). These representations on \( V_i/V_{i+1}, i = p, \ldots, 1 \) \( V_{p+1} = (0) \) so that \( V_p/(0) = V_p \) are called irreducible constituents or components of \( L \), and a chain of subspaces such as (102) is called a reduction of \( V \) (or \( L \)). Similarly, the irreducible matrix representations \( \Delta_{pp}, \Delta_{p-1,p-1}, \ldots, \Delta_{11} \) are called the irreducible constituents or components of \( \Delta \) and the matrix (101) is called a reduction of \( \Delta \) into irreducible components. The integer \( p \) is called the length of the reduction.

Theorem 1.17 Let \( \Delta : S \to \text{GL}(n,\mathbb{R}) \) and \( \kappa : S \to \text{GL}(m,\mathbb{R}) \) be two matrix representations of the group \( S \) and suppose that

\[
\chi_\Delta = \chi_\kappa.
\]

Then \( m = n \), any reduction of \( \Delta \) into absolutely irreducible components has the same length as any reduction of \( \kappa \) into absolutely irreducible components, and the components can be paired off into equivalent pairs.

Proof: Let a reduction of \( \Delta(s) \) be given by (101) and let

\[
K(s) = \begin{bmatrix}
\chi_{11}(s) & 0 & 0 & \cdots & 0 \\
\star & \chi_{22}(s) & 0 & \cdots & \star \\
\star & \star & \ddots & \ddots & \ddots \\
\star & \star & \cdots & \ddots & 0 \\
\star & \star & \cdots & \star & \chi_{qq}(s)
\end{bmatrix}
\]
be a corresponding reduction for \( K \). (We continue to denote the reductions of \( \Delta \) and \( K \) with these symbols.) The components \( \Delta_{ii} \) and \( K_{jj} \) are assumed here to be absolutely irreducible. Let \( \tau_1, \ldots, \tau_r \) be the complete list of absolutely irreducible inequivalent representations that appear in the cumulative list

\[ \Delta_{11}, \ldots, \Delta_{pp}, K_{11}, \ldots, K_{qq}, \]

and let \( \chi_i \) be the character of \( \tau_i \), \( i = 1, \ldots, r \). Suppose \( e_i \) of the representations in the list \( \Delta_{11}, \ldots, \Delta_{pp} \) and \( f_i \) in the list \( K_{11}, \ldots, K_{qq} \) are equivalent to \( \tau_i \), \( i = 1, \ldots, r \). Then

\[
\chi_{\Delta}(s) = \text{tr} \Delta(s) = \sum_{i=1}^{p} \text{tr} \Delta_{ii}(s) = \sum_{i=1}^{r} e_i \text{tr} \tau_i(s) = \sum_{i=1}^{r} e_i \chi_i(s)
\]

and similarly

\[
\chi_{K}(s) = \sum_{i=1}^{r} f_i \chi_i(s).
\]
Since \( \chi_\Delta = \chi_K \) we have
\[
\sum_{i=1}^{r} e_i \chi_i = \sum_{i=1}^{r} f_i \chi_i .
\]

By Theorem 1.16, \( e_i = f_i \), \( i = 1, \ldots, r \), and all the statements in the theorem follow immediately.

If we take \( \Delta = K \) in Theorem 1.17 we can conclude that the components in any reduction of \( \Delta \) into absolutely irreducible components are completely determined, including multiplicity, to within equivalence.

Example 1.4 (Burnside) Let \( S \subset \text{GL}(n, \mathbb{R}) \) be an absolutely irreducible group of nonsingular matrices. If every matrix \( A \in S \) satisfies
\[
A^r = I_n
\]
then \( S \) is said to have exponent \( r \). In this example we prove the celebrated result of Burnside to the effect that any such group of matrices \( S \) is finite and in fact the order of \( S \) satisfies
\[
|S| \leq n^3 .
\]

Every \( A \in S \) satisfies \( A^r = I_n \) so that the eigenvalues of \( A \) are roots of the polynomial \( \lambda^r - 1 \), and there are \( r \) possibilities for these roots. Let \( \lambda_i(A) \), \( i = 1, \ldots, n \), denote these eigenvalues so that
\[
\text{tr}(A) = \sum_{i=1}^{n} \lambda_i(A)
\]
can take on at most \(r^n\) values for \(A \in S\). Now \(S\) is an absolutely irreducible representation of itself so that by Exercise 4 (i.e., the extension of Theorem 1.11 to infinite groups)

\[
\dim\langle A, A \in S \rangle = n^2.
\]

Set \(N = n^2\), let \(A_1, \ldots, A_N\) be a basis of \(\langle A, A \in S \rangle\) chosen from \(S\), and notice that \(A_k A \in S\) for any \(A \in S\), \(k = 1, \ldots, N\). Then

\[
\text{tr}(A_k A) = t_k, \quad k = 1, \ldots, N,
\]

where \(t_k\) is one of the \(r^n\) possible values of \(\chi_0(A) = \text{tr} A\).

Clearly, there are at most \((r^n)^N = r^{n^3}\) possibilities for the \(N\)-tuple \(\tau = (t_1, \ldots, t_N)\). Regard the preceding equations as a system of \(N = n^2\) linear equations for the determination of the matrix \(A\). It is simple to see that by properly ordering the entries of \(A\) to form an \(N\)-tuple \(\alpha\), the coefficient matrix \(M\) has as its rows precisely the matrices \(A_k\), strung out as \(N\)-tuples.

Since the \(A_k\) are l.i., the equation is simply

\[
M \alpha = \tau
\]

where \(M\) is an \(N\)-square nonsingular matrix. Thus

\[
\alpha = M^{-1} \tau
\]
so that $A$ must be one of $r^{n^3}$ possible matrices.

The assumption in Example 1.4 that $S$ is absolutely irreducible is not necessary (see Exercise 11).

**Exercises**

1. Let $A_0, \ldots, A_p$ be $p+1$ matrices in $\mathbb{M}_N(R)$ that satisfy

$$A_i^2 = A_i, i = 0, \ldots, p, \quad A_iA_j = 0, \quad i \neq j.$$ 

Prove that there is a nonsingular matrix $B \in \mathbb{M}_N(R)$ such that

$$BA_iB^{-1} = \begin{bmatrix} \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
\end{bmatrix}, \quad i = 0, \ldots, p;$$

as $i$ runs from $0, \ldots, p$, the identity blocks appear down the main diagonal in immediate succession and are totally disjoint.

**Hint:** We can begin by assuming that none of the $A_i$ is zero since otherwise we can simply discard it. From $A_0^2 = A_0$ we know that the elementary divisors of $A_0$ (i.e., the characteristic matrix of $A_0$) are $\lambda - 1$ and $\lambda$. Clearly there are some of each, otherwise $A_0 = 0$ or $A_0 = I_N$, which is incompatible with $A_0A_j = 0, \quad j \neq 0$. Thus assume $p(A_0) = N_0$, $1 \leq N_0 < N$, and obtain a matrix $C \in \text{GL}(N,R)$ such that

$$CA_0C^{-1} = I_{N_0} \oplus 0.$$  

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Let $E_i = CA_i C^{-1}$, $i = 0, \ldots, p$, and observe that

$E_i^2 = E_i$, $E_i E_j = 0$, $i \neq j$. If we partition each $E_i$

conformally with $CA_i C^{-1}$, then by block multiplication we

conclude from $E_i E_j = E_0 E_j = 0$, $j \neq 0$, that

$$E_i = 0 + F_i$$

where $F_i$ is $(N - N_0)$-square, $i = 1, \ldots, p$. Now it is easy

to see that $F_i^2 = F_i$, $F_i F_j = 0$, $i \neq j$. By induction on

the size of the matrices there is an $(N - N_0)$-square nonsingular matrix $H$

such that $H^{-1} F_i H$ has an identity block on the main diagonal, is otherwise zero, and the identity blocks in $H^{-1} F_i H$ run down the main diagonal in immediate succession and are totally disjoint, $i = 1, \ldots, p$. Let $B = (T_{N_0 \rightarrow N}) C$.

If $N = 1$, then there can be only one matrix and the problem is trivial.

2. Prove that if $\Omega \subset M_m(R)$ is an irreducible set of matrices ,

then so is $\Omega' = \{A^T \mid A \in \Omega\}$.

3. Prove that $\langle C_m(T), T \in GL_n(V) \rangle = \langle C_m(T), T \in L(V,V) \rangle$.

Hint: Let $x_{ij}$ denote $n^2$ independent indeterminates over $R$ and let $X = [x_{ij}]$. Suppose $f$ is a linear functional on $M_n(R)$ which vanishes for all $C_m(A)$, $A \in GL(n,R)$, i.e., $f(C_m(A)) = 0$ whenever $\det A \neq 0$. Then obviously the polynomial

$$p(X) = f(C_m(X))\det X$$

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vanishes for all specializations of the indeterminates to elements of $R$. But then $p(X)$ is the 0 polynomial in the integral domain $R[x_{11}, \ldots, x_{nn}]$. Since $\det X \neq 0$ it follows that $f(C_m(X)) = 0$ so that $f(C_m(A)) = 0$ for all matrices $A \in M(R)$. In other words, any linear functional vanishing on $\langle C_m(A), A \in GL(n, R) \rangle$ vanishes on $\langle C_n(A), A \in M_n(R) \rangle$ and the asserted equality follows.

4. Prove the following result (Burnside's Theorem) valid for an arbitrary, not necessarily finite group.

Let $L : S \rightarrow GL_n(V)$ be an absolutely irreducible representation of the group $S$. Then

$$\dim(\text{Im} \ L) = n^2.$$  \hfill (103)

Hint: We shall switch over to a corresponding matrix representation $\Delta : S \rightarrow GL(n, R)$ and show that

$$\dim(\text{Im} \ \Delta) = n^2.$$  \hfill (104)

To simplify notation we let $A$ denote a typical matrix in $\text{Im} \ \Delta$. Assume then that $\dim(\text{Im} \ \Delta) = r < n^2$. Regard each matrix $A \in \text{Im} \ \Delta$ strung out as an $n^2$-tuple,

$$(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}).$$

There are $r$ matrices $A_1, \ldots, A_r$ that form a basis of $\langle \text{Im} \ \Delta \rangle$. Consider the $r \times n^2$ matrix $M$ whose $i^{th}$ row is $A_i$ written as an $n^2$-tuple. Since $A_1, \ldots, A_r$ form a basis of $\langle \text{Im} \ \Delta \rangle$, $M$ has rank $r < n^2$ and hence the set of vectors in $V^{n^2}(R)$ that comprise the null space of $M$ is not empty. In fact, we
know that its dimension is precisely \( s = n^2 - r \). Now \( s \geq 1 \) because we are assuming \( r < n^2 \). Let \( W \subseteq M_n(\mathbb{R}) \) denote the null space of \( M \) and let \( K_1, \ldots, K_s \) be a basis for \( W \). Then obviously \( \text{tr}(K_j A_i) = 0 \), \( i = 1, \ldots, r \), \( j = 1, \ldots, s \), or equivalently, since the \( A_i \) span \( \langle \text{Im} \Delta \rangle \) and the \( K_j \) span \( W \),

\[
\text{tr}(KA) = 0, \quad A \in \text{Im} \Delta, \quad K \in W. \tag{105}
\]

Also observe that if \( A \in \text{Im} \Delta \) and \( K \in W \), then for \( B \in \text{Im} \Delta \)

\[
\text{tr}((KA)B) = \text{tr}(K(AB)) = 0
\]

because \( AB \in \text{Im} \Delta \). Thus \( KA \in W \) for any \( K \in W, A \in \text{Im} \Delta \).

We can thus write

\[
K_i A = \sum_{q=1}^{s} u_{pq} K_q, \quad 1 \leq p \leq s, \tag{106}
\]

in which the coefficients \( u_{pq} \) depend on the matrix \( A \in \text{Im} \Delta \). Now setting

\[
K_p = \begin{bmatrix} k_{ij}^p \end{bmatrix} \in M_n(\mathbb{R}),
\]

and matching \((i,j)\) entries on either side of (106) we have
\[ \sum_{t=1}^{n} k_{it} a_{tj} = \sum_{q=1}^{s} u_{pq} k_{jq}^q, \quad 1 \leq p \leq s, \quad 1 \leq i, j \leq n. \quad (107) \]

Let \( U \in M_{s,n}(R) \) be the matrix \([u_{pq}]\), and for each \( i = 1, \ldots, n \), let \( P_i \in M_{s,n}(R) \) be the matrix whose \((p,t)\)-entry is \( k_{it}^p \), \( p = 1, \ldots, s \), \( t = 1, \ldots, n \). We can then rewrite (107) as

\[ (P_i A)_{pj} = (U P_i)_{pj}, \quad p = 1, \ldots, s, \quad j = 1, \ldots, n, \]

or

\[ P_i A = U P_i, \quad i = 1, \ldots, n. \quad (108) \]

In other words, each of the matrices \( P_i \in M_{s,n}(R) \) links \( \text{Im} \Delta \) and the set of matrices \( U \in M_{s,n}(R) \). We let \( \Gamma \subset M_{s,n}(R) \) denote the totality of \( U \) that appear in (108) (recall that for each \( A \in \text{Im} \Delta \) there is such a \( U \in \Gamma \)). Now not every \( P_i = 0 \), otherwise

\[ k_{it}^p = 0, \quad i = 1, \ldots, n, \quad t = 1, \ldots, n, \quad p = 1, \ldots, s, \]

i.e., \( K_1 = \cdots = K_s = 0 \), a contradiction to the fact that \( K_1, \ldots, K_s \) span \( W \) and \( W \neq 0 \). Thus suppose \( P_{i_0} \neq 0 \) so that

\[ P_{i_0} A = U P_{i_0}, \quad A \in \text{Im} \Delta. \quad (109) \]

There are two possibilities: \( \Gamma \) is irreducible over \( R \) or it is not.
Case 1. $\Gamma$ is irreducible $R$.

Then since $\text{Im} \Delta$ is absolutely irreducible we can conclude from Theorem 1.8(a) that $s = n$ and $P_\delta$ is nonsingular. Let $Q = P_{\delta\delta} \delta_0$ so that (109) becomes $QA = UQ$ and hence (108) can be written

$$P_i A = QA^{-1}P_i$$

or

$$(Q^{-1}P_i) A = A(Q^{-1}P_i), \quad A \in \text{Im} \Delta, \quad i = 1, \ldots, n. \quad (110)$$

By hypothesis $\text{Im} \Delta$ is absolutely irreducible over $R$ so we can conclude from Theorem 1.9(a)(ii) that $Q^{-1}P_i = \alpha_i I_n$, i.e.,

$$P_i = \alpha_i Q, \quad i = 1, \ldots, n. \quad (111)$$

Now $k^p_{it}$ is the $(p,t)$-entry of $P_i$ so that from (111)

$$k^p_{it} = \alpha_i q_{pt}$$

and thus the equation

$$\text{tr}(k A) = 0, \quad p = 1, \ldots, s \quad (= n)$$

becomes

$$0 = \sum_{i, t=1}^{n} k^p_{it} s_{ti}$$
\[ = \sum_{i, t=1}^{n} \alpha_i q_i p_t t_i \]

\[ = \sum_{i=1}^{n} (QA)_{p_i} \alpha_{i_1}, \quad p = 1, \ldots, n. \]

In other words, letting \( \alpha = (\alpha_1, \ldots, \alpha_n) \) we have

\[ QA\alpha = 0, \]

and since both \( Q \) and any \( A \in \text{Im} \Delta \) are nonsingular it follows that \( \alpha = 0 \), i.e., \( \alpha_1 = \cdots = \alpha_n = 0 \). But then from (111) \( P_1 = \cdots = P_n = 0 \) and as we saw before this is untenable because it implies that \( K_1 = \cdots = K_n = 0 \). Thus \( \Gamma \) cannot be irreducible over \( R \) and we are led to

**Case 2.** \( \Gamma \) is reducible over \( R \).

Then there is a nonsingular matrix \( N \in GL(s, R) \) such that for all \( U \in \Gamma \)

\[ NUN^{-1} = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \quad (112) \]

where \( X \in M_d(R) \) and \( d \) is fixed. We can assume also that either the totality \( \Omega \) of matrices \( X \) that appear in (112) is irreducible over \( R \) or that they are all the zero matrix. Let \( D = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \), and writing \( N = [n_{p,v}] \), define matrices
\[ K'_p = \sum_{\nu=1}^{s} n_{p\nu} K_{\nu}, \quad p = 1, \ldots, s. \quad (113) \]

Because of the nonsingularity of \( N \), \( K'_p \), \( p = 1, \ldots, s \), also comprise a spanning set for \( W \), and hence are not all 0. We compute for \( A \in \text{Im} \Delta \) and \( p = 1, \ldots, s \) that

\[
K'_p A = \sum_{\nu=1}^{s} n_{p\nu} K_{\nu} A
\]

\[
= \sum_{\nu=1}^{s} \sum_{q=1}^{s} n_{p\nu} u_{\nu q} K_q \quad \text{[from (106)]}
\]

\[
= \sum_{q=1}^{s} \left( \sum_{\nu=1}^{s} n_{p\nu} u_{\nu q} \right) K_q
\]

\[
= \sum_{q=1}^{s} (NU)_{pq} K_q
\]

\[
= \sum_{q=1}^{s} (DN)_{pq} K_q \quad \text{[from (112)]}
\]

\[
= \sum_{q=1}^{s} \sum_{\mu=1}^{s} d_{p\mu} u_{\mu q} K_q
\]

\[
= \sum_{\mu=1}^{s} d_{p\mu} \left( \sum_{q=1}^{s} n_{\mu q} K_q \right)
\]

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\[ \sum_{\mu=1}^{s} d_{n \mu} K'_{\mu} \quad \text{[from (113)]}. \]

Now \( d_{p \mu} = 0 \) when \( p = 1, \ldots, d \) and \( \mu = d+1, \ldots, s \), and \( d_{p \mu} = x_{p \mu} \) when \( p = 1, \ldots, d \), \( \mu = 1, \ldots, d \) because \( X \) is d-square in

\[
D = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}.
\]

Hence

\[
K'_A = \sum_{\mu=1}^{d} x_{p \mu} K'_\mu, \quad p = 1, \ldots, d. \tag{114}
\]

This is precisely the same situation that we had in (106) with the matrices \( X \in \Omega \) playing the role of the matrices \( U \). We also know that either every \( X \in \Omega \) is in fact 0 or \( \Omega \) is irreducible. If the first alternative were to hold, then, of course, (113) would imply that \( K'_A = 0, p = 1, \ldots, d \). This is a contradiction, of course, since \( K'_1, \ldots, K'_d \) are part of a basis of \( W \). If, on the other, the matrices \( \Omega \) are irreducible over \( \mathbb{R} \), we can repeat precisely the argument we gave in Case 1 to show that \( K'_1 = \ldots = K'_d = 0 \), which is a contradiction.

Thus the assumption that \( W \) is nonempty, i.e., that there exist nonzero matrices \( K \) such that
is untenable. We conclude that

\[ s = n^2 - r = 0 \quad \text{or} \quad \dim(\text{Im} \, \Delta) = r = n^2. \]

5. Prove the following extension of Theorem 1.11 to infinite groups.

Let \( L_k : S \to \text{GL}(V_k) \), \( k = 1, \ldots, m \), be absolutely irreducible pairwise inequivalent representations of the group \( S \). Let \( L : S \to \text{GL}(\bigoplus_{k=1}^m V_k) \) denote the representation defined by \( L(s) = \bigoplus_{k=1}^m L_k(s) \), \( n = n_1 + \cdots + n_m \). Then

\[ \dim(\text{Im} \, L) = n_1^2 + \cdots + n_m^2. \quad (115) \]

Hint: By choosing bases of \( V_1, \ldots, V_m \) we again switch over to a matrix representation of the form

\[ \Delta(x) = \sum_{k=1}^m \Delta_k(x), \quad x \in S \quad (116) \]

where \( \Delta_k : S \to \text{GL}(n_k, \mathbb{R}) \), \( k = 1, \ldots, m \). The problem then is to show that

\[ \dim(\text{Im} \, \Delta) = n_1^2 + \cdots + n_m^2. \quad (117) \]

If (117) does not hold, then exactly as in the preceding exercise there exists a nonzero matrix of the form

\[ K = \sum_{k=1}^m K^k, \quad K^k \in M(n_k, \mathbb{R}) \], \( k = 1, \ldots, m \), \( \quad (118) \)
such that

\[ \text{tr } K_\Delta(x) = \sum_{k=1}^{m} \text{tr}(k^k_\Delta_k(x)) \]

\[ = 0, \quad x \in S. \quad (119) \]

Observe that if (119) holds for a matrix \( K \) of the form (118), then as in the preceding exercise it holds for any matrix of the form

\[ \sum_{k=1}^{m} k^k_\Delta_k(x), \quad x \in S. \quad (120) \]

The totality \( W \) of solutions \( K \) of the form (118) to the equations (119) form a vector space of dimension \( s > 0 \). Now let

\[ K_p = \sum_{k=1}^{m} k^k_p, \quad p = 1, \ldots, s \quad (121) \]

be a basis of \( W \). Then from (120), for any \( x \in S \),

\[ \sum_{k=1}^{m} k^k_p \Delta_k(x) \in W, \quad p = 1, \ldots, s. \quad (122) \]

Hence for any \( x \in S \),

\[ \sum_{k=1}^{m} k^k_p \Delta_k(x) = \sum_{q=1}^{s} u_{pq} \left( \sum_{k=1}^{m} k^k_q \right), \quad p = 1, \ldots, s. \]
That is

$$\sum_{k=1}^{\cdot m} k_p \Delta_k(x) = \sum_{k=1}^{\cdot m} \sum_{q=1}^{\cdot s} u_{pq} k^k_q, \quad p = 1, \ldots, s, \quad (123)$$

or blockwise, for each \( k = 1, \ldots, m \), we have

$$\sum_{q=1}^{\cdot s} u_{pq} k^k_q = \sum_{q=1}^{\cdot s} U_p k^k_q \quad (124)$$

where \( U = \begin{pmatrix} u_{pq} \end{pmatrix} \in M_s(R) \) depends on \( x \in S \). We again introduce the set \( \Gamma \) of \( s \)-square matrices \( U \) that appear in (124).

If \( \Gamma \) were reducible over \( R \), we can argue precisely as in Exercise 4 to obtain a matrix \( N \in GL(s,R) \) for which

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} = D,$$

$$\begin{pmatrix} U \end{pmatrix}^{-1} = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

(125)

\( X \in M_s(R) \). Moreover, we can assume either that all \( X \)
appearing in (125) are \( 0 \) or that their totality, \( \Omega \), is irreducible. For each \( k = 1, \ldots, m \) we introduce matrices

$$k_p^k = \sum_{\nu=1}^{\cdot s} n_{p\nu} k^k_{\nu}, \quad p = 1, \ldots, s.$$  

(126)

We then compute, analogously to (114), that
\[ K_p^k' \Delta_k(x) = \sum_{\mu=1}^{d} x_{p\mu}^k' , \quad p = 1, \ldots, d. \quad (127) \]

If all the matrices \( X \) that appear in (125) are 0, then we conclude from (127) that

\[ K_p^k' = 0 , \quad p = 1, \ldots, d , \quad k = 1, \ldots, m . \quad (128) \]

But if we set

\[ K_p' = \sum_{k=1}^{m} K_p^k' , \quad p = 1, \ldots, s ; \]

then (126) is simply

\[ K_p' = \sum_{j=1}^{s} n_{p} \phi_j K_{\phi_j} , \quad p = 1, \ldots, s , \]

and since the \( K_1, \ldots, K_s \) are a basis of \( W \) and \( N \) is nonsingular so are the \( K_1', \ldots, K_s' \), a basis of \( W \). But (128) implies \( K_1' = \cdots = K_d' = 0 \), a contradiction. Thus the matrices in \( \Omega \) are not all 0, and \( \Omega \) is irreducible. We can then return to the equations (124) [instead of using (127)] and assume to begin with that \( \Gamma \) is irreducible.

If we proceed as in the argument in Case 1 in Exercise 4, we then conclude from (124) that for each fixed \( k = 1, \ldots, m \), either

\[ K_p^k = 0 , \quad p = 1, \ldots, s , \quad (129) \]
or that $\Delta_k$ and $\Gamma$ are equivalent. Since $\Delta_1, \ldots, \Delta_m$ are inequivalent, we conclude that there is at most one value of $k$, say $k = k_0$, for which (129) fails, i.e., not all of the matrices $K_p^{k_0}$ are 0, $p = 1, \ldots, s$. (There is at least one such value of $k$ since we are assuming that $W \neq 0$.) But then (119) becomes for $p = 1, \ldots, s$,

$$\text{tr}(K_p^{k_0} \Delta_{k_0}(s)) = 0, \ s \in S,$$

and we can apply Exercise 4 itself to conclude that $K_p^{k_0} = 0$, $p = 1, \ldots, s$. But then we have all of the $K_p^{k_0}$ equal to 0, again contradicting the assumption that $W \neq 0$.

6. Let $S$ and $S'$ be groups (denoted multiplicatively) and let $f : S \to S'$ be a group homomorphism. If $H = \ker f$, then $H \triangleleft S$; i.e., $H$ is a normal subgroup of $S$. Let $q : S \to S/H$ denote the quotient map, i.e., the canonical homomorphism that sends any element $s$ into the coset $sH$.

(a) Prove that $\bar{f} : S/H \to f(S)$ defined by $\bar{f}(q(s)) = f(s)$ is a group isomorphism, i.e., $S/H \cong f(S)$. (First isomorphism theorem for groups.)

(b) Prove that if $f$ is surjective and $K \triangleleft S$ then $f(K) \triangleleft S'$.

Hint: (a) Observe that $\bar{f}$ is well defined since $q(s) = q(t)$ implies $s^{-1}t \in H = \ker f$ and hence $f(s) = f(t)$. Also, if $\bar{f}(q(s)) = \bar{f}(q(t))$ then $f(s) = f(t)$ so that $s^{-1}t \in H$. But then $q(s) = q(t)$.

(b) If $f$ is surjective then any $s' \in S'$ is of the form
\[ s' = f(s), \quad s \in S. \] Then for \( k \in K, \quad s^{-1}ks \in K \) so that
\[ f(s^{-1}ks) \in f(K), \quad f(s)^{-1}f(k)f(s) \in f(K). \]

7. Let \( H \) and \( K \) be subgroups of \( S \), \( K \triangleleft S \), \( q: S \rightarrow S/K \) the quotient map. Prove:
   
   (a) \( HK \) is a group, \( K \triangleleft HK \), \( q(H) = HK/K \);
   
   (b) if \( h = q|H \) then \( \ker h = H \cap K \);
   
   (c) (Second isomorphism theorem for groups.) \( \overline{q}: H/H \cap K \rightarrow S/K \)
maps \( H/H \cap K \) isomorphically onto \( HK/K \), i.e.,
\[ H/H \cap K \cong HK/K. \]

(\( \overline{q} \) is defined as in Exercise 6, i.e., \( h(q)(s) = h(s). \))

Hint: (a) \( HK = KH \) because \( K \) is normal. Then \( (HK)^2 = H^2K^2 = HK \) and \( (HK)^{-1} = K^{-1}H^{-1} = KH = HK \) so \( HK \) is a subgroup of \( S \). If \( s \in S, \quad s^{-1}Ks = K \) so that this equation also holds for any \( s \in HK \). Thus \( K \triangleleft HK \). Also \( q(H) = \{q(a) \mid a \in H\} = \{aK \mid a \in H\} = HK/K \).

(b) \( s \in \ker h \iff s \in H \) and \( q(s) = sK = K \iff s \in H \) and \( s \in K \).

(c) From (a) and (b), \( h: H \rightarrow HK/K \) is surjective with kernel \( H \cap K \). Apply Exercise 6(a).

8. If \( H < S \), i.e., \( H \) is a subgroup of \( S \), and \( \nu: S \rightarrow \{sH, \quad s \in S\}, \quad i.e., \nu(s) = sH \), then \( |\text{Im } \nu| \) is called the index of \( H \) in \( S \) if this integer is finite. Prove that if \( H_1 \) and \( H_2 \) are of finite index in \( S \) then \( H_1 \cap H_2 \) is of finite index in \( S \).

Hint: By hypothesis, there are a finite number of left cosets
mod $H_1$, say $C_1, \ldots, C_p$, and a finite number of left cosets mod $H_2$, say $D_1, \ldots, D_q$, such that $S = \bigcup_{i=1}^{p} C_i = \bigcup_{j=1}^{q} D_j$. For fixed $i,j$ and any $s,t \in C_i \cap D_j$, we have $s^{-1}t \in H_1$, $s^{-1}t \in H_2$, $s^{-1}t \in H_1 \cap H_2$, i.e., the elements of $C_i \cap D_j$ are all in the same left coset mod $H_1 \cap H_2$.

Since $S = \bigcup_{i=1}^{p} \bigcup_{j=1}^{q} (C_i \cap D_j)$, we conclude that the index of $H_1 \cap H_2$ in $S$ is at most $pq$.

9. Let $\chi : S \to R$ be a group character. Show that $\chi(st) = \chi(ts)$ for any $s$ and $t$ in $S$.

Hint: $\chi(st) = \text{tr} \Delta(st) = \text{tr} \Delta(s)\Delta(t) = \text{tr} \Delta(t)\Delta(s) = \text{tr} \Delta(ts) = \chi(ts)$.

10. Let $\chi : S \to \mathbb{C}$ be a group character where $S$ is finite.

Prove that $\chi(s^{-1}) = \overline{\chi(s)}$.

Hint: Let $\chi(s) = \text{tr} \Delta(s)$. If $|S| = h$ then $\lambda_i^h(s) = 1$ where $\lambda_i(s)$ is any eigenvalue of $\Delta(s)$. Hence, since $\lambda_i(s^{-1}) = \lambda_i(s)^{-1} = \overline{\lambda_i(s)}$ we have $\chi(s^{-1}) = \text{tr} \Delta(s^{-1}) = \text{tr} \Delta(s)^{-1} = \sum_{i=1}^{n} \lambda_i(s)^{-1} = \sum_{i=1}^{n} \overline{\lambda_i(s)} = \overline{\text{tr} \Delta(s)} = \overline{\chi(s)}$.

11. Extend Example 1.4 to the general case of an arbitrary (i.e., not necessarily absolutely irreducible) subgroup $S \subset GL(n,R)$ of exponent $r$.

Hint: The argument here is by induction on $n$ with nothing to prove for $n = 1$. If $S$ is absolutely irreducible, we can apply Example 1.4. Assume then that $S$ is reducible over some extension field of $R$ (observe that we need only go to a
splitting field for \( \lambda^r - 1 \). Then there exists a nonsingular \( V \) such that

\[
V^{-1}AV = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}, \quad A \in S. \quad (130)
\]

Notice that \( \Psi_i = \{ A_{ii} \mid A \in S \} \), \( i = 1,2 \), are groups of exponent \( r \) and hence by induction are finite. Define \( S' \) to be the set of matrices on the right in (130), obviously a group, and define

\[
\varphi_i \left( \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix} \right) = A_{ii}, \quad i = 1,2.
\]

It is clear that \( \varphi_i \) is a surjective homomorphism of \( S' \) onto \( \Psi_i \), \( i = 1,2 \). Moreover,

\[
\ker \varphi_1 = \begin{bmatrix}
\mathbb{I}_p & 0 \\
A_{21} & A_{22}
\end{bmatrix},
\]

and
\[
\ker \varphi_2 = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & I_q
\end{bmatrix}.
\]

By Exercise 6(a), \( S'/\ker \varphi_i \cong \mathbb{U}_i \), \( i = 1, 2 \). Also by Exercise 7(c)

\[
\ker \varphi_1/\ker \varphi_2 \cap \ker \varphi_2 \cong \ker \varphi_1 \ker \varphi_2/\ker \varphi_2.
\]

But \( \ker \varphi_1 \ker \varphi_2/\ker \varphi_2 \subset S'/\ker \varphi_2 \cong \mathbb{U}_2 \) and by induction \( \mathbb{U}_2 \) is finite. Thus

\[
\ker \varphi_1/\ker \varphi_1 \cap \ker \varphi_2 \quad (131)
\]

is a finite group. Suppose \( X \in \ker \varphi_1 \cap \ker \varphi_2 \) so that

\[
X = \begin{bmatrix}
I_p & 0 \\
A_{21} & I_q
\end{bmatrix}.
\]

Now

\[
X^r = \begin{bmatrix}
I_p & 0 \\
rA_{21} & I_q
\end{bmatrix} = I_n
\]

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so \( rA_{21} = 0 \), i.e., \( A_{21} = 0 \) and \( X = I_n \), i.e.,

\[ \ker \varphi_1 \cap \ker \varphi_2 = 1. \]

But since the group (13) is finite it follows that \( \ker \varphi_1 \) is finite. Finally, since \( S' / \ker \varphi_1 \cong \mathbb{Z} \)

\( \cong \mathbb{Z} \) is finite and \( \ker \varphi_1 \) is finite we conclude that \( S' \) is finite.

12. Let \( A \in \text{GL}(n, \mathbb{R}) \) satisfy \( A^k = I_n \) for some positive integer \( k \), i.e., \( A \) has finite period \( k \). Show that \( A \) is similar to a diagonal matrix over an extension field of \( \mathbb{R} \).

Hint: Let \( r \) be the least such integer for which \( A^r = I_n \) and then \( S = \{ I_n, A, A^2, \ldots, A^{r-1} \} \) is a finite cyclic group. Regard \( S \) as a matrix representation of itself, i.e., \( \Delta(s) = s \) and by Theorem 1.6 (Maschke's theorem) \( \Delta \sim \Delta_{11} \uparrow \cdots \uparrow \Delta_{pp} \) where the \( \Delta_{ii} \) are absolutely irreducible over some extension field of \( \mathbb{R} \). Now the \( \Delta_{ii} \) are absolutely irreducible representations of the finite abelian (in fact, cyclic) group \( S \) so that each \( \Delta_{ii} \) is of degree 1 (prove this, using Theorem 1.9). Thus over some extension field (actually the splitting field of \( \chi^r - 1 \) will do) there is a nonsingular matrix \( P \) such that \( P^{-1}sP \) is a diagonal matrix for each \( s \in S \).

It should be observed that to establish the result in Exercise 12 does not really require the use of representation theory. For, \( \chi^r - 1 \) has distinct linear factors in its splitting field because \( r \chi^r - 1 \neq 0 \). Hence \( A \) is similar to a diagonal matrix because the minimal polynomial of \( A \) must divide \( \chi^r - 1 \).
Also note that if

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \in \text{GL}(4, \mathbb{R}) \]

then the minimal polynomial of \( A \) is \( \lambda^4 - 1 \) and to reduce \( A \) to a diagonal matrix we must go to the extension field \( \mathbb{C} \).

13. Let \( S \) be a group and \( V \) an \( S \)-module. Let \( e \) be the identity of \( S \). If \( \mathcal{L} : S \rightarrow \text{L}(V, V) \) is the representation function then show:

(a) \( \mathcal{L}(e)V \) is a proper \( S \)-module;

(b) \( V = \mathcal{L}(e)V = (I_V - \mathcal{L}(e))V \);

(c) \( \mathcal{L}(s) | (I_V - \mathcal{L}(e))V = 0 \) for all \( s \in S \).

(d) There exists a basis \( E \) of \( V \) such that if \( \Delta(s) = [\mathcal{L}(s)]_E^E \) then

\[
\Delta(s) = \begin{bmatrix}
K(s) & 0 \\
0 & 0
\end{bmatrix}
\]

where \( K : S \rightarrow \text{GL}(r, \mathbb{R}) \) is a proper matrix representation and \( r = \dim \mathcal{L}(e)V \).

Hint: (a) If \( s \in S \) then \( \mathcal{L}(s)(\mathcal{L}(e)v) = \mathcal{L}(se)v = \mathcal{L}(es)v = \mathcal{L}(e)\mathcal{L}(s)v \). Hence \( \mathcal{L}(s)\mathcal{L}(e)V \subset \mathcal{L}(e)V \). Also \( \mathcal{L}(e)\mathcal{L}(e)v = \mathcal{L}(e)v \) and thus \( \mathcal{L}(e)|_\mathcal{L}(e)V = I_{\mathcal{L}(e)V} \). We assert in general that if
M : S \rightarrow L(\mathcal{W}, W) and M(e) = I_{\mathcal{W}} \text{ then } M \text{ is proper. For,}
\begin{align*}
I_{\mathcal{W}} = M(e) &= M(s{s^{-1}}) = M(s)M(s^{-1}) \text{ and thus } M(s) \text{ has an inverse. Thus } L(s) \mid L(e)V \text{ has an inverse and it follows that } L(e)V \text{ is a proper } S\text{-module for which the representation function is simply } L \mid L(e)V. \\
(b) \text{ If } w \in L(e)V \cap (I_{\mathcal{V}} - L(e))V \text{ then } w = L(e)v = u - L(e)u \text{ for some } u, v \in V. \text{ Then as above } L(e)w = w \\
\text{and } L(e)w = L(e)(u - L(e)u) = 0. \text{ Hence } w = 0. \text{ Obviously } v = L(e)v + (v - L(e)v). \\
(c) \ L(s)(I_{\mathcal{V}} - L(e))v = L(s)v - L(s)L(e)v = 0. \\
(d) \text{ Let } [e_1, \ldots, e_n] \text{ be a basis of } V \text{ so chosen that } \\
[e_1, \ldots, e_r] \text{ is a basis of } L(e)V \text{ and } [e_{r+1}, \ldots, e_n] \text{ is a basis of } (I_{\mathcal{V}} - L(e))V. \text{ Obviously } [L(s)]^E \text{ has the indicated form. Observe that we also have proved that if } L : S \rightarrow L(V, V) \text{ is a representation of a group, then all the transformations } \\
L(s), \ s \in S, \text{ have the same rank } r.
\end{align*}

14. Let M be a group of matrices in \mathcal{M}_n(R) \text{ (not necessarily a group of nonsingular matrices). Prove:}
\begin{align*}
(a) \ \text{There exists an integer } r \text{ such that the rank of every } A \in M \text{ is } r. \\
(b) \ \text{There exists a nonsingular matrix } P \text{ such that for any } A \in M, \ P^{-1}AP = B + O_{n-r} \text{ where } O_{n-r} \text{ is an } (n-r)\text{-square zero matrix.} \\
(c) \ \text{If } H \text{ is the identity in } M, \text{ then } P^{-1}HP = I_r + O_{n-r} \text{ Hint: Let } V \text{ be an } n\text{-dimensional vector space over } R, \\
F \text{ a basis for } V, \text{ and } S \text{ the group of } T \in L(V, V) \text{ defined}
\end{align*}
by \( T \in S \) iff \( [T]_F^E = A \in M \). Then obviously \( V \) is an
\( S \)-module in which \( L(T) = T \) for \( T \in S \). According to
Exercise 13, there exists a basis \( E \) of \( V \) such that if
\( \Delta(T) = [L(T)]_E^E = [T]_E^E \) then \( \Delta(T) = K(T) \cdot 0_{n-r} \) and
\( K : S \rightarrow GL(r, \mathbb{R}) \) is a proper matrix representation. But
\( \Delta(T) = [T]_E^E = [I_\mathbb{V}]_F^E, (T) = [I_\mathbb{V}]_F^E \) or \( P^{-1}AP = B + 0_{n-r} \) where \( P = [I_\mathbb{V}]_E^F \), \( A = \Delta(T) \in M \) and \( B = K(T) \). Since \( K \) is proper we
know that \( K(T) = B \) has rank \( r \) and hence \( A \) has rank \( r \).
This proves (a) and (b). Also \( K(H) \) must be the identity
because \( K \) is proper, i.e., \( K(H) = I_r \). This proves (c).

15. Let \( M \) be the set of all matrices of the form \( c \begin{bmatrix} 1 & b & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \),
\( 0 \neq c \in \mathbb{R} \).
(a) Show that \( M \) is a group with respect to matrix
multiplication.
(b) Find a matrix \( P \) such that \( P^{-1}AP = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \) for
every \( A \in M \).

6.2 The Regular Representation
Recall the notation \( R^S_0 \) in Definition 2.1, Section 1.2, for
the totality of functions \( f : S \rightarrow \mathbb{R} \) whose function values in the
field \( \mathbb{R} \) are nonzero for at most a finite number of elements in
\( S \).

Definition 2.1 (Groupoid ring) If \( S \) is a groupoid, define
operations in \( R^S_0 \) as follows:
\[(f + g)(s) = f(s) + g(s), \quad (1)\]

\[(fg)(s) = \sum_{rt=s} f(r)g(t), \quad (2)\]

and

\[(cf)(s) = cf(s), \quad s \in S, \quad c \in R. \quad (3)\]

Then \(R^S_0\), together with these operations, is called the groupoid ring of \(S\) over \(R\).

We leave the verification of the following sequence of elementary remarks concerning the groupoid ring to the reader.

(a) The convolution multiplication defined in (2) is closed, i.e., \(f, g \in R^S_0\) implies that \(fg \in R^S_0\).

(b) Let \(\tau : S \to R^S_0\) be the function which associates with each \(s \in S\) the function defined by

\[\tau(s)(t) = \delta_{st}, \quad t \in S.\]

Then \(\tau\) is an injection. An element \(f \in R^S_0\) is written

\[f = \sum_{s \in S} f(s)\tau(s)\]

or simply

\[f = \sum_{s \in S} f(s)s.\]
(c) \( \tau : S \rightarrow R_0^S \) is a faithful representation. Any finite subset of \( \tau(S) \) is l.i. and

\[ \langle \tau(S) \rangle = R_0^S, \]

i.e., \( R_0^S \) is the set of all finite linear combinations

\[ \sum_{s \in S} c_s \tau(s). \]

If \( S \) has an identity \( e \) then \( \tau(e) \) is a multiplicative identity in \( R_0^S \).

(d) If \( L : S \rightarrow L(V,V) \) is a representation function then \( L \) can be extended to a representation of \( R_0^S \) by

\[ L(f) = \sum_{s \in S} f(s)\tau(s). \]  \hspace{1cm} (4)

In other words, the definition (4) has the following properties:

for any \( f \) and \( g \) in \( R_0^S \), \( c \in \mathbb{R} \),

\[ L(fg) = L(f)L(g), \]  \hspace{1cm} (5)

\[ L(f+g) = L(f) + L(g), \]  \hspace{1cm} (6)

and

\[ L(cf) = cL(f). \]

The vector space \( V \) will also be called an \( R_0^S \)-module.

(e) If \( S \) is a semi-group, then \( R_0^S \) is a linear associative
algebra called the semi-group algebra of $S$ over $R$.

Henceforth we shall assume that $S$ is a group (not necessarily finite). The group-algebra defined above will be denoted by $\mathbb{H}(S)$ rather than $R_0^S$, or sometimes by $\mathbb{H}$ once $S$ is understood. The group $S$ will be regarded as a generating subset of $\mathbb{H}(S)$ in view of (b). [Actually by a standard application of the ring extension theorem we can construct an isomorphic copy of $\mathbb{H}(S)$ that contains $S$.] If $L: S \rightarrow L(V, V)$ is a representation, then we will automatically regard $L$ as a representation of the group algebra $\mathbb{H}(S)$ as well. The concepts of reducibility, complete reducibility, absolute irreducibility, and so on carry over unaltered. One should observe, however, that $L: S \rightarrow L(V, V)$ is reducible, and so forth iff $L: \mathbb{H} \rightarrow L(V, V)$ satisfies the same property. To see this, choose a basis of $V$ and consider the corresponding matrix representation $\Delta: S \rightarrow L(V, V)$, which of course continues to be a matrix representation of $\mathbb{H}$, $\Delta: \mathbb{H} \rightarrow L(V, V)$. Then if

$$
\Delta(s) = \begin{bmatrix}
\Delta_{11}(s) & 0 \\
\Delta_{21}(s) & \Delta_{22}(s)
\end{bmatrix}
$$

it is obvious that $\Delta(f) = \sum_{s \in S} f(s)\Delta(s)$ (i.e., a finite sum) has the same form.

**Definition 2.2 (Regular representation)** Let $S$ be a group. The left regular representation of $S$ is the function $\rho: S \rightarrow L(\mathbb{H}(S), \mathbb{H}(S))$ defined by
\[ \rho(s)(f) = sf, \quad s \in S, \quad f \in \mathcal{U}(S). \]

The right regular representation, \( \rho_r : S \to \mathcal{U}(S, \mathcal{U}(S)) \) is defined by

\[ \rho_r(s)(f) = fs^{-1}, \quad s \in S, \quad f \in \mathcal{U}(S). \]

Observe that for any \( s_1, s_2 \) in \( S \), \( f \in \mathcal{U}(S) \),

\[ \rho(s_1s_2)(f) = (s_1s_2)f \]
\[ = s_1(s_2f) \]
\[ = \rho(s_1)(s_2f) \]
\[ = \rho(s_1)(s_2)(f) \]
\[ = \rho_r(s_1)(s_2)(f) \]
\[ = \rho_r(s_1s_2)(f). \]

Thus both \( \rho \) and \( \rho_r \) are representations of \( S \) and can be extended to representations of \( \mathcal{U}(S) \) as above:

\[ \rho(\sum_{s \in S} c_s s)(f) = \sum_{s \in S} c_s (sf), \]
\[ \rho_r(\sum_{s \in S} c_s s)(f) = \sum_{s \in S} c_s (fs^{-1}). \]
For most of our applications $S$ will be a finite group so that

\[(\text{verify})\]

\[\dim \mathcal{U}(S) = |S|.\]

However, whether or not $S$ is finite, both $\rho$ and $\rho^r$ are proper faithful representations of $S$ and in fact $\rho(s)$ and $\rho^r(s)$ are both bijective linear transformations on $\mathcal{U}(S)$. For, $\rho(s_1) = \rho(s_2)$ implies $s_1 = s_2 e = \rho(s_1)e = \rho(s_2)e = s_2$, and similarly for $\rho^r$. Also for $s, t \in S$,

\[\rho(s)t = st,\]

so that as $t$ runs over the basis $S$ of $\mathcal{U}(S)$, $\rho(s)t$ does also and in an obviously bijective way, i.e., $\rho(s)$ is a bijection on the basis $S$.

Observe that if $A \subset \mathcal{U}(S)$ is an invariant subspace of $\rho$ then

\[\rho(f)A \subset A\]

for all $f \in \mathcal{U}(S)$ so that $A$ is a left ideal in $\mathcal{U}(S)$. Conversely, any left ideal in $\mathcal{U}(S)$ is an invariant subspace of $\rho$. There is a bijective mapping

\[\psi: \mathcal{U}(S) \to \mathcal{U}(S)\]

that sends any left ideal into a right ideal and conversely:

\[\psi(f) = \sum_{s \in S} f(s^{-1})s,\]

(7)
[or equivalently, \( \nu \) is linear and \( \nu(s) = s^{-1} \)]. It is easy to confirm (see Exercise 1) that \( \nu \) is a linear bijection on \( \mathbb{H}(S) \) and an antiisomorphism, i.e.,

\[
\nu(fg) = \nu(g)\nu(f).
\]

(8)

Thus \( \nu \) provides a one-to-one correspondence between the left and right ideals in \( \mathbb{H}(S) \).

If \( \mathfrak{A} \) is any linear associative algebra over \( \mathbb{R} \), then the left regular representation of \( \mathfrak{A} \) is defined in precisely the same way:

\[
\rho: \mathfrak{A} \to \mathbb{L}(\mathfrak{A}, \mathfrak{A})
\]

where

\[
\rho(f)(g) = fg,
\]

(9)

i.e., the linear transformation \( \rho(f) \) is simply left multiplication by \( f \) in \( \mathfrak{A} \). Clearly \( \rho(f) \) is linear and \( \rho \) is indeed a representation by precisely the same calculations we made above. We will be most interested here in finite dimensional linear associative algebras \( \mathfrak{A} \) that are direct sums of complete matrix algebras. To clarify what is meant here consider the situation described in Theorem 1.13 in Section 6.1. The representations \( P^1, \ldots, P^r \) are all the absolutely irreducible pairwise inequivalent components of \( P: S \to \mathbb{GL}_N(\otimes V) \) over a simple algebraic extension \( F \) of \( \mathbb{R} \), in which \( P^t \) has degree \( n_t \) and occurs with multiplicity \( e_t \),

\[ t = 1, \ldots, r. \]

We found that by choosing an appropriate basis of
\( \otimes V \) it is possible to construct a matrix representation of \( S_m^1 \) corresponding to \( P \) of the form

\[
\Delta(\sigma) = \sum_{k=1}^{r} I_{e_k} \otimes \Delta_k(\sigma), \quad \sigma \in S_m,
\]

in which \( \Delta_1, \ldots, \Delta_r \) are pairwise inequivalent absolutely irreducible matrix representations of \( S_m \) and the degree of \( \Delta_t \) is \( n_t \), \( t = 1, \ldots, r \). Moreover, by applying Theorem 1.11, the linear closure of \( \text{Im} \Delta \) is seen to consist of all matrices of the form

\[
\sum_{k=1}^{r} I_{e_k} \otimes A_k, \quad A_k \in M_{n_k}(F), \quad k = 1, \ldots, r. \tag{10}
\]

We write

\[
\langle \text{Im} \Delta \rangle = \sum_{k=1}^{r} e_k M_{n_k}(F) \tag{11}
\]

which means simply that \( \langle \text{Im} \Delta \rangle \) consists precisely of the totality of matrices of the form (10). An algebra of matrices of the form (11) is called a direct sum of complete matrix algebras. Theorem 1.13 in Section 6.1 then states that for the matrix representation \( \otimes \) corresponding to \( m^1 \),

\[
\langle \text{Im} \Omega \rangle = \sum_{k=1}^{r} n_k M_{e_k}(F). \tag{12}
\]
In other words, a basis of \( \bigotimes_{i=1}^{m} V_i \) can be chosen so that the matrix algebra consisting of all the corresponding matrix representations of the bisymmetric transformations in \( B_m \) is the direct sum of complete matrix algebras (12).

Before proceeding we remark that the direct sum of complete matrix algebras

\[
\mathfrak{B} = \bigoplus_{k=1}^{r} \mathfrak{M}_{\mathfrak{e}_k}(F)
\]

has certain natural representations \( J_k \), \( k = 1, \ldots, r \),

\[
J_k : \mathfrak{B} \to \mathcal{L}(V(\mathfrak{e}_k(F)), V(\mathfrak{e}_k(F)))
\]

where \( V(\mathfrak{e}_k(F)) \) is the space of \( \mathfrak{e}_k \)-tuples over \( F \). The representation \( J_k \) is defined as follows for \( B \in \mathfrak{B} \) where

\[
B = \bigoplus_{k=1}^{r} \sum_{i=1}^{m_i} n_{ik} \otimes B_{ik}
\]

with \( B_{ik} \in \mathfrak{M}_{\mathfrak{e}_k}(F) \):

\[
J_k(B) : x \to B_{ik}x, \quad x \in V_{\mathfrak{e}_k}(F).
\]

**Theorem 2.1** Let \( \mathfrak{B} \) be a direct sum of complete matrix algebras over \( F \),

\[
\mathfrak{B} = \bigoplus_{k=1}^{r} \mathfrak{M}_{\mathfrak{e}_k}(F).
\]
Then any representation \( D : \mathcal{B} \to \mathbb{L}(U, U) \) for which \( D(I_N) = I_U \), is fully reducible. Moreover, any irreducible component of \( D \) is equivalent to one of the \( J_k \), \( k = 1, \ldots, r \). Thus to within equivalence the irreducible representations of \( \mathcal{B} \), a direct sum of complete matrix algebras, are completely determined by \( \mathcal{B} \) itself, and their number is finite.

**Proof:** We first examine the structure of the (left) regular representation \( \rho \).

By definition, the elements of \( \mathcal{B} \) are the matrices

\[
B = \sum_{k=1}^{r} I_{n_k} \otimes B_k, \quad B_k \in \mathbb{M}_{e_k}(F). \tag{13}
\]

For each \( k = 1, \ldots, r \), \( j = 1, \ldots, e_k \), define a subspace \( \mathcal{B}_{kj} \) of \( \mathcal{B} \) consisting of all matrices of the form

\[
X_{kj} = (I_{n_1} \otimes 0) + \cdots + (I_{n_{k-1}} \otimes 0) + (I_{n_k} \otimes E_j) + (I_{n_{k+1}} \otimes 0) + \cdots + (I_{n_r} \otimes 0),
\]

where \( E_j \in \mathbb{M}_{e_k}(F) \) is any matrix of the form

\[
E_j = \begin{bmatrix}
0 & \cdots & 0 & c_1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & c_2 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & c_{e_k} & 0 & \cdots & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix} e_k \end{bmatrix}
\]

\[
, \tag{15}
\]
i.e., the only nonzero entries of $E_j$ occur in column $j$. If $B \in \mathfrak{B}$ is given by (13) then

$$
\rho(B)(X_{kj}) = B_{kj}
= \left(I_{n_1} \otimes 0\right) \cdot \cdots \cdot \left(I_{n_{k-1}} \otimes 0\right) \cdot \left(I_{n_k} \otimes B_{j} E_j \right) \cdot \left(I_{n_{k+1}} \otimes 0\right)
\cdot \cdots \cdot \left(I_{n_r} \otimes 0\right)
$$

and clearly $B_{kj} E_j$ has the same structure as $E_j$ itself. Thus

$$
\rho(B)(\mathfrak{S}_{kj}) \subset \mathfrak{S}_{kj}
$$

so that each subspace $\mathfrak{S}_{kj}$ is an invariant subspace of $\rho$. Moreover, it is obviously absolutely irreducible since $B_k$ can be chosen arbitrarily in $M_{e_k}(F)$. This absolutely irreducible component of $\rho$ will be denoted by $\rho_{kj}$, i.e., $\rho_{kj}(B) = \rho(B)|\mathfrak{S}_{kj}$.

For fixed $k$ and $j$ let

$$
\zeta : V_{e_k}(F) \to \mathfrak{S}_{kj}
$$

be the linear bijection (why?) that maps each $e_k$-tuple $\xi = (c_1, \ldots, c_{e_k})$ into the matrix (14), i.e., $\zeta(\xi) = X_{kj}$. Then we assert that if $B$ is any matrix (13) then

$$
\zeta^{-1} \rho_{kj}(B) \zeta = J_k(B),
$$

i.e., the irreducible component $\rho_{kj}$ of $\rho$ is equivalent to $J_k$.

To confirm (18) simply observe that
\[ \rho_{kj}(B)\xi = \rho_{kj}(B)X_{kj} \]
\[ = BX_{kj} \]
\[ = \xi(B_k\xi), \]
or
\[ \xi^{-1}\rho_{kj}(B)\xi = J_k(B). \]

Now let \( D : \mathfrak{g} \to L(U, U) \) be any proper representation of \( \mathfrak{g} \) and let \( U_0 \) be an invariant subspace of \( D \). If \( u \notin U_0 \) and \( I_N \) is the identity matrix in \( \mathfrak{g} \), then
\[ u = I_N u \]
\[ = D(I_N)u \]
\[ \in D(\mathfrak{g})u \]
\[ = \sum_{k=1}^{r} \sum_{j=1}^{e_k} D(\mathfrak{g}_{kj})u. \]

Obviously not every \( D(\mathfrak{g}_{kj})u \subset U_0 \). Thus for some \( p \) and \( q \), \( D(\mathfrak{g}_{pq})u \) is not a subset of \( U_0 \), i.e.,
\[ D(\mathfrak{g}_{pq})u \notin U_0. \]
(19)

Observe that every \( D(\mathfrak{g}_{kj})u \) is an invariant subspace of \( D \). For, if \( X_{kj} \) is any matrix in \( \mathfrak{g}_{kj} \) and \( B \in \mathfrak{g} \), then as we have seen,
\[ BX_{kj} \in \mathfrak{g}_{kj}, \] so
\[ D(BD(X_{k_j})u = D(BX_{k_j})u \]

\[ \in D(\beta_{k_j})u. \]

We also assert that \( D(\beta_{pq})u \) is an irreducible invariant subspace of \( D \). For, suppose \( 0 \neq W \subset D(\beta_{pq})u \) is an invariant subspace of \( D \) and let \( \mathcal{J}_W \) denote the set of all \( X_{pq} \in \beta_{pq} \) for which \( D(X_{pq})u \in W \). Since \( 0 \neq W \subset D(\beta_{pq})u, \mathcal{J}_W \neq 0 \). Also, it is clear that \( \mathcal{J}_W \) is a subspace of \( \beta_{pq} \) and if \( B \in \beta \) and \( X_{pq} \in \mathcal{J}_W \) then

\[ D(BX_{pq})u = D(B)D(X_{pq})u \]

\[ \in D(B)W \quad (\text{i.e., } D(X_{pq})u \in W) \]

\[ \subset W \quad (\text{i.e., } W \text{ is invariant}), \]

so that \( BX_{pq} \in \mathcal{J}_W \). Thus \( \mathcal{J}_W \) is a nonzero subspace of \( \beta_{pq} \) invariant under all left multiplications by matrices in \( \beta \). As before it follows that \( \mathcal{J}_W = \beta_{pq} \) and hence

\[ W = D(\mathcal{J}_W)u \]

\[ = D(\beta_{pq})u. \]

Thus \( D(\beta_{pq})u \) is an irreducible invariant subspace. But

\[ U_0 \cap D(\beta_{pq})u \]

is invariant and a subspace of \( D(\beta_{pq})u \). Thus either the space \((20)\) is \( 0 \) or it is \( D(\beta_{pq})u \). This latter alternative is ruled out by \((19)\). Hence
\[ U_0 \cap D(\mathfrak{g}_{pq})u = 0. \]

Set \( U_1 = D(\mathfrak{g}_{pq})u \). If \( U = U_0 + U_1 \) we are finished. Otherwise replace \( U_0 \) by \( U_0 + U_1 \) in the preceding argument to obtain an invariant subspace \( U_2 \) such that \( U_2 \cap (U_0 + U_1) = 0 \). This process must terminate in a finite number of steps because \( U \) is finite dimensional. Thus

\[ U = U_0 + U_1 + \ldots + U_m \]

and each of \( U_1, \ldots, U_m \) is an invariant subspace of \( D \). It follows that \( D \) is fully reducible. (Thus in particular \( \rho \) itself is fully reducible.)

Our next task will be to prove that any absolutely irreducible component of \( D \) is equivalent to one of the components \( \rho_{kj} \) of the left regular representation described above. Since \( \rho_{kj} \) is equivalent to \( M_{ek} \) (\( \mathbb{F} \)) in the sense described immediately preceding the statement of the theorem, we will have obtained the result. Clearly we can assume that \( D \) itself is absolutely irreducible, otherwise simply proceed in the following argument with an absolutely irreducible component of \( D \). Thus let \( 0 \neq u \) be some fixed vector in \( U \). Then since \( D(I_N)u = u \) we conclude precisely as we did in the argument leading to (19) that for some \( p \) and \( q \)

\[ D(\mathfrak{g}_{pq})u \neq 0. \quad (21) \]

Again, as above, \( D(\mathfrak{g}_{pq})u \) is an invariant subspace of \( D \) so that
the irreducibility of $D$ implies that

$$D(\beta_{pq})u = U. \quad (22)$$

In other words, every vector $v \in U$ is of the form

$$D(X_{pq})u = v, \quad X_{pq} \notin \mathcal{G}_{pq}. \quad (23)$$

Suppose also that $D(Y_{pq})u = v, \quad Y_{pq} \in \mathcal{G}_{pq}$, so that

$$D(X_{pq} - Y_{pq})u = 0. \quad (24)$$

Set $Z_{pq} = X_{pq} - Y_{pq}$ so that $D(Z_{pq})u = 0$ and hence

$$D(B)D(Z_{pq})u = 0$$

or

$$D(BZ_{pq})u = 0, \quad B \in \mathcal{G}.$$

If $Z_{pq} \neq 0$, then as $B$ runs over $\mathcal{G}$ we know that $BZ_{pq}$ runs over $\mathcal{G}_{pq}$ so that

$$D(\theta_{pq})u = 0$$

contradicting (21). Thus if we define a mapping $\theta : \mathcal{G}_{pq} \to U$ by

$$\theta(X_{pq}) = D(X_{pq})u \quad (25)$$

then $\theta$ is injective. It is obvious that $\theta$ is linear and since every vector in $U$ is of the form (25), $\theta$ is bijective. We assert that

$$\theta^{-1}D(B)\theta = \rho_{pq}(B), \quad B \in \mathcal{G} \quad (26)$$
Let $X_{pq} \in \mathfrak{R}_{pq}$. Then

$$\rho_{pq}(B)(X_{pq}) = BX_{pq}$$

and

$$\theta^{-1}D(B)\theta(X_{pq}) = \theta^{-1}D(B)D(X_{pq})u$$

$$= \theta^{-1}D(BX_{pq})u$$

$$= \theta^{-1}\theta(BX_{pq})$$

$$= BX_{pq}.$$  \hspace{2cm} (27)

Suppose we take $\mathfrak{B}$ to be the algebra $\mathfrak{B}_m$ of bisymmetric transformations on $\bigotimes V$. This algebra is a direct sum of complete $1$-matrix algebras and can be regarded as a representation of itself. According to Theorem 2.1, any representation of a direct sum of complete matrix algebras is fully reducible. Of course

$$\mathfrak{B}_m = \langle \Pi'(T), T \in \text{GL}_n(V) \rangle,$$

so that the $m^{th}$ tensor power representation

$$\Pi^m : \text{GL}_n(V) \to \text{GL}_{n^m}(\bigotimes V), \quad N = n^m,$$

is fully reducible.

We have

**Theorem 2.2** Let $\varphi : \text{GL}_n(U) \to \text{GL}_p(V)$ be a rational representation. Then $\varphi$ is fully reducible over a suitable algebraic extension of $R$. 

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Proof: Assume that bases of $U$ and $V$ have been chosen and by Theorem 1.4, Section 6.1, (with $\varphi$ also denoting the corresponding matrix representation),

$$\varphi(X) = (\det X)^k \sum_{i=1}^r \theta_i(X),$$

where $\theta_1, \ldots, \theta_r$ are homogeneous polynomial representations.

Suppose $\theta_i$ has degree $m_i$, i.e., $\theta_i(tX) = t^{m_i} \theta_i(X)$, $i = 1, \ldots, r$.

By Theorem 1.5, Section 6.1, there is a unique algebra homomorphism $h_i : \beta_{m_i} \rightarrow M_{p_i}(R)$, $p_1 + \cdots + p_r = p$

such that

$$\theta_i = h_i \prod_{j=1}^{m_i}.$$

(28)

From Theorem 1.13, Section 6.1, there is a simple algebraic extension $F$ of $R$ such that over $F$ each $\prod_{i=1}^{m_i}$ is equivalent to a direct sum of absolutely irreducible components. By grouping together equivalent components in

$$\sum_{i=1}^r \prod_{j=1}^{m_i},$$

we conclude that the linear closure over $F$,

$$\beta = \left\langle \sum_{i=1}^r \prod_{j=1}^{m_i} \theta_i(X), \ X \in GL(n,R) \right\rangle,$$
is equivalent over $F$ to a direct sum of complete matrix algebras. Define an algebra homomorphism

$$h : \mathfrak{B} \rightarrow M_p(F)$$

by

$$h \left( \sum_{i=1}^{r} \prod_{i=1}^{m_i} (X) \right) = \sum_{i=1}^{r} h \left( \prod_{i=1}^{m_i} (X) \right).$$

Since $\mathfrak{B}$ is equivalent to a direct sum of complete matrix algebras, we can apply Theorem 2.1 to conclude that $h$ is fully reducible. Set $\theta(X) = \sum_{i=1}^{r} \theta_i(X)$ so that (28) implies

$$\theta(X) = h \left( \sum_{i=1}^{r} \prod_{i=1}^{m_i} (X) \right).$$

Suppose that over $F$,

$$S^{-1} \theta(X) S = \begin{bmatrix} \theta_{11}(X) & 0 \\ \vdots & \ddots & \vdots \\ \theta_{21}(X) & \cdots & \theta_{21}(X) \end{bmatrix} = S^{-1} h \left( \sum_{i=1}^{r} \prod_{i=1}^{m_i} (X) \right) S,$$

for all $X \in \text{GL}(n,R)$.

Since every element of $\mathfrak{B}$ is a linear combination over $F$ of matrices $\sum_{i=1}^{r} \prod_{i=1}^{m_i} (X)$, it follows that for any $B \in \mathfrak{B}$, $S^{-1} h(B) S$
has precisely the same form as the block matrix appearing in (29). But \( h \) is fully reducible so there exists a non-singular \( T \) over \( F \) such that

\[
T^{-1} S^{-1} h(B) S T, \quad B \in \mathcal{B},
\]

is a direct sum in which the partitioning is into blocks precisely as in (29). But then setting

\[
B = \mathbf{\sum}_{i=1}^{r} \prod_{i}^m (X)
\]

it follows that

\[
T^{-1} S^{-1} \Theta(X) S T = \begin{bmatrix}
\Theta'_{11}(X) & 0 \\
- & - \\
- & - \\
0 & \Theta'_{22}(X)
\end{bmatrix}, \quad X \in \text{GL}(n, \mathbb{R}),
\]

where \( \Theta'_{ii} \) is the same size as \( \Theta_{ii} \), \( i = 1, 2 \). Hence \( \Theta \) is fully reducible over \( F \) and so \( \phi(X) = (\det X)^k \Theta(X) \) is. In fact, the components of \( \Theta \) and hence of \( \frac{\phi(X)}{(\det X)^k} \) are equivalent to the complete matrix algebras \( M_{\mathbf{e}_k^m} \) appearing in the decomposition of the various \( \mathcal{B}_{m_i} \) (see formula (12)).

In Theorem 1.15, Section 6.1, we learned that if \( m \)

\[
\Pi' : \text{GL}_n(V) \rightarrow \text{GL}_m(\otimes^m V)
\]

is fully reducible and \( W \subset \otimes^m V \) is an invariant subspace of \( \Pi' \), then there exists a projection \( Y \in \mathcal{B}_m \) such that

\[
W = Y(\otimes^m V).
\]

(30)
Since we now know that $\Pi'$ is indeed fully reducible, we can state the following.

**Theorem 2.3** The tensor power representation $\Pi': GL_n(V) \to GL_n(\otimes^m V)$ is fully reducible. Moreover, the invariant subspaces of $\Pi'$ are the symmetry classes of tensors (30) in $\otimes^m V$ obtained as ranges of projections in $\mathcal{P}_m = \langle P(\sigma), \sigma \in S_m \rangle$.

We considered an instance of this remarkable result in Example 1.3, Section 6.1. There $\Pi': GL_n(V) = GL_n(V \otimes V)$ has precisely two invariant subspaces and these were the ranges of the two projection symmetry operators $S_\varepsilon$ and $S_1$. Moreover, $S_\varepsilon^2 = S_\varepsilon$, $S_1^2 = S_1$, $S_1 + S_\varepsilon = I_{V \otimes V}$, $S_1 S_\varepsilon = S_\varepsilon S_1 = 0$, the subspaces $\wedge^2 V = S_\varepsilon(V \otimes V)$, $V^{(2)} = S_1(V \otimes V)$ are absolutely irreducible and finally

$$\Pi'(T) | \wedge^2 V = C_2(T),$$

$$\Pi'(T) | V^{(2)} = P_2(T), \quad T \in L(V, V).$$

To identify the irreducible invariant symmetry classes (30) of $\Pi'$, we require a few elementary general facts concerning projections. Recall again that in general a projection operator on an arbitrary vector space $V$ is simply a transformation $P \in L(V, V)$ that satisfies $P^2 = P$, i.e., it is an idempotent in $L(V, V)$.

Suppose $V = W_1 + W_2$ so that each vector $v \in V$ has a unique decomposition $v = w_1 + w_2$, with $w_i \in W_i$, $i = 1, 2$. Then set $Pv = w_1$ so that $P^2v = Pw_1 = w_1$. Conversely if $P^2 = P$, then let
\[ W_1 = \text{Im } P, \ W_2 = \ker P \] and suppose \( v \in W_1 \cap W_2 \). Then \( v = Px \), \( x \in V \) and \( Pv = 0 \) because \( v \notin \ker P \). Hence \( 0 = Pv = P^2x = Px = v \). In other words \( W_1 \cap W_2 = 0 \) so the sum \( W_1 + W_2 \) is direct. Moreover, \( v = Pv + (I_V - P)v \) and \( Pv \in W_1 \), \( (I_V - P)v \in W_2 \). If we set \( Q = I_V - P \) then \( Q^2 = (I_V - P)^2 = I_V - 2P + P^2 = I_V - 2P + P = Q \) so that \( Q \) is also a projection, \( PQ = QP = 0 \), and \( I_V = P + Q \). Observe also that \( \text{Im } Q = \ker P \). These calculations are easily generalized to more than two projections. Thus suppose that

\[ I_V = \sum_{k=1}^{r} P_k \tag{31} \]

where \( P_k \) are projections and

\[ F_k P_j = 0 \text{ for } k \neq j. \tag{32} \]

Then

\[ V = \sum_{k=1}^{r} W_k \tag{33} \]

where \( W_k = \text{Im } P_k \). Conversely given a decomposition (33) of \( V \) define

\[ P_k v = w_k \]

where \( v = w_1 + \cdots + w_r \), \( w_k \in W_k \), \( k = 1, \ldots, r \), and the \( P_k \) satisfy (31) and (32). Thus for any decomposition of the space (33) there is a corresponding set of projections satisfying (31) and (32), and the converse is also true. We remark that if
$V$ is a unitary space and the decomposition (33) is into pairwise orthogonal subspaces, then the corresponding projections are hermitian. Conversely, if hermitian projections $P_1, \ldots, P_r$ satisfy (31) and (32) then the decomposition (33) is into pairwise orthogonal subspaces. The verifications of these assertions are easy and routine, e.g., if $w_i \in W_i$ and $P^*_i = P_i$, $i = 1, 2$, then $(w_1, w_2) = (P_1 w_1, P_2 w_2) = (P^*_1 P_1 w_1, w_2) = (P_2 P_1 w_1, w_2) = 0$, and so on. We leave similar calculations to the reader.

Let $L : S \rightarrow L(V, V)$ be a representation of the group $S$ and suppose (33) is a direct sum decomposition of $V$ into invariant subspaces with corresponding projections as in (31) and (32). Then for any $v \in V$, $v = w_1 + \cdots + w_r$, $w_k = P_k v \in W_k$, and thus

$$P_k L(s)v = P_k \sum_{i=1}^{r} L(s)w_i$$

$$= \sum_{i=1}^{r} P_k L(s)w_i$$

$$= L(s)w_k$$

$$= L(s)P_k v.$$  \tag{34}

The third equality in (34) follows from the facts that $L(s)w_i \in W_i$, $i = 1, \ldots, r$, $P_k | W_k = \mathbb{I}_W$, $P_k | W_i = 0$, $i \neq k$. Thus we see that every one of the projections $P_k$ commutes with each $L(s)$, $s \in S$. Conversely, if (34) is satisfied by projections satisfying (31) and (32) and $W_k = \text{Im} P_k$, $k = 1, \ldots, r$, then
it is easy to see that each $W_k$ is an invariant subspace of the representation $L$ (confirm this). A projection $P$ is said to be \textit{irreducible with respect to the representation} $L$ if

$$PL(s) = L(s)P, \ s \in S$$

but $P$ is not a sum of nonzero projections $P = P_1 + P_2$ where

$$P_1P_2 = P_2P_1 = 0,$$

and each of $P_1$ and $P_2$ commute with $L(s), \ s \in S$. By easy calculations entirely analogous to the above we prove the following result.

\textbf{Theorem 2.4} \ Let $L : S \to L(V,V)$ be a fully reducible representation of the group $S$. Let $W_1$ be a proper invariant subspace of $L$ and by the full reducibility let $V = W_1 \oplus W_2$ where $W_2$ is also an invariant subspace. Then $W_1$ is irreducible iff the projection $P$ along $W_1$ parallel to $W_2$ is irreducible with respect to $L$.

\textbf{Proof:} If $W_1$ were reducible, then by full reducibility

$$W_1 = W_3 \oplus W_4$$

where $W_3$ and $W_4$ are invariant subspaces. Hence

$$V = W_3 \oplus W_4 \oplus W_2.$$ 

Define $P_3$ to be the projection along $W_3$ parallel to $W_4 \oplus W_2$ and similarly for $P_4$. Then obviously $P = P_3 + P_4, \ P_3P_4 = P_4P_3 = 0$, and $P_3$ and $P_4$ both commute with each $L(s)$. The proof of the converse is also elementary and left to the reader. \hfill \blacksquare
Now suppose $S$ is a group, $\mathbb{H}(S)$ is the group algebra, $\rho : \mathbb{H}(S) \to L(\mathbb{H}(S), \mathbb{H}(S))$ is the left regular representation, and $\rho$ is fully reducible. Let $W$ be an invariant subspace of $\rho$ (i.e., a left ideal in $\mathbb{H}(S)$). Write

$$\mathbb{H}(S) = W + W'$$

where $W'$ is an invariant subspace, and let $P$ be the projection along $W$ parallel to $W'$. For any $s \in S$, $P$ commutes with $\rho(s)$, $\rho(s)P = P\rho(s)$, so that if $f \in \mathbb{H}(S)$ then

$$\rho(s)Pf = P\rho(s)f. \quad (35)$$

Let $e$ be the identity element in $\mathbb{H}(S)$, i.e., $e$ is the function whose value at the identity in $S$ is 1 and otherwise is 0, and set $e' = Pe$. Then for any $f \in \mathbb{H}(S)$

$$Pf = \sum_{t \in S} f(t)P_t$$

$$= \sum_{t \in S} f(t)P\rho(t)e \quad \text{[i.e., } \rho(t)e = te = t \text{]}$$

$$= \sum_{t \in S} f(t)\rho(t)Pe \quad \text{[from } (35)\text{]}$$

$$= \sum_{t \in S} f(t)\rho(t)e'$$

$$= \rho(f)(e')$$

$$= fe'. \quad (36)$$
Hence

\[ P^2 f = PPf \]

\[ = Pf'e' \]

\[ = (fe')e' \]

\[ = fe'e'^2. \quad (37) \]

Since \( P^2 = P \) we can combine (36) and (37) to obtain

\[ fe' = fe'^2 \quad (38) \]

so that setting \( f = e \) we conclude that

\[ e'^2 = e', \quad (39) \]

i.e., \( e' \) is an idempotent element in \( \mathbb{U}(S) \). Moreover

\[ W = P(\mathbb{U}(S)) \]

\[ = \mathbb{U}(S)e' \quad [\text{by (36)}]. \]

In other words, the left ideal \( W \) is of the form \( \mathbb{U}(S)e' \) where \( e' \) is an idempotent in the group algebra \( \mathbb{U}(S) \): \( e' \) is called a generating idempotent for \( W \). Conversely, suppose \( e' \) is an idempotent in \( \mathbb{U}(S) \), i.e., satisfies (39). Define \( P \in L(\mathbb{U}(S), \mathbb{U}(S)) \), by

\[ Pf = fe', \quad f \in \mathbb{U}(S). \quad (40) \]

Then obviously

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\[ P^2 f = P(Pf) \]
\[ = Pfe' \]
\[ = fe'^2 \]
\[ = fe' \]
\[ = Pf \]

so that \( P^2 = P \) and \( P \) is a projection. Moreover, for any \( f \) and \( g \) in \( \mathbb{W}(S) \)

\[ \rho(f)Pg = \rho(f)ge' \]
\[ = fge' \]
\[ = P\rho(f)g \]

so that

\[ \rho(f)P = \rho(P\rho(f)). \quad (41) \]

It follows that \( W = \text{Im } P = \mathbb{W}(S)e' \) is an invariant subspace of \( \rho \). Observe that \( e' = Pe' \) is determined by \( P \) and conversely \( P \) is determined [see (40)] by \( e' \). Hence there is a one-to-one correspondence between idempotents \( e' \) in the group algebra and projections \( P \) which commute with every \( \rho(f) \). Suppose that \( e_i \) is the idempotent associated with the projection \( P_i \);

\[ P_i f = fe_i, \quad i = 1,2, \]
and \( P_1P_2 = P_2P_1 = 0 \), \( I_{U(S)} = P_1 + P_2 \). Then clearly \( e = e_1 + e_2 \) and

\[
0 = P_1P_2 e \\
= P_1 e e_2 \\
= e e_2 e_1 \\
= e_2 e_1
\]

(42)

and of course

\[
e_1 e_2 = 0
\]

(43)

also.

An idempotent \( e' \) is said to be **primitive** if it is not expressible in the form

\[
e' = e_1 + e_2
\]

where \( e_1 \) and \( e_2 \) are idempotents satisfying (42) and (43). The preceding discussion has established the following result.

**Theorem 2.5** Let \( U(S) \) be the group algebra of \( S \). Any left ideal in \( U(S) \) is of the form \( U(S)e' \) where \( e' \) is an idempotent in \( U(S) \). The ideal is irreducible, i.e., it is an irreducible invariant subspace of \( \rho \), if and only if \( e' \) is a primitive idempotent.

We can apply Theorem 2.4 directly to the tensor power representation as follows:
Theorem 2.6  The irreducible invariant subspaces of the tensor power representation

\[ \Pi' : \text{GL}_n(V) \to \text{GL}_m(\otimes V) \]

are precisely the subspaces of the form

\[ W = Y(\otimes V) \]

where \( Y \in \mathcal{G}_m \) is an irreducible idempotent (i.e., projection) symmetry operator with respect to \( \Pi' \).

We only remark that since \( Y \in \mathcal{G}_m \) it automatically commutes with every \( \Pi'(T) \).

Now consider the group algebra of \( S_m \), \( \mathbb{H}(S_m) \), and in the usual way define an algebra homomorphism by extending the permutation operator mapping \( P : \sigma \to P(\sigma) \) to a mapping

\[ P : \mathbb{H}(S_m) \to \mathcal{G}_m \]  \hspace{1cm} (44)

by the formula

\[ P \left( \sum_{\sigma} a_{\sigma} \sigma \right) = \sum_{\sigma} a_{\sigma} P(\sigma) . \]  \hspace{1cm} (45)

Clearly, if \( \varepsilon \) is an idempotent in \( \mathbb{H}(S_m) \), \( Y = P(\varepsilon) \) will be an idempotent symmetry operator in \( \mathcal{G}_m \). Note that \( \ker P \) is a two-sided ideal in \( \mathbb{H}(S_m) \). (This is true of the kernel of any algebra homomorphism.)

Theorem 2.7  Assume that the group algebra \( \mathbb{H}(S_m) \) is a direct

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sum of \( \ker P \) and another two-sided ideal \( \mathcal{O} \), i.e.,

\[
\mathbb{U}(S_m) = \mathcal{O} + \ker P.
\]

(46)

Let \( P_0 = P|_\mathcal{O} \) and let \( \varepsilon \in \mathcal{O} \). Then \( Y = P_0(\varepsilon) \) is a projection symmetry operator in \( \mathcal{O}_m \) iff \( \varepsilon \) is an idempotent in \( \mathbb{U}(S_m) \).

Moreover \( Y \) is irreducible with respect to \( \mathbb{U} \) iff \( \varepsilon \) is primitive.

**Proof:** If \( Y^2 = Y \), then \( P_0(\varepsilon) = P_0(\varepsilon)^2 = P_0(\varepsilon^2) \). But \( P_0 \) is injective on \( \mathcal{O} \) so that \( \varepsilon = \varepsilon^2 \). Suppose that \( \varepsilon \) is primitive and \( P_0(\varepsilon) = Y = Y_1 + Y_2 = P_0(\varepsilon_1) + P_0(\varepsilon_2) \), \( \varepsilon_1, \varepsilon_2 \in \mathcal{O} \).

Then \( Y_i^2 = Y_i \), \( Y_1 Y_2 = Y_2 Y_1 = 0 \) imply that

\[
P_0(\varepsilon_i^2) = P_0(\varepsilon_i), \quad i = 1, 2
\]

and

\[
P_0(\varepsilon_1 \varepsilon_2) = P_0(\varepsilon_2 \varepsilon_1) = 0.
\]

But then \( \varepsilon_1^2 = \varepsilon_1 \), \( \varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1 = 0 \) and since \( P_0(\varepsilon) = P_0(\varepsilon_1) + P_0(\varepsilon_2) = P_0(\varepsilon_1 + \varepsilon_2) \), \( \varepsilon = \varepsilon_1 + \varepsilon_2 \). Thus if \( Y = P_0(\varepsilon) \), \( \varepsilon \in \mathcal{O} \), is reducible, \( \varepsilon \) is not primitive. To prove the converse simply reverse this argument.

By combining Theorems 2.6 and 2.7 we have

**Theorem 2.8** Assume that the decomposition (46) is possible, i.e.,

\[
\mathbb{U}(S_m) = \mathcal{O} + \ker P.
\]
Then the irreducible invariant subspaces of the tensor power representation
\[ \Pi^m : \text{GL}_n(\mathbb{C}) \to \text{GL}_N(\otimes^m V) \] (47)
are precisely the subspaces of the form
\[ P(\varepsilon)(\otimes^m V) \] (48)
where \( \varepsilon \) is a primitive idempotent of the group algebra \( \mathbb{U}(S_m) \)
lying in the two-sided ideal \( \mathcal{O} \).

We shall show in Theorem 2.10 that the decomposition (46) is always possible and in fact that \( \mathcal{O} \) is uniquely determined.

Thus the essential problem that remains is to find the precise structure of the primitive idempotents \( \varepsilon \in \mathbb{U}(S_m) \) that appear in (48). We shall do this in the next section.

**Example 2.1** The homomorphism \( P : \mathbb{U}(S_m) \to P_m \subset L(\otimes^m V, \otimes^m V) \) has a nonzero kernel under certain circumstances. In fact, \( \ker P = 0 \) iff \( m \neq n \) where \( \dim V = n \). For, suppose first that \( m > n \) and set \( f = \sum_{\sigma \in S_m} \varepsilon(\sigma)\sigma \). Then \( f \neq 0 \) but \( P(f)v_1 \otimes \cdots \otimes v_m = \sum_{\sigma \in S_m} \varepsilon(\sigma)v_{\sigma^{-1}} = m!v_1 \wedge \cdots \wedge v_m = 0 \) because \( v_1, \ldots, v_m \) must be linearly dependent. Thus \( P(f) = 0 \). On the other hand, suppose \( m \leq n \) and let \( g = \sum_{\sigma \in S_m} g(\sigma)\sigma \) be an arbitrary element of \( \mathbb{U}(S_m) \).

Let \( e_1, \ldots, e_m \) be l.i. vectors in \( V \) and observe that
\[ P(g)e^\otimes = \sum_{\sigma \in S_m} g(\sigma)e_{\sigma^{-1}}. \]
The tensors $g^\otimes \sigma \cdot 1, \sigma \in S_m$, are part of a basis of $\bigotimes_1^m V$ and hence must be i.i. It follows that if $P(g) e^\otimes = 0$, then $g(\sigma) = 0, \sigma \in S_m$, so that $\ker P$ must be 0.

At this point we examine some further interpretations of Theorem 2.1. Thus let $S$ be an arbitrary finite group of order $h$,

$$|S| = h,$$

and consider the left regular representation of the group algebra,

$$\rho : \mathcal{U}(S) \rightarrow L(\mathcal{U}(S), \mathcal{U}(S)),$$

$$\dim \mathcal{U}(S) = h.$$ We know that $\rho$ is faithful so that in fact

$$\rho : \mathcal{U}(S) \rightarrow \text{Im } \rho$$

is an algebra isomorphism. According to Theorem 1.6 in Section 6.1, $\rho|_S$ is fully reducible, but of course this means that $\rho$ is fully reducible as a representation of $\mathcal{U}(S)$. Moreover, by Theorem 1.7 in Section 6.1, there exists a simple algebraic extension $F$ of $R$ such that the space $\mathcal{U}(S)$ on which every $\rho(s), s \in S$, acts is a direct sum of absolutely irreducible subspaces and these remain absolutely irreducible subspaces for (49). Also, by Theorem 1.17 in Section 6.1, the corresponding components of \( \rho \) are determined uniquely to within equivalence. Since $\rho(f)$ is simply left multiplication by $f$, the absolutely irreducible spaces are minimal left ideals in $\mathcal{U}(S)$, i.e., they are left ideals that contain no proper left ideals. Suppose now that we
designate these minimal left ideals as follows: for each \( i = 1, \ldots, r \),

\[
\mathfrak{u}_{i1}, \ldots, \mathfrak{u}_{in_i}
\]  

(50)

are all those minimal left ideals for which \( \rho \mathfrak{u}_{i1}, \ldots, \rho \mathfrak{u}_{in_i} \) are equivalent. To be precise this means that for any fixed pair of integers \( j, k \) chosen from \( 1, \ldots, n_i \), there exists a linear bijection \( Q : \mathfrak{u}_{ij} \to \mathfrak{u}_{ik} \) such that

\[
(\rho(\xi) \mid \mathfrak{u}_{ik})Q = Q(\rho(\xi) \mid \mathfrak{u}_{ij}).
\]  

(51)

If \( i \neq t \), \( i, t \) integers chosen from \( 1, \ldots, r \), then \( \rho \mathfrak{u}_{ij} \) and \( \rho \mathfrak{u}_{tk} \) are inequivalent, \( j = 1, \ldots, n_i \), \( k = 1, \ldots, n_t \). We can write

\[
\mathfrak{U}(S) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \mathfrak{u}_{ij}
\]  

(52)

We refer to the ideals (50) as being \textit{equivalent} and the ideals \( \mathfrak{u}_{ij} \) and \( \mathfrak{u}_{tk} \) just discussed as being \textit{inequivalent}. The decomposition of \( \mathfrak{U}(S) \) into minimal left ideals in (52) is unique to within order and equivalence because as we noted above the \( \mathfrak{u}_{ij} \) are precisely the absolutely irreducible invariant subspaces of the left regular representation of \( S \).

Suppose a basis \( \mathcal{E} \) of \( \mathfrak{U}(S) \) is chosen so that the matrix representation corresponding to the left regular representation

\[
\rho : S \to L(\mathfrak{U}(S), \mathfrak{U}(S)),
\]  

(53)

is
\[ \Delta(s) = I_{n_1} \otimes \Delta^1(s) + \cdots + I_{n_r} \otimes \Delta^r(s). \]  

(54)

Then

\[ \Delta(f) = \sum_{s \in S} f(s) \Delta(s) \]

\[ = \sum_{t=1}^r I_{n_t} \otimes \left( \sum_{s \in S} f(s) \Delta^t(s) \right) \]

\[ = \sum_{t=1}^r I_{n_t} \otimes \Delta^t(f) \]  

(55)

is of course the matrix representation corresponding to (49). As we know, any matrix of the form \( \sum_{t=1}^r I_{n_t} \otimes A_t \) is obtainable as a value of \( \Delta(f) \), i.e., as a linear combination of matrices \( \Delta(s) \), \( s \in S \). In other words \( \text{Im} \Delta = \sum_{t=1}^r n_t M(F) \) where \( e_t \) is the degree of \( \Delta^t \), \( t = 1, \ldots, r \). We have proved then that

\[ \Delta: \mathcal{U}(S) \rightarrow \sum_{t=1}^r n_t M(F) = \mathcal{B} \]  

(56)

is an algebra isomorphism (i.e., it is a matrix version of (49) and \( \rho \) is faithful) mapping \( \mathcal{U}(S) \) onto a direct sum of complete matrix algebras according to the formula (55). Now let

\[ \delta: \mathcal{U}(S) \rightarrow \mathcal{L}(U,U) \]

be any absolutely irreducible representation of \( \mathcal{U}(S) \) into the
set of linear transformations of some vector space $U$ over $F$, 
$\delta(e) = I_U$, where $e$ is the identity in $S$. Then the isomorphism (56) provides a corresponding absolutely irreducible representation of the right side of (56) according to the formula

$$D(\Delta(f)) = \delta(f), \quad f \in \mathbb{U}(S),$$

(57)
i.e., $D : \mathcal{A} \rightarrow L(U, U)$ according to the formula (57). Observe that (57) makes sense: for, $\Delta$ is a bijection so that there is only one $f$ for a given $\Delta(f)$. According to Theorem 2.1, $D$ must be equivalent to one of the components $\rho_{kj}$ of the left regular representation $\rho$ of $\mathcal{A}$ described in the proof. We have proved the following interesting result.

**Theorem 2.9** If $\delta : S \rightarrow L(U, U)$ is an absolutely irreducible representation of the finite group $S$, then $\delta$ is equivalent to a component of the left regular representation acting on some minimal left ideal in the group algebra $\mathbb{U}(S)$.

We can also exploit the isomorphism (56) to show that the decomposition (46) is indeed possible.

**Theorem 2.10** Let $\mathcal{R}$ be a two-sided ideal in the group algebra $\mathbb{U}(S)$. Then there exists another two-sided ideal $\mathcal{R}'$ such that

$$\mathbb{U}(S) = \mathcal{R} + \mathcal{R}'.$$

(58)

**Proof:** Let $M = \Delta(\mathcal{R})$, i.e., $M$ is the image of $\mathcal{R}$ in (56). Then $M$ is clearly a two-sided ideal in $\mathcal{A}$, i.e., $M$ is a vector space of matrices of the form

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\[ B = \sum_{t=1}^{r} I_{n_t} \otimes B_t \]  
which is invariant under right and left multiplications by arbitrary matrices of the form
\[ \sum_{t=1}^{r} I_{n_t} \otimes A_t. \]  

For a fixed \( t \) let \( M^t \) be the totality of matrices \( B_t \) for which there exists a matrix \( B \in M \) such that \( B_t \) occurs in the \( t^{th} \) summand in (59). Obviously \( M^t \subset M_{\infty}^t(F) \) and \( M^t \) is a two-sided ideal. It follows (see Exercise 2) that either \( M^t = 0 \) or \( M^t = M_{\infty}^t(F) \). In other words, there exists some subset of \( 1, \ldots, r \) (which we may assume is \( 1, \ldots, s \) for simplicity of notation) such that \( M \) consists of the totality of matrices of the form
\[ \left( \sum_{t=1}^{s} I_{n_t} \otimes A_t \right) \circ \varphi, \quad p = \sum_{t=s+1}^{r} n_t e_t. \]

Then let \( M' \) be the totality of matrices of the form
\[ q \circ \left( \sum_{t=s+1}^{r} I_{n_t} \otimes A_t \right), \quad q = \sum_{t=1}^{s} n_t e_t, \]
so that \( M' \) is obviously a two-sided ideal in \( M \) and
\[ \beta = \Delta(\mathcal{W}(S)) = M + M'. \]
But then $\mathfrak{g}' = \Delta^{-1}(M')$ is a two-sided ideal in $\mathfrak{U}(S)$ and (62) implies that

$$\mathfrak{U}(S) = \mathfrak{g} + \mathfrak{g}'.$$  

Notice that we have also proved that the only two-sided ideals in $\mathfrak{U}(S)$ are images under $\Delta^{-1}$ of direct sums of complete matrix algebras as described above.

We can directly apply Theorem 2.10 to conclude that the decomposition of $\mathfrak{U}(S_m)$ in Theorem 2.7 and Theorem 2.8 is possible. Moreover the two-sided ideal $\mathfrak{g}$ for which

$$\mathfrak{U}(S_m) = \mathfrak{g} + \ker P$$  

(63)

must itself be a direct sum of two-sided ideals $\mathfrak{g}^t$ whose elements are

$$\Delta^{-1}((I_{n_1} \otimes 0_{e_1}) + \cdots + (I_{n_{t-1}} \otimes 0_{e_{t-1}}) + (I_{n_t} \otimes B_t) + \cdots + (I_{n_{r+1}} \otimes 0_{e_{r+1}}) + \cdots + (I_{n_r} \otimes e_r))$$

where $B_t \in M_{e_t}(k)$ is arbitrary. Observe that $\mathfrak{g}^t$ is a minimal two-sided ideal in $\mathfrak{g}$. Moreover, the $\mathfrak{g}^t$ are the only minimal two-sided ideals in $\mathfrak{U}(S_m)$ because any two-sided ideal must be a direct sum of some of the $\mathfrak{g}^t$ according to the remark at the end of the proof of Theorem 2.10. These remarks apply, of course, to an arbitrary $\mathfrak{U}(S)$. Thus we have:

**Theorem 2.11** If $S$ is a finite group, then the group algebra is a direct sum of minimal two-sided ideals and this decomposition is unique except possibly for order.
We next investigate somewhat more closely the choice of the idempotents \( \epsilon \) that determine the irreducible invariant symmetry classes

\[
P(\epsilon) \otimes V \quad \text{m}
\]

of \( \mathbb{M}^\prime \) in Theorem 2.8. We first remark that any right ideal in an arbitrary \( \mathbb{U}(S) \) has the form

\[
2 \mathbb{U}(S) \quad \text{(64)}
\]

where \( \epsilon \) is an idempotent. Moreover (64) is a minimal right ideal iff \( \epsilon \) is a primitive idempotent. This follows immediately by applying the operator \( \vee \) [see formula (7)] to (64) and using Theorem 2.5. Now if \( W \) is an invariant subspace of \( \mathbb{M}^\prime \) let

\[
\mathcal{J}_W = \{ f \in \mathcal{O} \mid P(f) \otimes V \subset W \} \quad \text{(65)}
\]

Observe that \( \mathcal{J}_W \) is a right ideal in \( \mathbb{U}(S_m) \). For, if \( g \in \mathbb{U}(S_m) \) and \( f \in \mathcal{J}_W \) then

\[
P(fg) \otimes V = P(f)P(g) \otimes V \quad \text{m}
\]

\[
\subset W.
\]

Moreover \( fg \in \mathcal{O} \) because \( \mathcal{O} \) is a two-sided ideal in \( \mathbb{U}(S_m) \). Thus

\[
\mathcal{J}_W = \epsilon \mathbb{U}(S_m)
\]
for some idempotent \( e \). Observe, since \( \mathcal{I}_W \subset \mathcal{O} \) and \( e \in \mathcal{I}_W \)
[i.e., \( \mathcal{I}(S_m) \) contains the multiplicative identity], that \( e \in \mathcal{O} \). Also

\[
\begin{align*}
e \mathcal{I}(S_m) & \subset \mathcal{O} \quad \text{(i.e., } e \in \mathcal{O} \text{ and } \mathcal{O} \text{ is two-sided)}, \\
e^2 \mathcal{I}(S_m) & \subset e\mathcal{O}, \\
e \mathcal{I}(S_m) & \subset e\mathcal{O},
\end{align*}
\]

and thus

\[
\mathcal{I}_W = e \mathcal{I}(S_m) = e\mathcal{O}.
\]

(66)

Conversely, suppose \( \mathcal{J} \) is a right ideal and \( \mathcal{J} \subset \mathcal{O} \). Define

\[
\mathcal{W}_\mathcal{J} = \mathcal{P}(\mathcal{J}) \otimes V.
\]

(67)

Then clearly

\[
\begin{align*}
\mathcal{P}(\mathcal{J}) (\mathcal{P}(\mathcal{J}) \otimes V) &= \mathcal{P}(\mathcal{J}) \mathcal{P}(\mathcal{J})' \otimes V \\
&\subset \mathcal{P}(\mathcal{J}) \otimes V
\end{align*}
\]

so that \( \mathcal{W}_\mathcal{J} \) is an invariant subspace of \( \mathcal{P}' \). The reader will confirm that if \( \mathcal{J} \) is a right ideal in \( \mathcal{O} \) and \( \mathcal{W} \) is an invariant subspace of \( \mathcal{P}' \), then

\[
\mathcal{W}_{\mathcal{J}_W} = \mathcal{W}
\]

(68)
and

\( \mathcal{J}_W = \mathcal{J} \).

(69)

Thus the correspondence \( W \rightarrow \mathcal{J}_W \) is bijective (see Exercise 4). Moreover \( \mathcal{J} \) is a minimal right ideal in \( \mathcal{O} \) iff \( W \) is an irreducible invariant subspace. To prove this last assertion recall that the idempotent \( e \) in (66) is primitive iff \( \mathcal{J}_W \) is minimal [see the remark following (64)]. But \( e \in \mathcal{O} \) so \( P(e) = P_0(e) \), and \( P_0 : \mathcal{O} \to \mathcal{P}_m \) is an algebra isomorphism. Thus clearly \( P_0(e) \) is an irreducible projection with respect to \( m \) iff \( e \) is a primitive idempotent in \( \mathcal{O} \) (see Theorem 2.7). Notice that if \( \mathcal{J} = e \mathcal{W}(S_m) \), then \( \mathcal{W}_e = P_0(e) \otimes V \) so that \( P_0(e) \) is irreducible with respect to \( m \) iff \( \mathcal{W}_e \) is an irreducible invariant subspace.

Hence if we are able to produce a set of primitive idempotents in \( \mathcal{O} \) generating minimal right ideals, these in turn will give rise to irreducible invariant symmetry classes for \( m \). However \( \mathcal{J} = e \mathcal{W}(S_m) \) is a minimal right ideal iff \( \mathcal{N}(\mathcal{J}) = \mathcal{W}(S_m) \cdot e \) is a minimal left ideal. It follows that if we decompose \( \mathcal{O} \) into a direct sum of minimal left ideals with corresponding primitive idempotents \( e \), then the irreducible invariant symmetry classes for \( m \) will simply be the nonzero subspaces

\[
\frac{P(\nu(e)) \otimes V}{1}
\]

The problem is to identify precisely the primitive idempotents \( e \in \mathcal{O} \) that generate the minimal left ideals in \( \mathcal{O} \). From the discussion immediately following the proof of Theorem 2.10 we know
that \( \mathcal{S} \) is a direct sum of minimal two-sided ideals in \( \mathbb{U}(S_m) \).

We also know from Theorem 2.11 that these minimal two-sided ideals in \( \mathbb{U}(S_m) \) are unique. The program that we wish to complete then is the following: We will show:

I. If

\[
\mathbb{U}(S_m) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} U_{ij},
\]

(70)
is a direct sum decomposition of \( \mathbb{U}(S_m) \) as described in (52), then

\[
\mathcal{S}_i = \sum_{j=1}^{n_i} U_{ij}, \quad i = 1, \ldots, r,
\]

(71)
is a minimal two-sided ideal in \( \mathbb{U}(S_m) \) and hence these are the unique such ideals (Theorem 2.11). Moreover the \( v(\mathcal{S}_i) \) are minimal two-sided ideals whose direct sum is \( \mathbb{U}(S_m) \) so these must be the \( \mathcal{S}_i \) in some order.

II. The two-sided ideal \( \mathcal{S} \) is then a direct sum of some of the two-sided ideals \( v(\mathcal{S}_i) \) as we just indicated. We can choose the notation so that

\[
\mathcal{S} = v(\mathcal{S}_1) + \cdots + v(\mathcal{S}_p).
\]

(72)

III. The exact form of the primitive idempotents \( e_{ij} \) that generate the \( U_{ij} \) in (70) will be determined in Section 6.3. Thus \( \mathcal{S} \) will be a direct sum of minimal right ideals generated
by the idempotents

\[ \nu(e_{ij}), \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, p. \]  

(73)

i.e., if \( \mathbb{W}_{ij} = \mathbb{W}(S_m) e_{ij} \) then \( \nu(\mathbb{W}_{ij}) = \nu(e_{ij}) \mathbb{W}(S_m) \). These idempotents \( \nu(e_{ij}) \) that generate the \( \nu(\mathbb{W}_{ij}) \) are split into two sets. In the first set are those that generate the \( \nu(\mathbb{W}_{ij}) \), \( j = 1, \ldots, p \) and hence in view of (72) are in \( \mathcal{O} \). Since \( \ker P \) is also a two-sided ideal it is a direct sum of some of the \( \mathcal{O}_i \) and in view of (71)

\[ \ker P = \nu(\mathcal{O}_{p+1}) + \cdots + \nu(\mathcal{O}_r). \]

The second set of \( \nu(e_{ij}) \) are those that generate the right ideals \( \nu(\mathbb{W}_{ij}) \), \( i = p+1, \ldots, r \), in \( \ker P \). It follows that a simple criterion for deciding if \( \nu(e_{ij}) \) generates one of the \( \nu(\mathbb{W}_{ij}) \) appearing in \( \ker P \) is obtained by evaluating \( P(\nu(e_{ij})) \):

if \( P(\nu(e_{ij})) = 0 \) then \( \nu(e_{ij}) \in \ker P \); if \( P(\nu(e_{ij})) \neq 0 \) then obviously \( \nu(e_{ij}) \in \mathcal{O} \).

IV. We will prove that for each fixed \( i \) the nonzero irreducible invariant subspaces

\[ P(\nu(e_{ij})) \otimes V \]

(74)
determine equivalent components of \( \Pi', \quad j = 1, \ldots, n_i \), and that for \( k \neq i \), if \( P(\nu(e_{kj})) \otimes V \) is not zero it determines a component of \( \Pi' \) not equivalent to (74). Anticipating the verification of these results we can therefore state
Theorem 2.12  The irreducible invariant subspaces of $\Pi'$ are the symmetry classes of tensors

$$P(e_{ij}) \otimes V$$

(75)

where the $e_{ij}$ generate the minimal right ideals $\nu(i_j)$ and

$$P(e_{ij}) \neq 0.$$  

(76)

Moreover the components of $\Pi'$ corresponding to the invariant subspaces (75) for $j = 1, \ldots, n_i$ are equivalent whereas those corresponding to $e_{ij}$ and $e_{ik}$ are inequivalent if $i \neq k$.

To confirm the assertions I - IV, we will develop an elementary calculus of ideals in the group algebra $\mathbb{U}(S)$ of a finite group. The results are true for arbitrary finite groups and will therefore be stated for general $S$.

If $A$ and $B$ are arbitrary subsets of $\mathbb{U}(S)$, then we use the notations:

$$A + B = \{a + b, a \in A, b \in B\},$$

$$AB = \{ab, a \in A, b \in B\},$$

$$\langle AB \rangle = \left\{ \sum_{k} a_k b_k, a_k \in A, b_k \in B \right\},$$

and finally

$$[A] = \langle A\mathbb{U}(S) \rangle.$$  

Theorem 2.13 If $A$ and $B$ are left ideals in $\mathbb{U}(S)$ then
(a) $A + B$ and $\langle AB \rangle$ are left ideals;
(b) the intersection of any family of two-sided ideals in $\mathbb{U}(S)$ is a two-sided ideal;
(c) $\langle A \rangle$ is the intersection of all two-sided ideals containing $A$;
(d) if $b \in B$ then $Ab \subseteq B$ and $Ab$ is a left ideal;
(e) if $B$ is minimal then $AB = B$ or $AB = 0$;
(f) if $A$ is minimal then $A^2 = AA = A$ or $A^2 = 0$;
(g) if $B$ is minimal then $AB = \langle AB \rangle$;
(h) if $A^2 = 0$ then $A = 0$.

Proof: Parts (a) - (g) are left as routine exercises (see Exercise 5). We prove (h). By Theorem 2.5 if $A \neq 0$ then $A = \mathbb{U}(S)e$ where $e$ is an idempotent. Then since $e \in A$, $e^2 \in A^2$ and if $A^2 = 0$, then $e = e^2 = 0$. But then $A = 0$, a contradiction. Thus if $A^2 = 0$ then $A = 0$. 

Observe that by applying (h) to (f) in Theorem 2.13 we can conclude that if $A$ is a minimal left ideal then

$$A^2 = A.$$ (77)

Theorem 2.14 Let $A$ and $B$ be minimal left ideals in $\mathbb{U}(S)$. Then

(a) $\rho|A$ and $\rho|B$ are equivalent iff $AB = B$;
(b) $\rho|A$ and $\rho|B$ are inequivalent iff $AB = 0$;
(c) if $\rho|A$ and $\rho|B$ are equivalent then $\{A\} = \{B\}$;
(d) if $\rho|A$ and $\rho|B$ are inequivalent then $\{A\} \cap \{B\} = 0$.

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and \((A)[B] = [B][A] = 0\).

**Proof:** (a) First note that by Theorem 2.13(e) \(AB = B\) or \(AB = 0\). Suppose then that \(\rho|A\) and \(\rho|B\) are equivalent and let \(T : B \rightarrow A\) be a linear bijection such that \((\rho(f)|A)T = T(\rho(f)|B)\), \(f \in \mathbb{U}(S)\). Then

\[
ft(b) = T(fb), \quad f \in \mathbb{U}(S), \quad b \in B, \quad (78)
\]

so that taking \(f \in A\) in (78) we conclude from (77) that

\[
T(AB) \subset A^2 = A.
\]

On the other hand, anything in \(A = A^2\) is of the form \(ft(b) = T(fb), f \in A, b \in B\), so that \(T(AB) = A^2 = A\). Hence \(AB \neq 0\) so that \(AB = B\). Conversely, suppose \(AB = B\) and let \(a \in A, b \in B\) be chosen so that \(ab \neq 0\). Define \(T : A \rightarrow AB = B\) by

\[
T_\alpha = \alpha b,
\]

set \(B_0 = \text{Im} \, T\) and \(A_0 = \text{ker} \, T\). Now \(Ta = ab \neq 0\) so \(A_0\) is a proper subspace of \(A\). Also if \(f \in \mathbb{U}(S), \alpha \in A_0\), then \(f_\alpha \in A\), \(\alpha b = T(\alpha) = 0\),

\[
T(f_\alpha) = (f_\alpha)b
= f(\alpha b)
= 0,
\]

and hence \(A_0 \subset A\) is a left ideal in \(\mathbb{U}(S)\). Since \(A\) is minimal
we conclude that $A_0 = 0$ and hence $T$ is injective. Also observe that for any $\alpha \in A$, $f \in W(S)$,

$$fT(\alpha) = f(\alpha b)$$

$$= (f\alpha)b$$

$$= T(f\alpha)$$

$$\in \text{Im } T = B_0,$$

i.e., $W(S)B_0 \subseteq B_0$ so that $B_0 \subseteq B$ is a left ideal. Since $Ta = ab \neq 0$ we know $B_0 \neq 0$ so that the minimality of $B$ implies $B_0 = B$. It follows that $T : A \to B$ is bijective. Finally if $f \in W(S)$, $\alpha \in A$, then

$$(\rho(f)|B)T\alpha = (\rho(f)|B)\alpha b$$

$$= f(\alpha b)$$

$$= (f\alpha)b$$

whereas

$$T(\rho(f)|A)\alpha = T(f\alpha)$$

$$= (f\alpha)b.$$ 

(b) Since $AB \neq B$ is equivalent to $AB = 0$ by Theorem 2.13(e), (b) is simply a restatement of (a).

(c) From Theorem 2.13(c), $[A]$ and $[B]$ are the smallest two-sided ideals containing $A$ and $B$ respectively. If $\rho|A$
and \( p | B \) are equivalent, then by (a), \( AB = B \) so that by Theorem 2.13(c) and the definition of \([A]\),

\[ B \subseteq AB \subseteq A \cap (A \cap B) = [A]. \]

Thus

\[ [B] \subseteq [A] \]

and similarly

\[ [A] \subseteq [B], \]

so that

\[ [A] = [B]. \]

(d) From (b) we have \( AB = 0 \). Let \( \alpha \in [A], \beta \in [B], \)

\[ \alpha = a_0 + \sum_{i=1}^{m} a_i f_i, \]

\[ \beta = b_0 + \sum_{i=1}^{n} b_i g_i, \]

\( a_i \in A, b_i \in B, f_i, g_i \) in \( \mathbb{U}(S) \), so that

\[ \alpha \beta = a_0 b_0 + \sum_{i=1}^{n} (a_0 b_i) f_i + \sum_{i=1}^{m} a_i (f_1 b_0) + \sum_{i,j} (a_i (f_1 d_j) g_j. (79) \]

Now \( AB = 0 \) so that every summand in (79) is 0 and thus \( [A][B] = 0 \). Similarly \([B][A] = 0\). If \( C = [A] \cap [B], \) then \( C \) is a
two-sided ideal and since \( C \subseteq \{ A \}, \ C \subseteq \{ B \} \),

\[
C^2 \subseteq [A][B] = 0.
\]

By Theorem 2.13(h), \( C = 0 \).

Now write the group algebra \( \mathcal{U}(S) \) as a direct sum of minimal left ideals:

\[
\mathcal{U}(S) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} A_{ij} \quad (80)
\]

precisely as in (52). (We use the notation \( A_{ij} \) in dealing with a general group algebra rather than \( Y_{ij} \) to conform to the preceding two theorems.) For each \( i = 1, \ldots, r \), the ideals in the list

\[
A_{i1}, \ldots, A_{in_i} \quad (81)
\]

are equivalent, i.e., the regular representation restricted to each of the ideals in (81) are all equivalent. Also, if \( s \neq t \) then \( A_{sj} \) and \( A_{tj} \) are inequivalent.

As we observed following (52) the left ideals \( A_{ij} \) are unique to within equivalence and order. We have

**Theorem 2.15 (a) Each of the ideals**

\[
A_i = \sum_{j=1}^{n_i} A_{ij}, \quad i = 1, \ldots, r, \quad (82)
\]

is a minimal two-sided ideal, i.e., does not properly contain any nonzero two-sided ideal.
(b) If \( A \) is any minimal left ideal, then there exists an
\( i \) such that \( A \subseteq A_i \) and \( \rho\vert A \) is equivalent to each \( \rho\vert A_{ij} \),
\( j = 1, \ldots, n_i \).

(c) The decomposition of \( S(S) \) into a direct sum of minimal
two-sided ideals

\[ S(S) = A_1 + \cdots + A_r \]

is unique to within order.

Proof: (a) By Theorem 2.14(c)

\[ \{A_{i1}\} = \cdots = \{A_{in_i}\} \]

and since \( A_{ij} \subseteq \{A_{ij}\} \) we have

\[ A_i = \sum_{j=1}^{n_i} A_{ij} \]

\[ \subseteq \{A_{i1}\}, \ i = 1, \ldots, r. \quad (83) \]

Also, by Theorem 2.14(d),

\[ \{A_{i1}\} \cap \{A_{kl}\} = 0, \ i \neq k. \quad (84) \]

Since

\[ S(S) = A_1 + \cdots + A_r, \]

(83) and (84) imply that

\[ A_i = \{A_{i1}\} = \cdots = \{A_{in_i}\}, \ i = 1, \ldots, r, \]
so that $A_i$ is a two-sided ideal. To see that it is minimal suppose $0 \neq C$ is a two-sided ideal, $C \subseteq A_i$. Now $CA_{ij} \subseteq A_{ij}$ for $j = 1, \ldots, n_i$, and

$$CA_i \subseteq \sum_{j=1}^{n_i} CA_{ij}.$$ 

Thus if $CA_{ij} = 0$ for $j = 1, \ldots, n_i$, it would follow that $CA_i = 0$. In particular $C^2 = 0$ so that $C = 0$ from Theorem 2.13(h). Since we are assuming $C \neq 0$ it follows that for some $k$, $1 \leq k \leq n_i$,

$$0 \neq CA_{ik} \subseteq A_{ik}.$$ 

The product $CA_{ik}$ is a nonzero left ideal in $W(S)$ contained in the minimal left ideal $A_{ik}$ and hence

$$CA_{ik} = A_{ik}.$$ 

Since $C$ is a right ideal, $CA_{ik} \subseteq C$, and hence from the preceding equality,

$$A_{ik} \subseteq C.$$ 

But then

$$A_i = \{A_{ik}\} \subseteq C$$ 

so that

$$A_i = C.$$ 

Thus any nonzero two-sided ideal contained in $A_i$ is equal to $A_i$.  

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(b) By Theorem 2.13 (e), $A_{ij} A = A$ or $A_{ij} A = 0$ for all $i, j$. If the latter alternative held for all $i, j$ it would follow that

$$\mathbb{A}(S)A = 0$$

and hence $A^2 = 0$. But then $A = 0$ by Theorem 2.13(h). Thus for some $i_0, j_0$,

$$A_{i_0 j_0} A = A$$

and hence by Theorem 2.14(a), $\rho|A_{i_0 j_0}$ and $\rho |A$ are equivalent. But then by Theorem 2.14(c)

$$[A] = [A_{i_0 j_0}] = A_{i_0}$$

and hence

$$A \subset A_{i_0}.$$

(c) Although we have proved this part of the theorem by other methods, the following elementary argument is of some interest. We shall show in fact that if $B$ is any two-sided ideal then $B$ is a direct sum of some of the $A_i$. For, by the discussion immediately following the proof of Theorem 2.10, $B$ is a direct sum of minimal two-sided ideals. Thus it suffices to prove that if $B$ is a minimal two-sided ideal, then $B$ is equal to some $A_i$.

Consider $\langle BA_j \rangle$, clearly a two-sided ideal contained in both $B$ and $A_j$. Since both $B$ and $A_j$ are minimal (by (a)), we must have $\langle BA_j \rangle = 0$ or
\[ \langle BA_j \rangle = B = A_j. \] \hspace{1cm} (85)

But if \( \langle BA_j \rangle = 0 \) for all \( j \), it would follow that

\[ B\mathbb{U}(S) = 0, \]

\[ B^2 = 0, \]

and by Theorem 2.13(h) that \( B = 0 \). Thus by (85) \( B \) must be some \( A_i \).

If

\[ \mathbb{U}(S) = B_1 \dagger \cdots \dagger B_p \]

is a direct sum of minimal two-sided ideals, then each \( B_i \) must be an \( A_j \) and the uniqueness follows.

**Example 2.2** The results in Theorem 2.14 can be easily applied to obtain a simple criterion for two minimal left ideals \( A \) and \( B \) in \( \mathbb{U}(S) \) to be equivalent. Thus let \( e_1 \) and \( e_2 \) be generating idempotents for \( A \) and \( B \), i.e.,

\[ A = \mathbb{U}(S)e_1, \]

and

\[ B = \mathbb{U}(S)e_2. \]

Then \( \rho|A \) and \( \rho|B \) are equivalent iff there is an \( f \in \mathbb{U}(S) \) such that

\[ e_1fe_2 \neq 0. \] \hspace{1cm} (86)
Hence if \( x = e_1 f e_2 \), then \( 0 \neq x = e_1 f e_2 e_1^{-1} = e_1 x e_2 \). To confirm (86) we know from Theorem 2.14(a) that \( A \) and \( B \) are equivalent iff \( AB = B \). Observe that

\[
e_1 \mathcal{W}(S)e_2 = e_1 B
\]

so that if \( e_1 \mathcal{W}(S)e_2 = 0 \), then \( e_1 B = 0 \) and \( B = AB = \mathcal{W}(S)e_1 B = 0 \). Thus \( e_1 \mathcal{W}(S)e_2 \neq 0 \) so that (86) is verified. Conversely, if (86) holds then \( e_1 \mathcal{W}(S)e_2 \neq 0 \) and thus \( AB \neq 0 \). Hence \( AB = B \) and \( A \) and \( B \) are equivalent.

We can now apply some of these general results to \( S_m \) to obtain

**Theorem 2.16** Let

\[
\mathcal{W}(S_m) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} u_{ij}
\]

be the decomposition of \( \mathcal{W}(S_m) \) into minimal left ideals as in (70) and let \( v_{ij} \) be a primitive generating idempotent for \( u_{ij} \), i.e.,

\[
u_{ij} = \mathcal{W}(S_m) v_{ij}.
\]

Let

\[
W_{ij} = P(\nu v_{ij}) \otimes V
\]

where \( P(\nu v_{ij}) \neq 0 \). Then for a fixed \( i \) the subspaces \( W_{ij} \) determine equivalent irreducible components of \( \mathcal{W} \) while for \( k \neq i \), \( W_{ij} \) and \( W_{kt} \) determine inequivalent irreducible components of \( \mathcal{W} \).
Proof: As we saw in Theorem 2.7 and in the discussion following formulas (68) and (69), the nonzero subspaces $W_{ij}$ generated by the projections $P(\nu(e_{ij}))$, $\nu(e_{ij}) \in \mathcal{O}$, are irreducible invariant subspaces of $\Pi$. Moreover $P(\nu(e_{ij})) \neq 0$ iff $\nu(e_{ij}) \in \mathcal{O}$ whereas $P(\nu(e_{ij})) = 0$ iff $\nu(e_{ij}) \in \ker P$ (see III). Now we turn our attention to $e_{ij} = e_1$ and $e_{it} = e_2$ where $j$ and $t$ are fixed and $P(\nu(e_1)) = P(\nu(e_2)) \neq 0$. It will automatically follow that $P(\nu(e_2)) \neq 0$. For, $\nu(W_{ij})$ and $\nu(W_{it})$ lie in the same minimal two-sided ideal $\nu(R_{i}) = \bigoplus_{s=1}^{n} \nu(W_{is})$. [The two-sided ideals $\nu(R_{i})$, $i = 1, \ldots, r$, and $\nu(R_{i})$, $i = 1, \ldots, r$, are the same to within order as we saw in I.] Since $\mathcal{O}$ is a direct sum of some of the $\nu(R_{k})$, $k = 1, \ldots, r$, and $\nu(e_{1}) \in \nu(R_{i})$, it follows that $\nu(R_{i})$ must be a direct summand in this representation of $\mathcal{O}$. Thus $\nu(e_{2}) \in \nu(R_{i}) \subset \mathcal{O}$ so $P(\nu(e_{2})) \neq 0$.

Since $e_1$ and $e_2$ determine equivalent minimal left ideals, we can use Example 2.2 to produce a nonzero $x$ such that

$$x = e_1xe_2.$$

Observe that $\nu(x) = \nu(e_2)\nu(x)e_1 \in \nu(e_2)\nu(S_m) = \nu(W_{it}) \subset \nu(R_{i}) \subset \mathcal{O}$. Thus in fact $\nu(x)$ is in the domain $\mathcal{O}$ of the bijection $P_0: \mathcal{O} \rightarrow P_m$. Set $W_1 = W_{ij}$, $W_2 = W_{it}$ and compute that

$$P(\nu(x))W_1 \subset P(\nu(x)) \otimes V$$

$$= P(\nu(e_2))P(\nu(x))P(\nu(e_1)) \otimes V$$

$$\subset P(\nu(e_2)) \otimes V$$

$$= W_2.$$

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Thus \( P(\nu(x)) : W_1 \rightarrow W_2 \) is a linear map and of course it commutes
with every \( \Pi'(T) \), i.e., it is in \( \Pi_m \). It follows from Theorem
1.8(b) in Section 6.1 that either \( P(\nu(x)) = 0 \) or
\[ \Pi'_m | W_1 \sim \Pi'_m | W_2. \]  \hspace{1cm} (90)

But we just observed that \( \nu(x) \in G \) so \( P(\nu(x)) \neq 0 \) and (90)
obtains. That is, the components
\[ \Pi'_m | W_{ij}, \hspace{0.5cm} j = 1, ..., n_i \]
are equivalent.

To prove the second part of the theorem suppose that \( W_{ij} \)
and \( W_{kt} \) determine equivalent components, i.e., \( P(\nu(\epsilon_{ij})) \neq 0 \)
and \( P(\nu(\epsilon_{kt})) \neq 0 \) so that \( \nu(\epsilon_{ij}) \in G \) and \( \nu(\epsilon_{kt}) \in G \). Then
there is a linear bijection
\[ \mathcal{L} : W_{ij} \rightarrow W_{kt} \]
such that
\[ \Pi'_m | W_{ij} = \Pi'_m | W_{kt} \mathcal{L}, \hspace{0.5cm} T \in \text{GL}_m(V). \]
The representation \( \Pi' \) is fully reducible so we can find an in-
variant subspace \( W' \) of \( \Pi' \) such that
\[ \otimes V = W_{ij} \oplus W'. \]

Extend \( \mathcal{L} \) to all of \( \otimes V \) by setting
\[ \mathcal{L}(w + w') = \mathcal{L}w \]
where \( w \in W_{ij} \), \( w' \in W' \). Clearly the extended operator \( L_m \) commutes with every \( \prod^*(T) \) and thus by Theorem 1.14, Section 6.1

\[ L_m \in P_m. \]

Of course \( L \neq 0 \) so that \( L \) must be a value of the bijection

\[ P_0 = P \mid \mathcal{O}, \ P_0 : \mathcal{O} \rightarrow P_m, \text{ i.e., there exists some } f \in \mathcal{O} \text{ such that} \]

\[ P_0(f) = L. \]

Hence

\[ W_{kt} = \sigma(W_{ij}) \]

\[ = P_0(f)W_{ij}, \]

or equivalently

\[ P_0(f)P_0(\nu(e_{ij})) \otimes V = P_0(\nu(e_{kt})) \otimes V. \quad (91) \]

Apply the idempotent \( P_0(\nu(e_{kt})) \) to each side of (91) to obtain

\[ P_0(\nu(e_{kt})P_0(f)P_0(\nu(e_{ij})) \otimes V = P_0(\nu(e_{kt}))P_0(\nu(e_{kt})) \otimes V \]

or

\[ P_0(\nu(e_{kt})f\nu(e_{ij})) \otimes V = P_0(\nu(e_{kt})) \otimes V \neq 0. \quad (92) \]

Thus \( \nu(e_{kt})f\nu(e_{ij}) \neq 0 \). Let \( f = \nu(f') \). Then

\[ \nu(e_{ij}f' e_{kt}) = \nu(e_{kt})f\nu(e_{ij}) \neq 0 \]
\[ e_{ij} e'_{ij} e_{kt} \neq 0. \quad (93) \]

Hence by Example 2.2, \( W_{ij} \) and \( W_{kt} \) are equivalent, i.e.,
\[ p|W_{ij} \sim p|W_{kt}. \]
Since for \( i \neq k \), \( W_{ij} \) and \( W_{kt} \) are inequivalent, it follows that \( W_{ij} \) and \( W_{kt} \) determine inequivalent components of \( \Pi^m \).

Theorem 2.16 completes the program outlined in the discussion contained in I, II, III, IV. Thus Theorem 2.12 is established and the remaining problem is to determine the precise structure of the primitive idempotents in the group algebra \( \mathbb{U}(S_m) \) that generate the minimal left ideals \( W_{ij} \) in the decomposition (87). This problem will be solved in Section 6.3 and the exact structure of the irreducible invariant symmetry classes of tensors for \( \Pi^m \) will be determined. Before going on, however, it is interesting to observe that for any \( f \in \mathbb{U}(S_m) \), \( \mathbb{P}(\varphi(f)) = 0 \) iff \( \mathbb{P}(f) = 0 \), i.e.,
\[ \varphi(\ker \mathbb{P}) = \ker \mathbb{P}. \]
For, let \{ \( e_1, \ldots, e_n \) \} be a basis of \( V \) and
\[ E = \{ e^\alpha, \alpha \in \Gamma_n \} \]
be the induced basis in \( \otimes \, V \). Notice that for any \( \sigma \in S_m \), \( e^\alpha \), \( \alpha \in \Gamma_n^m \),
\[ \mathbb{P}(\sigma) e^\alpha = e^\sigma \]
and \( \alpha^{-1} = \beta^{-1} \) iff \( \alpha = \beta \), i.e., \( \mathbb{P}(\sigma) \) induces a permutation of the basis \( E \) and hence
\[ [\mathbb{P}(\sigma)]_E^E \]

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is an $n^m$-square permutation matrix. Since $P(\sigma^{-1}) = P(\sigma)^{-1}$, it follows that

$$[P(\sigma^{-1})]_E^\otimes = ([P(\sigma)]_E^\otimes)^T.$$ 

Thus

$$P(\nu(f)) = P\left(\nu\left(\sum_{\sigma \in S_m} f(\sigma)\sigma\right)\right)$$

$$= P\left(\sum_{\sigma \in S_m} f(\sigma)\nu(\sigma)\right)$$

$$= P\left(\sum_{\sigma \in S_m} f(\sigma)\sigma^{-1}\right)$$

$$= \sum_{\sigma \in S_m} f(\sigma)P(\sigma^{-1}).$$

By taking matrix representations it follows that

$$[P(\nu(f))]_E^\otimes = \sum_{\sigma \in S_m} f(\sigma)\left([P(\sigma)]_E^\otimes\right)^T$$

$$= \left(\sum_{\sigma \in S_m} f(\sigma)P(\sigma)\right)_E^\otimes^T$$

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\[
(P(f))^E \otimes \otimes
\]

so that clearly \( P(\nu(f)) = 0 \) iff \( P(f) = 0 \). Thus in the statement of Theorem 2.16 we can replace \( P(\nu(s)) \neq 0 \) by \( P(s) \neq 0 \).

The remainder of this section is devoted to briefly surveying a number of interesting and important results in general representation theory to be used in Section 6.3. We shall assume henceforth that \( S \) is a finite group of order \( h \), \( |S| = h \), and all representations are proper.

**Theorem 2.17** Let \( \chi \) and \( \psi \) be distinct irreducible characters. Then

\[
(\chi, \psi) = 0.
\]  

(94)

If \( \chi \) is an absolutely irreducible character then

\[
(\chi, \chi) = 1.
\]  

(95)

**Proof:** Let \( \chi(s) = \text{tr} \Delta(s), \psi(s) = \text{tr} \kappa(s) \). Then

\[
\chi = \sum_{i=1}^{m} \Delta_{ii}, \quad \psi = \sum_{i=1}^{m} \kappa_{ii}
\]

so that

\[
(\chi, \psi) = \left( \sum_{i=1}^{m} \Delta_{ii}, \sum_{i=1}^{n} \kappa_{ii} \right)
\]

\[
= \sum_{i=1, j=1}^{m, n} \left( \Delta_{ii}, \kappa_{jj} \right)
\]

\[
= 0.
\]

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This last equality is an application of Theorem 1.10 [formula (77)] in Section 6.1. On the other hand, setting $p = q = j = i$ in formula (78) in the preceding reference, we have $\langle \Delta_{ii}, \Delta_{ii} \rangle = \frac{1}{m}$ so that

$$\langle \chi, \chi \rangle = \sum_{i=1}^{m} \langle \Delta_{ii}, \Delta_{ii} \rangle = \frac{1}{m} = 1.$$  

Formulas (94) and (95) are called the character orthogonality relationships of the first kind.

**Theorem 2.18** If $\chi$ is an absolutely irreducible character of degree $m$, then for any $t \in S$

$$\frac{1}{h} \sum_{s \in S} \chi(st) \chi(s^{-1}) = \frac{1}{m} \chi(t), \quad (96)$$

and

$$\frac{1}{h} \sum_{s \in S} \chi(ts^{-1}) \chi(s) = \frac{1}{m} \chi(t). \quad (97)$$

**Proof:** If we write out formula (78) in Section 6.1, we have

$$\frac{1}{h} \sum_{s \in S} \Delta_{ip}(s) \Delta_{jq}(s^{-1}) = \frac{1}{m} \delta_{ij} \delta_{pq}.$$
so that multiplying both sides of this last equation by $\Delta_{p k}(t)$ and summing on $p$ we have

$$\frac{1}{h} \sum_{s \in S} \left( \sum_{p=1}^{m} \Delta_{i p}(s) \Delta_{p k}(t) \right) \Delta_{q j}(s^{-1}) = \frac{1}{m} \sum_{p=1}^{m} \Delta_{p k}(t) \delta_{p q},$$

$$\frac{1}{h} \sum_{s \in S} (\Delta(s) \Delta(t))_{i k} \Delta_{q j}(s^{-1}) = \frac{1}{m} \delta_{i j} \Delta_{q k}(t).$$

In the preceding equation, set $k = i$, $q = j$, to obtain

$$\frac{1}{h} \sum_{s \in S} \Delta_{i i}(s t) \Delta_{j j}(s^{-1}) = \frac{1}{m} \delta_{i j} \Delta_{i i}(t)$$

and then sum both sides over all $i$ and $j$ to obtain

$$\frac{1}{h} \sum_{s \in S} \chi(st) \chi(s^{-1}) = \frac{1}{m} \chi(t)$$

[i.e., $\sum_{i=1}^{m} \left( \sum_{j=1}^{m} \delta_{i j} \right) \Delta_{i i}(t) = \sum_{i=1}^{m} \Delta_{i i}(t) = \chi(t)$, and (96) is proved.]

To prove (97) replace $s$ by $s^{-1}$ in (96) to obtain

$$\frac{1}{h} \sum_{s \in S} \chi(s^{-1} t) \chi(s) = \frac{1}{m} \chi(t)$$

and then use the fact that $\chi(s^{-1} t) = \chi(ts^{-1})$.

**Theorem 2.19** If $\chi$ is the principal character of $S$ then
\[ \sum_{s \in S} \chi(s) = h. \quad (98) \]

If \( \psi \) is a character of an irreducible representation, \( \psi \neq \chi \), then

\[ \sum_{s \in S} \psi(s) = 0. \quad (99) \]

Proof: The equality (98) is obvious because \( \chi(s) \equiv 1 \).

If \( \psi \) is the character of an irreducible representation different from the principal character, then obviously the two representations must be inequivalent. Thus we can apply (94) to yield (99).

Theorem 2.20 (a) Let \( \chi_1, \ldots, \chi_k \) be distinct absolutely irreducible characters of \( S \) and let \( \chi = \sum_{i=1}^{k} a_i \chi_i \), \( \psi = \sum_{i=1}^{k} b_i \chi_i \).

Then

\[ (\chi, \psi) = \sum_{i=1}^{k} a_i b_i. \quad (100) \]

(b) The character \( \chi \) is absolutely irreducible iff

\[ (\chi, \chi) = 1. \quad (101) \]

Proof: (a) The proof of (a) follows immediately from (94), (95) and the properties of the scalar product.

(b) Suppose \( \chi(s) = \text{tr} \Delta(s) \). Then \( \Delta \) is fully reducible (Theorem 1.6, Section 6.1) so that by extending the field \( \mathbb{R} \) if necessary, we can assume
\[ \Delta \sim \Delta^1 + \cdots + \Delta^k \]  

(102)

where the \( \Delta^i \) are absolutely irreducible. If we group the equivalent components in (102) together, we can change the notation slightly as follows:

\[ \Delta \sim m_1 \Delta_1 + \cdots + m_r \Delta_r \]  

(103)

where \( \Delta_1, \ldots, \Delta_r \) are absolutely irreducible, pairwise inequivalent, and the \( m_i \) are positive integers (i.e., \( m_\ell \Delta_\ell \) means \( \Delta_\ell + \cdots + \Delta_\ell \)). Then if \( \chi_i \) is the character corresponding to \( \Delta_i \) we have

\[ \chi = \sum_{i=1}^{r} m_i \chi_i, \]

and from (94)

\[ (\chi, \chi) = \sum_{i=1}^{r} m_i^2. \]  

(104)

Now if \( \chi \) is absolutely irreducible, then \( r = 1 \) and \( m_1 = 1 \) so that \( (\chi, \chi) = 1 \). Conversely, if \( (\chi, \chi) = 1 \), then

\[ \sum_{i=1}^{r} m_i^2 = 1. \]  

(105)

Hence some \( m_i = 1 \) and the rest are 0. But then \( \Delta \sim \Delta_i \).  

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The next result explicitly sets out the central role played by the regular representation of a group. Although the proofs of some parts of this theorem have occurred elsewhere, we briefly recapitulate them here.

**Theorem 2.21** (a) If \( L : S \to GL_n(V) \) is an irreducible representation of \( S \) then \( L \) is equivalent over \( R \) to some irreducible component of \( \rho \).

(b) There are only a finite number of pairwise inequivalent absolutely irreducible representations of \( S \). More precisely: if \( L \) is an absolutely irreducible representation of \( S \) over some field \( D \), \( R \subset D \), then \( L \) is equivalent over a finite algebraic extension \( K \) of \( D \) to an absolutely irreducible component of \( \rho \) and moreover this component can be chosen to be over a finite algebraic extension \( F \) of \( R \).

(c) Let \( L_1, \ldots, L_r \) be a complete list of pairwise inequivalent absolutely irreducible representations of \( S \) over \( F \) as given by (b) (i.e., \( L_1, \ldots, L_r \) are the distinct absolutely irreducible components of \( \rho \) over \( F \), a finite algebraic extension field of \( R \)). Suppose that the degree of \( L_i \) is \( n_i \), \( i = 1, \ldots, r \). Then \( L_i \) is equivalent over \( F \) to precisely \( n_i \) absolutely irreducible components of \( \rho \) and

\[
\sum_{i=1}^{r} n_i^2 = h. \tag{106}
\]

In other words, \( L_i \) occurs \( n_i \) times among all the absolutely irreducible components of \( \rho \).
(d) If $L$ is any representation of $S$, then

$$L \sim \sum_{i=1}^{r} c_i L_i$$

where $c_i = (x_{L_i}, x_{L_i})$, $i = 1, \ldots, r$.

(e) Let $\Delta^1, \ldots, \Delta^r$ be matrix representations over $F$ corresponding to $L_1, \ldots, L_r$, respectively. Then

$$\dim(\Delta^t_{i,j}, \ i,j = 1,\ldots,n_t, \ t = 1,\ldots,r) = h \quad (107)$$

and the $u_1^2 + \cdots + u_r^2$ functions $\Delta^t_{i,j}$, $i,j = 1,\ldots,n_t$, $t = 1,\ldots,r$ are a basis of $\mathfrak{u}(S)$ over $F$.

Proof: (a) First notice that if $0 \neq v_0 \in V$, then $L(\mathfrak{u}(S))v_0 = V$. For obviously $L(\mathfrak{u}(S))v_0$ is a nonzero invariant subspace of $L$ and the irreducibility of $L$ implies it cannot be a proper subspace of $V$. Let

$$A = \{ f \in \mathfrak{u}(S) \mid L(f)v_0 = 0 \}. \quad (108)$$

It is easy to confirm that $A$ is a left ideal in $\mathfrak{u}(S)$ and thus an invariant subspace of $\rho$. Now $\rho$ is fully reducible so that there exists a left ideal $B$ such that

$$\mathfrak{u}(S) = A + B.$$ 

Define $T : V \to B$ by $Tv = b$ where, $L(f)v_0 = v$ and $f = a + b$, $a \in A$, $b \in B$. We confirm that $T$ is unambiguously defined:

If $L(g)v_0 = v$, then $L(f - g)v_0 = 0$, $f - g \in A$, so that the projection along $B$ of both $f$ and $g$ must be the same, i.e., $b$. 

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Observe next that $T$ is linear. For suppose $Tu = b_1, Tv = b_2$
where $L(a_i + b_i)v_0 = L(b_i)v_0 = u$ and $L(a_2 + b_2)v_0 = L(b_2)v_0 = v$,
$a_i \in A, b_i \in B, i = 1,2$. Then for arbitrary $r$ and $k$ in $R$

$$L(r(a_1 + b_1) + k(a_2 + b_2))v_0 = rL(b_1)v_0 + kL(b_2)v_0$$

$$= ru + kv.$$ 

Thus

$$L(rb_1 + kb_2)v_0 = ru + kv.$$ 

It follows from the definition of $T$ that

$$T(ru + kv) = rb_1 + kb_2$$

$$= rTu + kTv.$$ 

To see that $T$ is injective suppose $Tu = Tv$. Then $b_1 = b_2$
and hence $u = L(b_1)v_0 = L(b_2)v_0 = v$. Finally, let $s \in S, v \in V$,
$L(f)v_0 = v, f = a + b, a \in A, b \in B, \rho_1 = \rho|B$, and compute that

$$TL(s)v = TL(s)L(f)v_0$$

$$= TL(sf)v_0$$

$$= TL(sa + sb)v_0$$

$$= TL(sb)v_0$$

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= sb

= \rho(s)(b)

= \rho_1(s)(b)

= \rho_1(s)Tv \quad [i.e., \ L(b)v_0 = v].

In other words, T is a linear bijection linking L and \rho_1.

(b) We know that over some appropriate finite algebraic extension field \( F = R(\theta) \) where \( \theta \) is algebraic over \( R \), the regular representation has a reduction into absolutely irreducible components, and these components are uniquely determined to within order and equivalence (Theorem 1.17, Section 6.1). Let L be an absolutely irreducible representation of S over some extension field \( D \) of \( R \). Now regard both \( \rho \) and L over the extension field \( K = D(\theta) \) of \( R \). Then by (a) we know that L is equivalent over K to one of the absolutely irreducible components of \( \rho \). These components of \( \rho \) are uniquely determined to within order and equivalence, as we have just noted, and are over \( F \).

(c) Suppose that \( L_i \) occurs \( d_i \) times among the absolutely irreducible components of \( \rho \) over \( F \), \( i = 1, \ldots, r \). Then clearly

\[
\chi_\rho = \sum_{i=1}^{r} d_i \chi_i
\]

where \( \chi_i = \chi_{L_i} \), \( i = 1, \ldots, r \). Thus by (94) and (95)
\[
\langle \chi_p, \chi_k \rangle = \sum_{i=1}^{r} d_i \langle \chi_i, \chi_k \rangle
\]
\[
= d_k.
\]

But
\[
\langle \chi_p, \chi_k \rangle = \frac{1}{\hbar} \sum_{s \in S} \chi_p(s) \chi_k(s^{-1}).
\]

We assert that
\[
\chi_p(s) = \begin{cases} 
\hbar, & s = e \\
0, & s \neq e
\end{cases}
\]

For, we know that the group algebra \( \mathbb{G}(S) \) has dimension \( \hbar \), i.e., \( S \) is a basis, and since \( \rho(e) = I_{\mathbb{G}(S)} \) it follows that \( \chi_p(e) = \hbar \). On the other hand, let \( E = \{s_1, \ldots, s_N\} \) be a basis for \( \mathbb{G}(S) \) consisting of the elements of \( S \) in some fixed order. Then \( \rho(s_k) s_j = s_j \) means simply that \( s_k s_j = s_j \) i.e., \( s_k = e \). Hence \( [\rho(s_k)]_E \) has zeros along the main diagonal if \( s_k \neq e \). It follows that \( \chi_p(s) = 0 \) if \( s \neq e \). Returning to (109), we have
\[
d_k = \langle \chi_p, \chi_k \rangle
\]
\[
= \frac{1}{\hbar} \chi_p(e) \chi_k(e)
\]
\[
= \frac{\hbar}{\hbar} \chi_k(e)
\]
\[
= \chi_k(e).
\]

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The degree of $L_k$ is $n_k$ so that $\chi_k(e)$ is the trace of the identity transformation on a space of dimension $n_k$. Hence $d_k = n_k$, $k = 1, \ldots, r$. Thus we have

$$\rho \sim \sum_{i=1}^{r} n_i L_i$$

and $L_i$ has degree $n_i$, $i = 1, \ldots, r$. We compute that

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{i,j=1}^{r} n_i n_j \langle \chi_i, \chi_j \rangle$$

$$= \sum_{i=1}^{r} n_i^2 \langle \chi_i, \chi_i \rangle \quad \text{[by (94)]}$$

$$= \sum_{i=1}^{r} n_i^2 \quad \text{[by (95)]}.$$  

But we just saw that $\chi_\rho(e) = h$ and otherwise $\chi_\rho(s)$ is 0 so that

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{h} \chi_\rho(e) \chi_\rho(e)$$

$$= \frac{h^2}{h}$$

$$= h.$$

(d) Over some extension field of $\mathbb{R}$ we know that $L$ is equivalent to a direct sum of absolutely irreducible components. According to (b), each of these components is equivalent to one
of the absolutely irreducible components \( L_1, \ldots, L_r \) of \( \rho \).

Let \( c_i \) be the number of components of \( L \) equivalent to \( L_i \), \( i = 1, \ldots, r \). Then clearly

\[
\chi_L = \sum_{i=1}^{r} c_i \chi_i
\]

and thus

\[
(\chi_L, \chi_k) = \sum_{i=1}^{r} c_i (\chi_i, \chi_k)
\]

\[
= c_k \quad \text{[by (94) and (95)].}
\]

(e) The group algebra has dimension \( h \). Also by Theorem 1.11 in Section 6.1 the entry functions \( \Delta^t_{ij}, i, j = 1, \ldots, n_t \), \( t = 1, \ldots, r \) are l.i. in \( \mathcal{U}(S) \) and hence the dimension of the set on the left in (107) is precisely \( n_1^2 + \cdots + n_r^2 \). But (106) then tells us that \( n_1^2 + \cdots + n_r^2 = h \) and that these entry functions comprise a basis of \( \mathcal{U}(S) \). We remark that whether \( \mathcal{U}(S) \) is regarded as a vector space over \( R \) or \( F \) its dimension is still \( h \), since \( S \) is always a basis for \( \mathcal{U}(S) \).

Let \( S \) be a group. If two elements \( x \) and \( y \) in \( S \) are related by \( zxz^{-1} = y \) for some \( z \in S \), then \( x \) and \( y \) are said to be conjugate elements. It is easy to see that conjugacy is an equivalence relation that separates \( S \) into equivalence classes called conjugacy classes. If \( X \subset S \), then the normalizer of \( X \) is the set
\[ N_X = \{ s \in S \mid Xs = sX \} \].

If \( X = \{ x \} \), we write \( N_X = N_x \).

We summarize some elementary properties of these items in the following result. The proof is left as Exercise 6.

**Theorem 2.22** (a) If \( C_e \) is the conjugacy class to which \( e \) belongs, then \( C_e = \{ e \} \).

(b) If \( C \) is a conjugacy class, then \( C^{-1} = \{ s^{-1} \mid s \in C \} \) is a conjugacy class.

(c) If \( X \subseteq S \), then \( N_X \subseteq S \), i.e., \( N_X \) is a subgroup of \( S \).

(d) If \( x \in S \), then the number of distinct conjugates of \( x \), i.e., the number of elements in the conjugacy class to which \( x \) belongs, is the index of \( N_x \) in \( S \).

(e) If \( C \) is a conjugacy class in \( S \), then \( |C| \mid h \).

(f) If \( \chi \) is a character of a representation of \( S \), then \( \chi \) is constant on each conjugacy class in \( S \).

The following result is of fundamental importance because it shows precisely the role played by the characters in the group algebra of a finite group.

**Theorem 2.23** Let \( C_1, \ldots, C_p \) be the conjugacy classes in \( S \) and define \( C = \{ f \in \mathbb{C}(S) \mid f(z^{-1}sz) = f(s) \text{ for all } s \text{ and } z \text{ in } S \} \). Then

(a) \( C \) is a subspace of \( \mathbb{C}(S) \) and \( \dim C = p \).

(b) If \( \chi^1, \ldots, \chi^r \) are the characters of the representations...
Let \( L_1, \ldots, L_r \) in Theorem 2.21(c), then \( r = p \) and \( \chi^1, \ldots, \chi^r \) form a basis of \( C \). Thus the number of pairwise inequivalent absolutely irreducible representations of \( S \) is the number of conjugacy classes in \( S \).

(c) \( f \in C \) iff \( fg = gf \) for all \( g \in \mathbb{G}(S) \), i.e., \( C \) is the center of \( \mathbb{G}(S) \).

**Proof:** (a) That \( C \) is a subspace of \( \mathbb{G} \) is obvious. Let \( \gamma_t = \sum_{s \in C_t} s \), i.e., \( \gamma_t \in C \) takes the value 1 on each \( s \in C_t \) and 0 otherwise. Obviously the disjointness of the \( C_t \) imply that the \( \gamma_t \) are l.i. in \( \mathbb{G}(S) \). Moreover if \( f \in C \), then \( f \) is constant on each \( C_t \), say \( f = \sum_{t=1}^{p} a_t \sum_{s \in C_t} s \). But this is simply another way of writing

\[
 f = \sum_{t=1}^{p} a_t \gamma_t .
\]

Hence \( \dim C = p \).

(b) It is clear from properties of the trace that \( \chi^t \in C \), \( t = 1, \ldots, r \). Moreover (94) and (95) immediately imply that \( \chi^1, \ldots, \chi^r \) are l.i. in \( \mathbb{G}(S) \). Hence by (a), \( \dim C = p \geq r \).

We show now that \( C \) is spanned by \( \chi^1, \ldots, \chi^r \). By Theorem 1.11 in Section 6.1 any element of \( \mathbb{G}(S) \) is a unique linear combination of the entry functions \( \Delta_{ij}^t \), \( i,j = 1, \ldots, n_t \), \( t = 1,\ldots,r \), where \( \Delta_{ij}^t \) is the matrix representation over \( F \) corresponding to \( L_t \), \( t = 1, \ldots, r \). Now suppose that \( f \in C \). Then
\[ f = \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ij}^{t} \Delta_{ij}^{t} \]

and we compute

\[
f(s) = \frac{1}{h} \sum_{y \in \mathcal{S}} f(y^{-1} sy) \quad \text{[i.e., } f \in \mathcal{C} \text{ means that } f(y^{-1} sy) = f(s)]
\]

\[
= \frac{1}{h} \sum_{y \in \mathcal{S}} \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ij}^{t} \Delta_{ij}^{t} (y^{-1} sy)
\]

\[
= \frac{1}{h} \sum_{y \in \mathcal{S}} \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ij}^{t} \sum_{\alpha,\beta=1}^{n_t} \Delta_{i\alpha}^{t} (y^{-1}) \Delta_{\alpha\beta}^{t} (s) \Delta_{\beta j}^{t} (y)
\]

\[
= \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ij}^{t} \Delta_{i\alpha}^{t} (s) \left( \frac{1}{h} \sum_{y \in \mathcal{S}} \Delta_{\alpha j}^{t} (y) \Delta_{i\alpha}^{t} (y^{-1}) \right)
\]

\[
= \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ij}^{t} \delta_{\alpha j}^{t} (s) \delta_{\alpha i}^{t} (s) \quad \text{(by Theorem 1.10, Section 6.1)}
\]

\[
= \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} c_{ii}^{t} \Delta_{i\alpha}^{t} (s) \frac{1}{n_t} \delta_{\alpha \beta} \delta_{ij}
\]

\[
= \sum_{t=1}^{r} \sum_{i,j=1}^{n_t} \frac{1}{n_t} c_{ii}^{t} \Delta_{i\alpha}^{t} (s) \sum_{\alpha=1}^{n_t}
\]

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\[
\sum_{t=1}^{r} \left( \frac{1}{n_t} \sum_{i=1}^{c} t_i \right) \chi^t(s).
\]

Thus \( f \) is a linear combination of \( \chi^1, \ldots, \chi^r \). In other words, \( C \) is spanned by \( \chi^1, \ldots, \chi^r \) and we also know that \( \dim C = p \).

Hence \( p = r \).

(c) Suppose \( fx = xf \) for all \( x \in S \). Then writing
\[
f = \sum_{s \in S} a_s s
\]
we have
\[
f = xfx^{-1}
\]
\[
= \sum_{s \in S} a_s xsx^{-1}
\]
\[
= \sum_{t \in S} a_{x^{-1}tx} t
\]
\[
= \sum_{s \in S} a_{x^{-1}sx} s
\]
\[
= \sum_{s \in S} a_s s.
\]

Thus \( a_{x^{-1}sx} = a_s \) for any \( x \) and \( s \) in \( S \) so that \( f \in C \).

Conversely, if \( f \in C \), then the same argument in reverse shows that \( f \) commutes with everything in \( \Psi(S) \).

The second set of orthogonality relations for the characters
\[ \chi^1, \ldots, \chi^r \text{ [see (94) and (95)] can now be derived.} \]

**Theorem 2.24** In the notation of Theorem 2.23, let \( |C_t| = h_t \),
\( t = 1, \ldots, p \), and let \( \chi^k_t \) be the (constant) value of \( \chi^k \) on
\( C_t \), \( t, k = 1, \ldots, p \). Let \( \chi^k_t \) be the value of \( \chi^k \) on \( C_t^{-1} \)
[which by Theorem 2.22(b) is also a conjugacy class]. Then

\[ \sum_{t=1}^{p} \frac{h_t}{h} \chi^k_t \chi^L_t = \delta_{kL} \quad \text{(110)} \]

and

\[ \sum_{t=1}^{p} \frac{h_t}{h} \chi^k_t \chi^s_t = \delta_{ts} \quad \text{(111)} \]

**Proof:** Suppose we have established (110). If \( A \) is the
\( p \)-square matrix whose \((k,t)\) entry is \( \frac{h_t}{h} \chi^k_t \) and \( B \) is the \( p \)-
square matrix whose \((t,s)\) entry is \( \frac{1}{h} \chi^s_t \), then (110) is the
assertion \( AB = I_p \). But this implies \( BA = I_p \) or

\[ \delta_{ts} = (BA)_{st} \]

\[ = \sum_{k=1}^{p} B_{sk} A_{kt} \]

\[ = \sum_{k=1}^{p} \frac{k}{h} \frac{h_t}{h} \chi^k_t \chi^s_t \]

\[ = \sum_{t=1}^{p} \frac{h_t}{h} \chi^k_t \chi^s_t \]

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precisely (111). Thus our only task is to prove (110). Recall (94) and (95), the orthogonality relations of the first kind, which state that

\[(\chi^k, \chi^l) = \delta_{k,l},\]

or

\[
\frac{1}{h} \sum_{x \in S} \chi^k(x) \overline{\chi^l(x^{-1})} = \delta_{k,l}.
\]

The group \( S \) is the disjoint union of the conjugacy classes \( C_1, \ldots, C_p \) and each \( \chi^k \) is constant on each \( C_t \) with value \( \chi^k_t \). The value \( \chi^l_t \) on \( C_t^{-1} \) is \( \overline{\chi^l_t} \). Thus

\[
\delta_{k,l} = \frac{1}{h} \sum_{t=1}^{p} \chi^k_t(\sum_{x \in C_t} \chi^l(x) \overline{\chi^l(x^{-1})})
\]

\[
= \frac{1}{h} \sum_{t=1}^{p} h_t \chi^k_t \overline{\chi^l_t},
\]

precisely (110).

The formulas (110) and (111) are called the character orthogonality relationships of the second kind. To diminish the number of words we shall refer to the characters \( \chi^1, \ldots, \chi^p \) (\( p=r \)) in Theorem 2.23 as a complete list of characters for the group \( S \). The orthogonality relationships (110) and (111) permit us to prove a result of considerable value in computing the degrees of the absolutely irreducible representations of \( S \).
Theorem 2.25  Let $\eta_1, \ldots, \eta_p$ be the degrees of the complete list of representations (or characters) of $S$. Then

$$\eta_k \mid h,$$

$k = 1, \ldots, p$.

Proof: As in the proof of Theorem 2.23(a), let $\gamma_t = \sum_{s \in C_t} s$, and as we saw, $\gamma_1, \ldots, \gamma_p$ is a basis of the center $C$ of the group algebra $\mathbb{F}(S)$. Let $L_1, \ldots, L_p$ be the complete list of absolutely irreducible pairwise inequivalent representations of $S$ over $F$. Thinking of the $L_k$ extended to $\mathbb{F}(S)$, it is easy to check that $L_k(\gamma_t)$ commutes with every $L_k(s), s \in S$, because $\gamma_t \in C$. Hence by Theorem 1.9(b), Section 6.1

$$L_k(\gamma_t) = \eta_t^k I$$

where $I$ is the identity transformation on some appropriate space of dimension $n_k$ over $F$. Now in the notation of Theorem 2.24 we have

$$\eta_t^k n_k = \operatorname{tr}(\eta_t^k I)$$

$$= \gamma_k(\gamma_t)$$

$$= \gamma_k\left(\sum_{s \in C_t} s\right)$$

$$= h_t \chi_t^k,$$

so that
\[ n_k \eta_t^k = h_t \chi_t^k, \quad k, t = 1, \ldots, p. \quad (112) \]

Since \( n_k \neq 0 \) we can divide in (112) to obtain

\[ \eta_t^k = \frac{h_t}{n_k} \chi_t^k. \quad (113) \]

Observe that if \( t = 1 \) (we choose the notation so that \( c_1 = [e] \)) then

\[ \eta_1^k = \frac{h_1}{n_k} \chi_1^k. \quad \text{But} \quad \chi_1^k = \chi_k(e) = n_k \quad \text{and hence in fact} \]

\[ \eta_1^k = 1. \quad (114) \]

Since \( \gamma_s \gamma_t \in \mathbb{C} \) for any \( s, t = 1, \ldots, p \), we can write

\[ \gamma_s \gamma_t = \sum_{j=1}^{p} c_{tj}^s \gamma_j, \quad s, t = 1, \ldots, p. \quad (115) \]

Evaluating \( L_k \) at both sides of (115) we have

\[ \eta_s^k \eta_t^k = \sum_{j=1}^{p} c_{tj}^s \eta_j^k, \quad s, t = 1, \ldots, p. \quad (116) \]

Let \( \Gamma^s = [c_{tj}^s], \quad t, j = 1, \ldots, p \), so that (116) then becomes

\[ \Gamma^s \eta^k = \eta_s^k \eta^k \quad (117) \]

where \( \eta^k = (\eta_1^k, \ldots, \eta_p^k) \). Notice from (114) that \( \eta^k \) is not the zero \( p \)-tuple. Also notice that the entries of \( \Gamma^s \), i.e., the numbers \( c_{tj}^s \), are nonnegative integers. This follows from the fact that if \( x \in \mathbb{C} \) then

\[ (\gamma_s \gamma_t)(x) = \sum_{j=1}^{p} c_{tj}^s \gamma_j(x) = c_{t0}^s. \]

But \( (\gamma_s \gamma_t)(x) \) is a convolution of values of \( \gamma_s \) and \( \gamma_t \) and
these are 0's and 1's. Hence $\Gamma^s$ has nonnegative integer
entries (i.e., integral multiples of the multiplicative identity).
Next, apply $\Gamma^k$ to both sides of (117) to obtain
\[
\Gamma^T \Gamma \eta^k = \eta^k \Gamma \eta^k
\]
\[
= (\eta_s \eta^k \eta_t \eta^k)^k .
\]  (118)

Let $\phi \in S_p$ be chosen so that $C_{\phi(t)}$ consists of the inverses
of the elements of $C_t$, $t = 1, \ldots, p$. Set $s = \phi(t)$ in (118),
multiply both sides by $\frac{h_t}{h_t}$ and sum on $t = 1, \ldots, p$, to obtain
\[
\left( \sum_{t=1}^{p} \frac{h_t}{h_t} \Gamma^T \phi(t) \eta_t \right)^k = \left( \sum_{t=1}^{p} \frac{h_t}{h_t} \eta_s \eta^k \eta_t \eta^k \right)^k .
\]  (119)

We compute that
\[
\sum_{t=1}^{p} \frac{h_t}{h_t} \eta_s \phi(t) \eta_t \eta^k = \sum_{t=1}^{p} \frac{h_t}{h_t} \frac{\phi(t)}{n_k} \eta_t \eta^k \chi^{\phi(t)} \chi_t^k \chi_t^k
\]
\[
= \sum_{t=1}^{p} \frac{h_t}{h_t} \frac{2}{n_k} \chi_t^k \eta_t \eta^k \chi_t^k
\]
\[(i.e., h_{\phi(t)} = h_t \text{ and } \chi_{\phi(t)} = \chi_t^k)\]
\[
= \frac{h_t}{n_k} \sum_{t=1}^{p} \frac{1}{2} \chi_t^k \eta_t \eta^k
\]
\[
= \frac{h_t}{n_k} \cdot h
\]
\[
= \left( \frac{n_t}{n_k} \right)^2 .
\]
Thus from (119) we have
\[
\left( \sum_{t=1}^{p} h_t \Gamma^t \varphi(t) \right) \eta^k = \left( \frac{h}{n_k} \right)^2 \eta^k, \quad \eta^k \neq 0.
\] (120)

But \( h_t \mid h \), so the matrix on the left in (120) has nonnegative integer entries. Moreover \( \left( \frac{h}{n_k} \right)^2 \) is an eigenvalue of this matrix. But \( \left( \frac{h}{n_k} \right)^2 \) must then satisfy a monic polynomial with integral coefficients and hence must be an integer itself (see Exercise 7). It follows that \( \frac{h}{n_k} \) is an integer. \( \square \)

**Example 2.3** As an example of the use of Theorem 2.25, we prove the following result: If \( |S| = h \) and \( h = m^2 \) where \( m \) is a prime then \( S \) is abelian, e.g., any group of order 49 is abelian. For, let \( n_1, \ldots, n_p \) be the complete list of degrees of the irreducible characters of \( S \) and take the field to be \( \mathbb{F} \). Now \( n_k \mid m^2 \) by Theorem 2.25 so \( n_k \mid m \). Hence \( n_k = m \) or \( n_k = 1 \), \( k = 1, \ldots, p \). Suppose \( n_k = m \) for some \( k \). Then from (106) and Theorem 2.23(b),
\[
n_1^2 + \cdots + n_{k-1}^2 + m^2 + n_{k+1}^2 + \cdots + n_p^2 = m^2,
\]
and hence \( n_1^2 + \cdots + n_{k-1}^2 + n_{k+1}^2 + \cdots + n_p^2 = 0 \). But the principal representation has degree 1, and this last equality is therefore impossible. Hence \( n_k = 1, k = 1, \ldots, p \), i.e., the absolutely irreducible components of the regular representation \( \rho \) are all of degree 1. It follows that \( \text{Im} \rho \) is commutative. But \( \rho \) is faithful so \( S \) is commutative.

**Exercises**

1. Prove the assertions concerning that mapping \( \nu : \mathbb{U}(S) \rightarrow \mathbb{U}(S) \)
defined in (7).

Hint: Clearly \( \nu \) is a bijection on the basis \( S \) of \( \mathbb{U}(S) \), i.e., \( \nu(s) = s^{-1} \), \( s \in S \). Also \( \nu \) is linear from (7).

To confirm (8) we need only check it for \( f \) and \( g \) in \( S \).

But then \( \nu(fg) = (fg)^{-1} = g^{-1}f^{-1} = \nu(g)\nu(f) \). Let \( A \subset \mathbb{U}(S) \) be a left ideal and let \( h \in \mathbb{U}(S) \). Since \( \nu \) is a bijection \( h = \nu(f) \) so that if \( \nu(g) \in \nu(A) \), \( g \in A \), then \( \nu(g)h = \nu(g)\nu(f) = \nu(fg) \) and \( fg \in A \). Thus \( \nu(fg) \in \nu(A) \). In other words, \( \nu(A) \) is a right ideal. Similarly if \( A \) is a right ideal \( \nu(A) \) is a left ideal.

2. Prove that a complete matrix algebra \( M_n(R) \) is simple. That is, \( M_n(R) \) contains no proper two-sided ideal.

Hint: Suppose \( \mathcal{J} \) is a two-sided ideal in \( M_n(R) \) and \( 0 \neq A \in \mathcal{J} \). Then obtain \( P \) and \( Q \) such that \( PAQ = E_{ij} \), the matrix with 1 in position \( i,j \), 0 elsewhere (how?). But then a basis of \( M_n(R) \) is contained in \( \mathcal{J} \) so \( \mathcal{J} = M_n(R) \).

3. Let \( C \in M_n(R) \). Prove that if \( (XC)^2 = 0 \) for all \( X \in M_n(R) \), then \( C = 0 \).

Hint: First let \( X = E_{ii} \) so that \( (XC)^2 = 0 \) implies that \( c_{ii} = 0 \). Then replacing \( X \) by \( E_{11}P \) where \( P \) is a permutation matrix that interchanges row 1 with row \( k \) implies that \( c^{(1)} = 0 \). Then setting \( X = QY \) where \( Q \) is a permutation matrix and \( Y \) is arbitrary we have \( 0 = (QYC)^2 = QYCQYC \) so that \( Y(CQ)Y(CQ) = 0 \) or \( (Y(CQ))^2 = 0 \) for all \( Y \). By what was just proved \( (CQ)^{(1)} = 0 \) and hence any column of \( C \) is 0.
4. Prove (68) and (69).

Hint: Proof of (68): By definition, \( w \in \mathcal{W}_W \) means that
\[
\bigoplus_{1}^{m} w = P(f) v \quad f \in \mathcal{W}, \quad v \in \bigotimes_{1}^{m} V.
\]
But \( f \in \mathcal{W}_W \) implies \( P(f) \bigotimes_{1}^{m} V \subset \mathcal{W} \) [see (65)] so \( w \in \mathcal{W} \). Thus \( \mathcal{W}_W \subset \mathcal{W} \).

Conversely suppose \( w \in \mathcal{W} \) and \( \mathcal{W} \) is an invariant subspace of \( \mathcal{W}_W \). By definition, \( \mathcal{W}_W = \{ f \in \mathcal{O} \mid P(f) \bigotimes_{1}^{m} V \subset \mathcal{W} \} \). We first assert that \( \mathcal{W}_W \neq \{0\} \). For, \( P: \mathcal{W}(S_m) \rightarrow P_m \) is surjective (by the definition of \( P_m \)) and \( P_m = C(\mathcal{O}_m) \) [see (99) in Section 6.1]. In other words, any linear transformation on \( \bigotimes_{1}^{m} V \) that commutes with all \( \mathcal{W}(T) \) is a value of \( P \). Thus if we let \( P_1 \) be a projection along the invariant subspace \( \mathcal{W} \) parallel to some \( \mathcal{W}_1 \) (another invariant subspace by the full reducibility of \( P \)), we can find an \( f \in \mathcal{W}(S_m) \) such that \( P(f) = P_1 \). Moreover, we can take \( f \in \mathcal{O} \) since \( \mathcal{W}(S_m) = \mathcal{O} + \ker P \) [see (63)]. Thus \( 0 \neq f \in \mathcal{W}_W \). Also, \( P(f)w = P_1 w = w \) so that \( w \in P(\mathcal{W}_W) \bigotimes_{1}^{m} V = \mathcal{W}_{\mathcal{W}_W} \).

Proof of (69): Let \( \mathcal{J} \) be a right ideal contained in \( \mathcal{O} \). If \( f \in \mathcal{J} \), then obviously \( P(f) \bigotimes_{1}^{m} V \subset P(\mathcal{J}) \bigotimes_{1}^{m} V = \mathcal{W}_J \) so that \( f \in \mathcal{W}_J \). Thus \( \mathcal{J} \subset \mathcal{W}_J \). Conversely, if \( f \in \mathcal{W}_J \), then \( f \in \mathcal{J} \) and \( P(f) \bigotimes_{1}^{m} V \subset \mathcal{W}_J = P(\mathcal{J}) \bigotimes_{1}^{m} V \) so that for any \( v \in \bigotimes_{1}^{m} V \), \( P(f)v = P(g)u \), for some appropriate \( g \in \mathcal{J} \), \( u \in \bigotimes_{1}^{m} V \). Now let \( \epsilon \) be a generating idempotent for \( \mathcal{J} \), i.e., \( \mathcal{J} = \epsilon \mathcal{W}(S_m) \).

Then \( P(f)v = P(g)u = P(\epsilon f')u \), \( f' \in \mathcal{W}(S_m) \), so that \( P(\epsilon)P(f)v = P(\epsilon^2 f')u = P(\epsilon f')u = P(f)v \), or \( P(\epsilon f)v = P(f)v \).
for all $v \in \Theta V$. Hence $P(\sigma f - f) = 0$. Since both $\sigma f$ and $f$ are in $\Theta$ and $P$ is a bijection on $\Theta$, it follows that $f = \sigma f \in \Theta$ and hence $\Theta \cup \Theta = \Theta$.

5. Prove Theorem 2.13(a) - (g).

6. Prove Theorem 2.22.

Hint: (a) trivial; (b) clearly $zx^{-1}y = y$ iff $zx^{-1}z^{-1} = y^{-1}$;
(c) $sX = Xs$ and $tX = Xt$ imply $X^{-1} = s^{-1}X$ and $(st)X = s(tX) = (st)X = (sX)t = (Xs)t = X(st)$ so $N_X$ is a group;
(d) let $u_1 N_X, \ldots, u_k N_X$ be all the left cosets of $N_X$ in $S$.

The $k$ conjugates $u_j x u_j^{-1}$ are distinct. For, $u_j x u_j^{-1} = u_j x u_j^{-1}$ implies $u_j u_i \in N_X$, i.e., $u_i \in u_j N_X$. So the number of distinct conjugates of $x$ is at least $k$. On the other hand, if $z^{-1}x$ is any conjugate of $x$, then $z^{-1}$ must lie in some left coset, say $z^{-1} = u y^{-1}$, $y^{-1} \in N_X$. Hence $z^{-1}xz = u y^{-1}xyu^{-1}_y = u x u^{-1}_y$. Thus there are at most $k$ conjugates of $x$;
(e) Let $C$ be the conjugacy class of $x$, i.e., $C$ consists of all the distinct conjugates of $x$. According to part (d), $|C|$ is the index of $N_X$ in $S$ and hence must divide $h$.

7. Suppose $p$ and $q$ are relatively prime integers and $1$ is the multiplicative identity in $R$. Show that if $\frac{p}{q}$ is a root of the monic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ in which the $a_i$ are integral multiples of $1$, then $\frac{p}{q}$ is an integer.

Hint: $\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \cdots + a_1\frac{p}{q} + a_0 = 0$ so that
\[ p^n + q(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_0q^{n-1}) = 0. \]

Then \( p^n + qm = 0 \), where \( m \) is an integer. It follows that \( q \mid p^n \) and hence since \( q \) and \( p \) have no common factors that \( q = \pm 1 \).

8. The prime subfield \( \pi \) of a field \( R \) is the smallest subfield of \( R \) containing 0 and 1. Prove that if \( S \) is a finite group, \( |S| = h \), then any absolutely irreducible component of a representation \( \Delta \) is equivalent to an absolutely irreducible representation over a simple algebraic extension of \( \pi \).

Hint: First reduce \( \Delta \) into irreducible components over \( R \), so we may as well start by assuming \( \Delta \) itself is irreducible over \( R \). By Theorem 2.21(a), \( \Delta \) is equivalent over \( R \) to a component \( \rho_1 \) of the regular representation \( \rho \). The entries in the matrices in \( \text{Im} \rho \) are 0 and 1 when \( S \) itself is used as a basis of the group algebra \( \mathbb{F}(S) \). Thus \( \rho_1 \) is over \( \pi \), i.e., the proof of Maschke's theorem (Theorem 1.6, Section 6.1) shows that the components are obtained using an equivalence over the field in which the entries lie. Thus \( \Delta \sim \rho_1 \) and \( \rho_1 \) is over \( \pi \). Now apply Theorem 1.7, Section 6.1 to conclude that the absolutely irreducible components of \( \rho_1 \) can be chosen to be over a finite algebraic extension of \( \pi \).

9. If \( L : S \to L(V,V) \), \( V \) is a unitary (or Euclidean) space and \( \text{Im} L \) consists of unitary transformations, then \( V \) is called a unitary (or Euclidean) representation module for \( S \). If
Δ: S → Mₙ(C) is a matrix representation in which Im Δ consists of unitary (or orthogonal) matrices, then Δ is called a unitary (or orthogonal) matrix representation. Prove:

Any set M of unitary matrices is fully reducible. Thus if S is a groupoid, and V is a unitary representation module for S, then V is fully reducible.

Hint: Let E be an o.n. basis of the unitary space V and let Ω be the set of all unitary transformations T defined by [T]₉ = U for U ∈ M. Clearly if M is reducible, there exists a subspace W that is invariant under all T ∈ Ω.

Choose an o.n. basis vₕ₊₁, ..., vₙ of W and augment it to an o.n. basis F = {w₁, ..., wₚ, vₚ₊₁, ..., vₙ} of V. Then

$$[T]_F^F = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix}.$$  

But

$$[T]_F^F = [I_V T I_V]_F^F = [I_V]_F^F [T]_E^E [I_V]_F^E.$$  

It is easy to see that $F = [I_V]_F^E$ is unitary. Thus $[T]_F^F$ is unitary so that

$$\begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} U_{11}^* & U_{21}^* \\ 0 & U_{22}^* \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$  

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Hence $U_{11}U_{21}^* = 0$ and since $U_{11}$ is nonsingular, $U_{21} = 0$.

It follows that

$$p^{-1}_{\text{AP}}$$

is a direct sum (i.e., $U_{21} = 0$) for each $A \in M$.

10. Let $S = \{g_1, g_2, \ldots, g_n\}$ be a finite group and let $\rho: S \rightarrow \mathcal{L}(\mathfrak{U}(S), \mathfrak{U}(S))$ be the regular representation, where $\mathfrak{U}(S)$ is the group algebra over either $\mathbb{R}$ or $\mathbb{C}$. Prove, using Exercise 9, that $\rho$ is a fully reducible representation.

Hint: Let $E = \{e_1, \ldots, e_n\}$ be the basis of $\mathfrak{U}(S)$ and define an inner product $\beta$ on $\mathfrak{U}(S)$ by

$$\beta\left(\sum_{k=1}^{n} a_k e_k, \sum_{k=1}^{n} b_k e_k\right) = \sum_{k=1}^{n} a_k \overline{b_k}.$$ 

Then $E$ is clearly an o.n. basis with respect to $\beta$. Let $\sigma_r \in S_n$ be defined by $g_r g_k = g_{\sigma_r(k)}$, $k = 1, \ldots, n$. Then

$$\beta\left(\rho(g_r) \sum_{k=1}^{n} a_k e_k, \rho(g_r) \sum_{k=1}^{n} b_k e_k\right) = \beta\left(\sum_{k=1}^{n} a_k g_{\sigma_r(k)} e_k, \sum_{k=1}^{n} b_k g_{\sigma_r(k)} e_k\right)$$

$$= \beta\left(\sum_{k=1}^{n} a_k g_{\sigma_r(k)}(k), \sum_{k=1}^{n} b_k g_{\sigma_r(k)}(k)\right)$$

$$= \sum_{k=1}^{n} a_k \overline{b_k}.$$ 

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\[ = \beta \left( \sum_{k=1}^{n} a_k g_k, \sum_{k=1}^{n} b_k g_k \right). \]

Thus \( \rho(g_r) \) is unitary, \( r = 1, \ldots, n \). It follows from Exercise 9 that \( \rho \) is fully reducible.

11. Let \( S \) be a finite group and \( L : S \to L(V,V) \) be a proper representation in which the underlying field is either \( \mathbb{R} \) or \( \mathbb{C} \).

Let \( \beta \) be an inner product on \( V \) (we are not assuming that \( V \) is a unitary \( S \)-module, i.e., that the transformations in the representation are unitary with respect to \( \beta \)). Show that an inner product can be defined on \( V \) so that \( \text{Im} \ L \) consists of unitary transformations.

Hint: Let \( T \) be any positive definite hermitian transformation and define

\[ H = \sum_{s \in S} L(s)^* TL(s). \]

Then if \( t \in S \), we compute

\[
L(t)^* HL(t) = \sum_{s \in S} L(t)^* L(s)^* TL(s)L(t)
\]

\[ = \sum_{s \in S} (L(s)L(t))^* TL(s)L(t) \]
\[ \sum_{s \in S} L(st) \ast_{T} L(st) \]

= \[ H. \]

Thus for each \( s \in S \), \( L(s) \ast_{T} H L(s) = H \) and \( H \) is positive definite hermitian (i.e., it is a sum of positive definite hermitian transformations). Now define a new inner product in \( V \) by

\[ (x,y) = \beta(Hx,y). \]

Then for any \( s \in S \)

\[ (L(s)x,L(s)y) = \beta(HL(s)x,L(s)y) \]

\[ = \beta(L(s) \ast_{T} HL(s)x,y) \]

\[ = \beta(Hx,y) \]

\[ = (x,y). \]

In other words, each \( L(s) \) is unitary with respect to \( (\cdot, \cdot) \).

12. Let \( \Delta : S \rightarrow GL(n,C) \), \( S \) a finite group. Describe a procedure for constructing a matrix \( P \) such that for each \( s \in S \), \( P^{-1} \Delta(s)P \) is a unitary matrix.

Hint: Let

\[ H = \sum_{s \in S} \Delta(s) \ast \Delta(s) \]

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and define $P = H^{-\frac{1}{2}}$, i.e., $P$ is the inverse of the positive definite hermitian square root of $H$. Then

$$(P^{-1}\Delta(s)P)^\ast (P^{-1}\Delta(s)P) = P^\ast \Delta(s)P \cdot P^{-1} \cdot P^{-1}\Delta(s)P$$

$$= H^{-\frac{1}{2}}\Delta(s)^\ast H\Delta(s)H^{-\frac{1}{2}}$$

$$= H^{-\frac{1}{2}}HH^{-\frac{1}{2}} \quad (i.e., \Delta(s)^\ast H\Delta(s) = H)$$

$$= I_n.$$ 

Hence $P^{-1}\Delta(s)P$ is a unitary matrix for each $s \in S$.

13. Let $S$ be the group of $2 \times 2$ matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$ 

(a) Show that the matrices in $S$ form a faithful irreducible representation of $S_3$.

(b) Find a matrix $P$ such that $P^{-1}\Delta(s)P$ is a unitary matrix for each matrix $\Delta(s)$ in $S$.

Hint: (a) To see that the matrices form a faithful representation of $S_3$ construct the $6 \times 6$ table for each group and compare the two. Suppose $S$ is reducible and $Q^{-1}\Delta(s)Q$ is lower triangular for each $\Delta(s)$. Then it would follow that $Q^{(2)}$ is a common eigenvector of each $\Delta(s)$. The eigenvectors of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $(1,1)$ and $(1,-1)$. However,
\[
\begin{bmatrix}
-1 & -1 \\
0 & 1
\end{bmatrix}
\]
(1,1) = (-2,1)

and

\[
\begin{bmatrix}
-1 & -1 \\
0 & 1
\end{bmatrix}
\]
(1,-1) = (0,-1).

(b) First form the sum \( H = \sum_{s \in S} \Delta(s)^{\otimes n} \Delta(s) \). Then find a \( 2 \times 2 \) unitary matrix \( U \) such that \( U^{-1}HU = \text{diag}(\alpha_1, \alpha_2) \).

Next define \( P = U \text{diag}(\alpha_1^{-\frac{1}{n}}, \alpha_2^{-\frac{1}{n}})U^{-1} \).

6.3 The Symmetric Group

Before we begin a systematic study of the representations of the symmetric group it is instructive to examine representations and characters of several relatively simple groups, each of which is a subgroup of an appropriate \( S_m \).

At this point we have a substantial amount of information about representations of a finite group \( S \) of order \( h \). We assemble this information here and show how it can be used to compute general representations and characters. To fix the notation we have:

(a) \( |S| = h \);

(b) \( L_1, \ldots, L_p \) is a complete list of pairwise inequivalent absolutely irreducible representations of \( S \), \( \deg L_i = n_i \),
\[ i = 1, \ldots, p; \]

(c) \( C_1, \ldots, C_p \) are the conjugacy classes in \( S \),
\[ |C_t| = h_t, \quad \gamma_t = \sum_{x \in C_t} x, \quad t = 1, \ldots, p. \]

(d) \( \chi^1, \ldots, \chi^p \) is a complete list of characters, \( \chi^k_t \) is the value of \( \chi^k \) on \( C_t \), \( \chi^{-1}_t \) is the value of \( \chi^k \) on \( C_t^{-1} = \{ s^{-1} : s \in C_t \} \);

(e) \( L_k(\gamma_t) = \eta_t^k I \), where \( I \) is the identity on some space of dimension \( \eta_k \), \( k, t = 1, \ldots, p \), \( \eta^k = (\eta_1^k, \ldots, \eta_p^k) \), \( k = 1, \ldots, p \).

We have proved that:

(i) \( \eta_1^2 + \cdots + \eta_p^2 = h \quad \text{[Theorem 2.21(c), Section 6.2]} \)

(ii) \( \eta_k | h, \quad k = 1, \ldots, p \quad \text{[Theorem 2.25, Section 6.2]} \)

(iii) \( \gamma_1, \ldots, \gamma_p \) comprise a basis for the commutator algebra \( C \) of the group algebra \( \mathbb{H}(S) \) and

\[ \gamma_s \gamma_t = \sum_{j=1}^{p} c_{tj}^s \gamma_j, \quad s, t = 1, \ldots, p, \quad (1) \]

where the \( c_{tj}^s \) are nonnegative integers. The equations (1) are called the class constant equations. If \( \tau^s = [c_{tj}^s] \), \( t, j = 1, \ldots, p \), then

\[ \tau^s \eta^k = \eta_s^k \eta^k, \quad k, s = 1, \ldots, p. \quad (2) \]

Also,

\[ \eta_t^k = \frac{h_t}{n_k} \chi^k_t, \quad (3) \]
\[ \eta_1^k = 1, \quad k = 1, \ldots, p. \]

(see Theorem 2.25, Section 6.2).

(iv) From formula (116) in Theorem 2.25, Section 6.2 we have

[in view of (3)]

\[ h_1 h_2 \chi_s t \chi_t = \eta_k \sum_{j=1}^{p} c_{tj}^s h_j \chi_j. \quad (4) \]

Of course (4) is only a restatement of (2). In fact, the problem

of computing the characters \( \chi_k \), \( k = 1, \ldots, p \), reduces to

solving the eigenvalue-eigenvector problem posed in (2). However,

in order to obtain the matrices \( P^s \) we must compute the coeffi-

cients \( c_{tj}^s \) in the class constant equations (1) directly from the

group. In view of (2) and (3) then we need to compute: \( h_1, \ldots, h_p \);

\( \eta_1, \ldots, \eta_p \); \( c_{tj}^s \), \( s, t, j, = 1, \ldots, p \), from the group in order

to compute the \( \chi_k \).

There is a convenient tabular form for assembling the data

about the characters. It is called a character table and has the

following form:

\[
\begin{array}{c|cccccc}
\eta_1 & h_1 & h_2 & h_3 & h_p \\
\eta_2 & c_1 & c_2 & c_3 & \cdots & c_p \\
\eta_3 & \chi_1 & \chi_2 & \chi_3 & \cdots & \chi_p \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n_p & \chi_1^p & \chi_2^p & \chi_3^p & \cdots & \chi_p^p \\
\end{array}
\]  

(5)
Theorem 2.24, Section 6.2 tells us that we have the following orthogonality conditions on the rows and columns:

\[ \sum_{t=1}^{p} \sum_{k=1}^{h} \chi_t^k \chi_t^{k'} = \delta_{k,k'}, \quad (6) \]

and

\[ \sum_{k=1}^{p} \sum_{t=1}^{h} \chi_t^{k-k} \chi_t^{k} = \delta_{ts}. \quad (7) \]

Of course, if the underlying field is \( \mathbb{C} \) then in fact each \( \chi_t^k \) is the complex conjugate of \( \chi_t^k \). For, we know that \( L_{\chi} \) is equivalent to a unitary representation (Exercises 11, 12, Section 6.2) and \( \chi^k(s^{-1}) = \chi^k(s) \) for a unitary representation (why?).

**Example 3.1** Consider the group \( S \) of all symmetries of the square:

![Diagram of a square with labels and arrows indicating symmetries](image)

\( e = \text{identity}; \)

(1 2 3 4) = counterclockwise rotation about \( \mathcal{O} \) through \( \frac{\pi}{2} \);

(1 3)(2 4) = counterclockwise rotation about \( \mathcal{O} \) through \( \pi \);

(1 4 3 2) = counterclockwise rotation about \( \mathcal{O} \) through \( \frac{3\pi}{2} \);
(1 4)(2 3) = reflection through the x axis;
(1 2)(3 4) = reflection through the y axis;
(2 4) = reflection through the diagonal \(d_1\); .
(1 3) = reflection through the diagonal \(d_2\).

Thus \(|S| = 8\). Now \((2 4)(1 4)(2 3)(2 4) = (1 2)(3 4)\) so that
(1 4)(2 3) and (1 2)(3 4) are in the same class. By similar
calculations we compute that \(C_1 = \{e\}, \ C_2 = \{(1 3)(2 4)\}, \ C_3 =
\{(1 2 3 4), (1 4 3 2)\}, \ C_4 = \{(1 4)(2 3), (1 2)(3 4)\}, \ C_5 =
\{(1 3), (2 4)\}. \) Hence \(p = 5\), i.e., the number of classes and
the number of representations is 5. We next compute the class
constant equations. Set \(\gamma_t = \sum x\). Then we calculate:

\[
\begin{align*}
\gamma_1\gamma_t &= \gamma_t, \quad t = 1, \ldots, 5; \\
\gamma_2 &= \gamma_1; \\
\gamma_2\gamma_3 &= \gamma_3; \\
\gamma_2\gamma_4 &= \gamma_4; \\
\gamma_2\gamma_5 &= \gamma_5; \\
\gamma_3 &= 2\gamma_1 + 2\gamma_2; \\
\gamma_3\gamma_4 &= 2\gamma_5; \\
\gamma_3\gamma_5 &= 2\gamma_4; \\
\gamma_4 &= 2\gamma_1 + 2\gamma_2;
\end{align*}
\]
\[ \gamma_4 \gamma_5 = 2\gamma_3; \]
\[ \gamma_5^2 = 2\gamma_1 + 2\gamma_2. \]

Before trying to use (2) [or the equivalent formulas (4)] we have from (1) that

\[ n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8. \]

Also \( n_1 = 1 \). Clearly, there can be only one \( n_j > 1 \) so \( n_1 = n_2 = n_3 = n_4 = 1 \) and \( n_5 = 2 \).

We next compute \( \Gamma^3 \) directly from (8):

\[ \gamma_3 \gamma_1 = \gamma_3, \]
\[ \gamma_3 \gamma_2 = \gamma_3, \]
\[ \gamma_3 \gamma_3 = 2\gamma_1 + 2\gamma_2, \]
\[ \gamma_3 \gamma_4 = 2\gamma_3, \]
\[ \gamma_3 \gamma_5 = 2\gamma_4. \]

Hence

\[ \Gamma^3 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 2 & 0 \\
\end{bmatrix}. \]
Then from (2) we have for $s = 3, k = 5$

$$I^3 \eta^5 = \eta_3^5 \eta_1^5.$$  

(9)

It is simple to see that the eigenvalues of $I^3$ are $0, 2, 2, -2, -2$, i.e., $\eta_3^5$ must be one of these numbers. The equation (9) becomes

$$\eta_3^5 = \eta_3^5 \eta_1^5,$$

$$\eta_3^5 = \eta_3^5 \eta_2^5,$$

$$2\eta_1^5 + 2\eta_2^5 = (\eta_3^5)^2,$$  

(10)

$$2\eta_3^5 = \eta_3^5 \eta_4^5,$$

$$2\eta_4^5 = \eta_3^5 \eta_5^5.$$

There are several possibilities: if $\eta_3^5 = 0$ then since $\eta_1^5 = 1$ [see (3)] the third of the equations (10) implies $\eta_2^5 = -1$. Then $\eta_5^5 = 0$. Thus

$$\eta^5 = (1, -1, 0, 0, 0).$$  

(11)

Next, from (3)

$$\chi_t^5 = \frac{n_5}{h_t} \eta_t^5$$

$$= \frac{2}{h_t} \eta_t^5.$$
Moreover, $(\chi^5, \chi^5) = 1$ because $\chi^5$ is irreducible. Hence

\[ 1 = (\chi^5, \chi^5) = \frac{1}{h}(h_1 |\chi^5_1|^2 + h_2 |\chi^5_2|^2 + h_3 |\chi^5_3|^2 + h_4 |\chi^5_4|^2 + h_5 |\chi^5_5|^2) \]

\[ = \frac{1}{h} \left( \sum_{t=1}^{5} h_t \left( \frac{n_t}{h_t} \right) \eta_t^{5,2} \right) \]

\[ = \frac{1}{h} \sum_{t=1}^{5} \frac{n_t^2}{h_t} (\eta_t^{5,2}) \]

\[ = \frac{1}{2} (\eta_1^{5,2} + \eta_2^{5,2} + \frac{1}{2}(\eta_3^{5,2}) + \frac{1}{2}(\eta_4^{5,2}) + \frac{1}{2}(\eta_5^{5,2}) \) \]

(12)

Consider $\eta_1^5 = 1$, $\eta_2^5 = 1$, $\eta_3^5 = \pm 2$, and (12) becomes

\[ 1 = \frac{1}{2} (1 + 1 + 2 + \frac{1}{2}(\eta_4^{5,2}) + \frac{1}{2}(\eta_5^{5,2}) \), \]

clearly impossible. Thus $\eta^5$ is the vector (11) and

\[ \chi^5 = \left( \frac{n_5}{h_1} \eta_1^5, \frac{n_5}{h_2} \eta_2^5, \frac{n_5}{h_3} \eta_3^5, \frac{n_5}{h_4} \eta_4^5, \frac{n_5}{h_5} \eta_5^5 \right) \]

\[ = (2, -2, 0, 0, 0). \]

This allows us to fill in the last row of the character table (5) for this example.
<table>
<thead>
<tr>
<th>$h_1 = 1$</th>
<th>$h_2 = 1$</th>
<th>$h_3 = 2$</th>
<th>$h_4 = 2$</th>
<th>$h_5 = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
<td>$c_4$</td>
<td>$c_5$</td>
</tr>
<tr>
<td>$n_1 = 1$</td>
<td>$L_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$n_2 = 1$</td>
<td>$L_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$n_3 = 1$</td>
<td>$L_3$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$n_4 = 1$</td>
<td>$L_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$n_5 = 2$</td>
<td>$L_5$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>

The first row is filled in trivially since $L_1$ is the principal representation. To obtain the second row of the table (13) we can try the equation

$$\Gamma^3 \eta^2 = \eta_3^2 \eta^2,$$

which componentwise becomes

$$\eta_3^2 = \eta_3^2 \eta_1^2,$$

$$\eta_3^2 = \eta_3^2 \eta_2^2,$$

$$2\eta_4^2 + 2\eta_2^2 = (\eta_3^2)^2,$$

$$2\eta_5^2 = \eta_3^2 \eta_4^2,$$

$$2\eta_4^2 = \eta_3^2 \eta_5^2.$$

Again $\eta_3^2$ is an eigenvalue of $\Gamma^3$ and hence is 0 or ±2. If
\[ \eta_3^2 = 0, \text{ then } \eta_2^2 = -1, \eta_5^2 = 0, \eta_4^2 = 0 \text{ and} \]
\[ \eta^2 = (1, -1, 0, 0, 0). \quad (14) \]

If \( \eta_3^2 = \pm 2 \), then \( \eta_1^2 = 1, \eta_2^2 = 1, \eta_4^2 = \pm \eta_5^2 \). From 
\[ l = (x^2, x^2) \] we again compute

\[ 1 = (x^2, x^2) \]

\[ = \frac{1}{h} \sum_{t=1}^{5} h_t \left( \frac{\eta_2^2}{\eta_t^2} \right)^2 \]

\[ = \frac{1}{8} \sum_{t=1}^{5} \left( \frac{\eta_t^2}{h_t} \right)^2 \]

\[ = \frac{1}{8} \left( \left( \eta_1^2 \right)^2 + \left( \eta_2^2 \right)^2 + \left( \eta_3^2 \right)^2 + \frac{\left( \eta_4^2 \right)^2}{2} + \frac{\left( \eta_5^2 \right)^2}{2} \right). \quad (15) \]

Clearly (14) cannot satisfy (15). Thus \( \eta_3^2 = \pm 2 \) and from (15) we have

\[ 1 = \frac{1}{8} \left( 1 + 1 + \frac{4}{2} + \frac{\left( \eta_4^2 \right)^2}{2} + \frac{\left( \eta_5^2 \right)^2}{2} \right), \]

\[ 1 = \frac{1}{8} \left( 4 + \left( \eta_4^2 \right)^2 \right) \quad \text{(since } \eta_4^2 = \pm \eta_5^2 \text{).} \]

Thus
\[
\left(\eta_4^2\right)^2 = 4,
\]
\[
\eta_4^2 = \pm 2,
\]
and
\[
\chi^2 = (1, 1, \pm 1, \pm 1, \pm 1).
\]

We immediately compute from (8) that
\[
\Gamma^4 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 \\
2 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0
\end{bmatrix},
\]
and it is easy to check that the eigenvalues of this matrix are 0 and ±2. Now
\[
\Gamma^4 \eta^3 = \eta_4^3 \eta_4^3
\]
and hence \(\eta_4^3 = 0\) or \(\eta_4^3 = \pm 2\). If \(\eta_4^3 = 0\) we can immediately verify from the component equations of (16) that \(\eta_1^3 = 1, \eta_2^3 = -1, \eta_3^3 = 0, \eta_4^3 = 0, \eta_5^3 = 0\). But this is easily seen to contradict \((\chi^3, \chi^3) = 1\). Thus \(\eta_4^3 = \pm 2, \eta_1^3 = 1, \eta_2^3 = 1, \eta_3^3 = \pm \eta_5^3\). From \((\chi^3, \chi^3) = 1\) we again conclude as before that \(\eta_3^3 = \pm 2, \eta_5^3 = \pm 2\). Thus
\[
\chi^3 = (1, 1, \pm 1, \pm 1, \pm 1).
\]
Precisely as before we compute that \(\Gamma^3 \eta^4 = \eta_3^4 \eta_4^4\) implies that
\[ \chi^4 = (1,1,\pm 1,\pm 1,\pm 1). \]

Now \(\chi^1, \chi^2\) = 0 implies that \(2 \pm 2 \pm 2 = 0\) and hence \(\chi^2\) must have two negative signs and one positive sign. Similarly for \(\chi^3\) and \(\chi^4\). But since \(\chi^2, \chi^3, \text{ and } \chi^4\) are distinct it follows that the positive sign must be in distinct components.

Thus we can take

\[ \chi^2 = (1,1,1,-1,-1) \]
\[ \chi^3 = (1,1,-1,1,-1) \]
\[ \chi^4 = (1,1,-1,-1,1). \]

This allows us to complete the character table as in (13).

If we wish, we can actually construct a representation of degree 2 as follows. Take the unit vectors \(e_1 = (1,0)\) and \(e_2 = (0,1)\). Then by noting what happens to these as we perform the operations of the group we have

\[
L_5(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad
L_5((1 2 3 4)) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
\[
L_5((1 3)(2 4)) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad
L_5((1 4 3 2)) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]
\[
L_5((1 4)(2 3)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad
L_5((1 2)(3 4)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},
\]

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\[ L_5((2,4)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_5((1,3)) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \]

**Example 3.2** We find the character table for \( S_3 \), the symmetric group of degree 3. As we know, two permutations are in the same conjugacy class iff they have the same cycle structure. Thus the classes in \( S_3 \) are \( C_1 = \{ e \} \), \( C_2 = \{ (1,2), (1,3), (2,3) \} \), \( C_3 = \{ (1,2,3), (1,3,2) \} \). Hence \( h = 6 \), \( h_1 = 1 \), \( h_2 = 3 \), \( h_3 = 2 \), \( p = 3 \). It follows that \( S_3 \) has three pairwise inequivalent absolutely irreducible representations. Now \( n_1 = 1 \) and \( n_1^2 + n_2^2 + n_3^2 = 6 \). The only solution is \( n_2 = 1 \), \( n_3 = 2 \). The two obvious representations of degree 1 are the principal and alternating representations. To compute the character of the degree 2 representation we calculate the class constant equations:

\[ \gamma_1^t \gamma_t = \gamma_t, \quad t = 1, 2, 3 \]

\[ \gamma_2^2 = 3\gamma_1 + 3\gamma_3 \]

\[ \gamma_2 \gamma_3 = 2\gamma_2 \]

\[ \gamma_3^2 = 2\gamma_1 + \gamma_3. \]

It is clear that

\[ \Gamma^2 = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \]

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and

\[ \Gamma^2 \eta^3 = \eta_2^3 \eta^3. \]

The eigenvalues of \( \Gamma^2 \) are 0 and \( \pm 3 \). If \( \eta_2^3 = 0 \), then \( \Gamma^2 \eta^3 = 0 \) so \( \eta^3 = (1,0,-1) \). However

\[ \chi^3_t = \frac{n_3}{h_t} \eta^3_t, \quad t = 1, 2, 3, \]

and hence

\[ \chi^3 = (2,0,-1). \quad (17) \]

Now it is clear that \( \chi^3 \) is determined by the orthogonality conditions \( \langle \chi^3, \chi^1 \rangle = \langle \chi^3, \chi^2 \rangle = 0 \), both of which are easily verified for (17). Thus the character table for \( S_3 \) is:

<table>
<thead>
<tr>
<th></th>
<th>( h_1=1 )</th>
<th>( h_2=3 )</th>
<th>( h_3=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( C_1 )</td>
<td>( C_2 )</td>
<td>( C_3 )</td>
</tr>
<tr>
<td>( n_1=1 )</td>
<td>( L_1 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( n_2=1 )</td>
<td>( L_2 )</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( n_3=2 )</td>
<td>( L_3 )</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

**Example 3.3** Consider the representation of \( S_n \) by \( n \)-square permutation matrices, i.e., \( L: \sigma \to A(\sigma) = [\delta_{iG(j)}] \). Now let
\[
M = \begin{bmatrix}
1 & -1 & -1 & \cdots & -1 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & \ddots \\
0 & \cdots & & & & 0 & 1 
\end{bmatrix}.
\]

Then since \( M \) is obtained from \( I_n \) by successively subtracting rows 2, \( \ldots, n \) from row 1, it is obvious that \( M^{-1} \) is obtained from \( I_n \) by successively adding rows 2, \( \ldots, n \) to row 1.

Hence

\[
M^{-1} = \begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & \ddots \\
\vdots & & & & \ddots \\
0 & \cdots & & & 0 & 1 
\end{bmatrix}.
\]

We conclude that \( M^{-1}A(\sigma)M \) adds rows 2, \( \ldots, n \) to row 1 in \( A(\sigma) \) and then subtracts column 1 from columns 2, \( \ldots, n \). Since the first set of operations results in a first row of all 1's it is clear that the matrix \( M^{-1}A(\sigma)M \) is of the form

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
* & * & \cdots & \cdots \\
* & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & * \\
\end{bmatrix}.
\]

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It follows from Maschke's theorem (Theorem 1.6, Section 6.1) that the representation $L$ has one component of degree 1. Note that we could have seen this directly without computing the matrix $M$ by simply observing that each $A(\sigma)$ holds the 1-dimensional space spanned by the n-tuple $(1, \ldots, 1)$ fixed. Let $\chi$ be the character of $L$ and write
\[
\chi = \sum_{i=1}^{p} m_i \chi^i
\]
where $\chi^1, \ldots, \chi^p$ is a complete list of characters for $S_n$. By formula (104) in Section 6.2,
\[
\langle \chi, \chi \rangle = \sum_{i=1}^{p} m_i^2.
\]
Of course, $m_i$ is just the number of times the $i^{th}$ irreducible representation of $S_n$ occurs in $L$. Now
\[
\langle \chi, \chi \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} |\chi(\sigma)|^2
\]

\[
= \frac{1}{n!} \sum_{\sigma \in S_n} |\text{tr } A(\sigma)|^2. \quad (18)
\]

Also $\text{tr } A(\sigma)$ is just the number of fixed points of $\sigma$. There are $\binom{n}{k}$ ways in which we can choose $k$ fixed points for a permutation. With each of these there are $d(n-k)$ ways of de-
fining the permutation so that it has no other fixed points, i.e.,
the permutation is a derangement of \( n - k \) items. Thus the number
of \( \sigma \in S_n \) for which \( \text{tr } A(\sigma) = k \) is \( \binom{n}{k} d(n - k) \). It follows
from (18) that

\[
\langle \chi, \chi \rangle = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} d(n - k) k^2. \tag{19}
\]

We outline the steps necessary to evaluate (19) and leave the de-
tails to the reader.

**Step 1:** We recapitulate briefly some properties of the per-
manent function [see Section 1.1, Example 1.1(c)]. The permanent
of an \( n \)-square matrix is defined by

\[
\text{per}(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i \sigma(i)}.
\]

The permanent satisfies the Laplace expansion theorem,

\[
\text{per}(X) = \sum_{\beta \in Q_{m,n}} \text{per } X[\alpha|\beta] \text{per } X(\alpha|\beta), \quad \alpha \in Q_{m,n},
\]

is linear in each row and column, and is unchanged by permutation
of rows and columns. It follows that

\[
\text{per}(A + B) = \text{per}(A_{(1)} + B_{(1)}, \ldots, A_{(n)} + B_{(n)})
\]

\[
= \sum_{r=0}^{n} \sum_{\alpha \in Q_{r,n}} \text{per}(\ldots, A_{\alpha(1)}, \ldots, A_{\alpha(r)}, \ldots)
\]

\[
= \sum_{r=0}^{n} \sum_{\alpha \in Q_{r,n}} \text{per}(A_{\alpha(1)}, \ldots, A_{\alpha(r)}, B_{\alpha'(1)}, \ldots, B_{\alpha'(n-r)}).
\]
In the second equality above, rows of $B$ numbered complementary to $\alpha$ (i.e., $\alpha'$) appear in the remaining $n-r$ positions.

Continuing by Laplace,

$$\text{per}(A+B) = \sum_{r=0}^{n} \sum_{\alpha \in Q_{r,n}} \sum_{\beta \in Q_{r,n}} \text{per} A[\alpha|\beta] \text{per} B(\alpha|\beta), \quad (20)$$

(see Section 4.3, Exercise 6). In the formula (20) take $A = J = [1]$, $B = xI_n$ and compute that

$$\begin{align*}
\text{per}(J + xI_n) &= \sum_{r=0}^{n} \sum_{\alpha, \beta \in Q_{r,n}} r! \delta_{\alpha\beta} x^{n-r} \\
&= \sum_{r=0}^{n} \binom{n}{r} r! x^{n-r}.
\end{align*}$$

Now set $x = -1$ so that

$$\begin{align*}
\text{per}(J - I_n) &= \sum_{r=0}^{n} \binom{n}{r} r! (-1)^{n-r} \\
&= n! \sum_{r=0}^{n} \frac{(-1)^{n-r}}{(n-r)!} \\
&= n! \sum_{r=0}^{n} \frac{(-1)^{r}}{r!}.
\end{align*} \quad (21)$$

A moment's reflection shows that $\text{per}(J - I_n) = d(n)$. 

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Step 2. Observe that

$$\sum_{n=0}^{\infty} \frac{d(n)}{n!} x^n = \frac{e^{-x}}{1-x}, \quad |x| < 1,$$

(22)

where we have set \(d(0) = 1\). For, from calculus,

$$\frac{e^{-x}}{1-x} = (1+x+x^2+\cdots)(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots)$$

$$= 1 + (1-1)x + \left(\frac{1}{2!} - 1 + 1\right)x^2$$

$$+ \left(-\frac{1}{3!} + \frac{1}{2!} - 1 + 1\right)x^3$$

$$+ \left(\frac{1}{4!} - \frac{1}{3!} + \frac{1}{2!} - 1 + 1\right)x^4$$

$$+ \cdots$$

$$= \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \frac{(-1)^r}{r!} \right) n^r x^n$$

$$= \sum_{n=0}^{\infty} \left( n! \sum_{r=0}^{n} \frac{(-1)^r}{r!} \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{d(n)}{n!} x^n.$$

Also, from calculus,

$$\sum_{n=0}^{\infty} \frac{n^2}{n!} x^n = (x^2+x)e^x,$$

(23)
which is easily obtained by multiplying the series for $e^x$ by $x^2 + x$ and comparing coefficients.

**Step 34.** From (23) and (22)

\[
\sum_{n=0}^{\infty} \frac{n^2}{n!} x^n \sum_{n=0}^{\infty} \frac{d(n)}{n!} x^n = (x^2 + x)e^x \cdot \frac{e^{-x}}{1 - x}
\]

\[
= \frac{x^2 + x}{1 - x}
\]

\[
= x + \frac{2x^2}{1 - x}
\]

\[
= x + 2x^2(1 + x + x^2 + \cdots)
\]

\[
= x + 2(x^2 + x^3 + x^4 + \cdots).
\]

Thus matching coefficients for $n > 1$ on both sides we have

\[
\sum_{k=0}^{n} \frac{k^2}{k!} \frac{d(n-k)}{(n-k)!} = 2
\]

or

\[
\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} d(n-k) k^2 = 2.
\]

Comparing this formula with (19) we have

\[
(\chi, \chi) = 2.
\]

Hence
Thus \( p = 2, m_1 = m_2 = 1 \). In other words, \( L \) is composed of two of the irreducible components. Since one of these is one-dimensional as we saw above, the other is \((n-1)\)-dimensional.

Thus \( S_n \) possesses an irreducible representation of degree \( n - 1 \).

Also from Theorem 1.11, Section 6.1,

\[
\dim\langle A(\sigma), \sigma \in S_n \rangle = 1^2 + (n-1)^2 = n^2 - 2n + 2.
\]

We turn our attention now to the systematic study of the general symmetric group \( S_n \) consisting of all \( n! \) permutations of the integers 1, \ldots, \( n \). As before, it is assumed about the underlying field \( \mathbb{R} \) that the left regular representation \( \rho : \mathfrak{S}_n \to \mathbb{L}(\mathfrak{S}_n, \mathfrak{S}_n) \) can be split into absolutely irreducible components over \( \mathbb{R} \). In view of Exercise 8, Section 6.2, in general we need only make a simple algebraic extension of the prime subfield of \( \mathbb{R} \) in order for this assumption to be valid.

The number \( p \) of pairwise inequivalent absolutely irreducible representations of any finite group \( S \) is precisely the number of conjugacy classes in \( S \) [Theorem 2.23(b), Section 6.2]. Moreover, two permutations in \( S_n \) are conjugate iff they have the same cycle structure. Thus for \( S = S_n \), \( p \) is the number of different cycle structures possible for an element of \( S_n \). A
cycle structure can be specified by a sequence of positive integers \( \alpha = (\alpha_1, \ldots, \alpha_k) \) satisfying

\[
\alpha_1 \equiv \cdots \equiv \alpha_k
\]

(24)

and

\[
\alpha_1 + \cdots + \alpha_k = n .
\]

(25)

The \( \alpha_j \) are the cycle lengths. We have proved the following result.

**Theorem 3.1** The number of pairwise inequivalent absolutely irreducible representations of \( S_n \) is precisely the number of partitions of \( n \).

We shall lexicographically order the partitions of \( n \):

\( \alpha > \beta \) means as usual that the first nonvanishing difference \( \alpha_t - \beta_t \) is positive. With a partition \( \alpha \) of \( n \) we shall construct a frame or a table \( F_\alpha \) consisting of \( \alpha_1 \) boxes in the first row, \( \alpha_2 \) boxes in the second row, etc.:

\[
\begin{array}{ccccccc}
\alpha_1 : & & & & & & \\
\alpha_2 : & & & & & & \\
\alpha_3 : & & & & & & \\
\vdots & & & & & & \\
\alpha_k : & & & & & & \\
\end{array}
\]

(26)

A Young tableau or diagram associated with \( \alpha \) is a table \( F_\alpha \).
together with an arrangement of the integers $1, \ldots, n$ in the
table. More precisely, a diagram $D_{\alpha, \sigma}$ is a pair of objects
$(F_{\alpha}, \sigma)$ where $F_{\alpha}$ is a table, $\sigma \in S_n$, and the $t^{th}$ row of
$F_{\alpha}$ is occupied by the integers

$$\sigma(\alpha_1 + \cdots + \alpha_{t-1} + 1), \ldots, \sigma(\alpha_1 + \cdots + \alpha_t), \ t = 1, \ldots, k.$$ 

For example, if $n = 5$, $\alpha = (2,2,1)$, $\sigma = (2\ 3) \in S_5$, then
$D_{\alpha, \sigma}$ is the following diagram:

$$
\begin{array}{ccc}
1 & 3 \\
2 & 4 \\
5 & \\
\end{array}
$$

In other words, the integer appearing in the $(t, j)$ position in
the diagram $D_{\alpha, \sigma}$ is

$$\sigma(\alpha_1 + \cdots + \alpha_{t-1} + j), \ j = 1, \ldots, \alpha_t,$$

in which $t$ is the row number and $j$ is the column number in
the table.

A standard diagram $D_{\alpha, \sigma}$ is one in which the integers appear-
ing in each row and each column increase. Thus for the fixed
frame $F_{\alpha}$ the standard diagrams in the case $n = 5$, $\alpha = (2,2,1)$,
are

$$
\begin{array}{ccc}
1 & 2 & 1 \\
3 & 4 & 3 \\
5 & 4 & \\
\end{array}, \quad
\begin{array}{ccc}
1 & 2 & 1 \\
3 & 5 & 2 \\
4 & 5 & \\
\end{array}, \quad
\begin{array}{ccc}
1 & 3 & 1 \\
2 & 4 & 2 \\
5 & 3 & \\
\end{array}, \quad
\begin{array}{ccc}
1 & 3 & 1 \\
2 & 5 & 2 \\
4 & 3 & \\
\end{array}. 
$$
For a fixed $\alpha$ we shall arrange the diagrams $D_{\alpha,\sigma}$ lexicographically according to the sequences $(\sigma(1), \ldots, \sigma(n))$.

Next, if $D_{\alpha,\sigma}$ is a diagram and $\varphi \in S_n$, then $\varphi D_{\alpha,\sigma}$ denotes the diagram obtained from $D_{\alpha,\sigma}$ as follows: replace the entry $\nu$ appearing in the $(t,j)$ position by $\varphi(\nu)$. In other words,

$$\varphi D_{\alpha,\sigma} = D_{\alpha,\varphi \sigma}.$$  

(28)

**Definition 3.1 (Row and column groups)** Let $D_{\alpha,\sigma}$ be a given diagram. The totality of permutations $p \in S_n$ that have the property that $p$ maps the set of integers appearing in any row of $D_{\alpha,\sigma}$ onto itself is denoted by $R(D_{\alpha,\sigma})$. Similarly $C(D_{\alpha,\sigma})$ is the totality of permutations $q \in S_n$ that map the set of integers appearing in any column of $D_{\alpha,\sigma}$ onto itself.

Both $R(D_{\alpha,\sigma})$ and $C(D_{\alpha,\sigma})$ are clearly subgroups of $S_n$ and are known as the row and column groups of $D_{\alpha,\sigma}$.

It is an easy task to confirm that $R(D_{\alpha,\sigma}) \cap C(D_{\alpha,\sigma})$ contains only the identity permutation (see Exercise 1).

We have the following elementary computational rules: if $\varphi \in S_n$ then

$$R(\varphi D_{\alpha,\sigma}) = \varphi R(D_{\alpha,\sigma}) \varphi^{-1}.$$  

(29)

and

$$C(\varphi D_{\alpha,\sigma}) = \varphi C(D_{\alpha,\sigma}) \varphi^{-1}.$$  

(30)

To verify (29) observe that since $\varphi D_{\alpha,\sigma} = D_{\alpha,\varphi \sigma}$, we see that
\[ p \in \mathcal{R}(\varphi_{D_{\alpha, \sigma}}) \iff \]
\[ p(\mathcal{U}^t) = \mathcal{U}^t, \quad t = 1, \ldots, k, \]

where
\[ \mathcal{U}^t = (\varphi \sigma(\alpha_1 + \cdots + \alpha_{t-1} + 1), \ldots, \varphi \sigma(\alpha_1 + \cdots + \alpha_t)). \]

But \( p(\mathcal{U}^t) = \mathcal{U}^t \iff \)
\[ \varphi^{-1} \rho \varphi(\varphi^{-1}(\mathcal{U}^t)) = \varphi^{-1}(\mathcal{U}^t), \quad (31) \]

and (31) simply states that
\[ \varphi^{-1} \rho \varphi(\sigma(\alpha_1 + \cdots + \alpha_{t-1} + 1), \ldots, \sigma(\alpha_1 + \cdots + \alpha_t)) \]
\[ = [\sigma(\alpha_1 + \cdots + \alpha_{t-1} + 1), \ldots, \sigma(\alpha_1 + \cdots + \alpha_t)]. \quad (32) \]

Clearly (32) is equivalent to
\[ \varphi^{-1} \rho \varphi \in \mathcal{R}(D_{\alpha, \sigma}), \]

i.e.,
\[ p \in \varphi \mathcal{R}(D_{\alpha, \sigma}) \varphi^{-1}. \quad (33) \]

This proves (29).

A similar computation will establish (30) (see Exercise 2).

**Definition 3.2 (Young symmetrizers)** Let \( D_{\alpha, \sigma} \) be a diagram and define the following elements of \( \mathcal{U}(S_n) \):
\[ \varepsilon(D_{\alpha,\sigma}) = \sum_{p \in R(D_{\alpha,\sigma})} \varepsilon(q)pq , \quad (34) \]

\[ r(D_{\alpha,\sigma}) = \sum_{p \in R(D_{\alpha,\sigma})} p , \quad (35) \]

\[ c(D_{\alpha,\sigma}) = \sum_{q \in C(D_{\alpha,\sigma})} \varepsilon(q)q ; \quad (36) \]

then \( \varepsilon(D_{\alpha,\sigma}) \) is called the **Young symmetrizer** associated with \( D_{\alpha,\sigma} \) and (35) and (36) are called the **row** and **column symmetrizers** respectively.

It is an immediate consequence of the definition that for any \( \varphi \in S_n \),

\[ \varepsilon(\varphi D_{\alpha,\sigma}) = \varphi \varepsilon(D_{\alpha,\sigma}) \varphi^{-1} \]

(see Exercise 3).

We have the following computational rules concerning these items.

**Theorem 3.2** If \( p_1 \in R(D_{\alpha,\sigma}) \) and \( q_1 \in C(D_{\alpha,\sigma}) \), then

\[ p_1 r(D_{\alpha,\sigma}) = r(D_{\alpha,\sigma}) \]

\[ = r(D_{\alpha,\sigma}) p_1 , \quad (37) \]

and
\[ q_1c(D_{\alpha,\sigma}) = \varepsilon(q_1)c(D_{\alpha,\sigma}) = c(D_{\alpha,\sigma})q_1. \]  

(38)

Moreover,

\[ \varepsilon(D_{\alpha,\sigma}) = r(D_{\alpha,\sigma})c(D_{\alpha,\sigma}), \]  

(39)

and hence

\[ p_1\varepsilon(D_{\alpha,\sigma})q_1 = \varepsilon(q_1)\varepsilon(D_{\alpha,\sigma}). \]

Moreover, each \( pq \) in \( \varepsilon(D_{\alpha,\sigma}) = r(D_{\alpha,\sigma})c(D_{\alpha,\sigma}) \) appears precisely once with coefficient \( \varepsilon(q) \).

**Proof:** All of (37) and (38) and (39) are obvious from the definitions. To prove the last assertion first notice that if

\[ pq = p_1q_1 \]

then

\[ p_1^{-1}p = q_1q^{-1}. \]

But \( p_1^{-1}p \in R(D_{\alpha,\sigma}) \) and \( q_1q^{-1} \in C(D_{\alpha,\sigma}) \), and by Exercise 1, \( p_1^{-1}p \) and \( q_1q^{-1} \) must both be the identity. Thus \( p_1 = p, q_1 = q \). If \( \theta \in S_n \), then the coefficient on \( \theta \) in the product

\[ r(D_{\alpha,\sigma})c(D_{\alpha,\sigma}) \]

is

\[ \sum \varepsilon(q)\delta_{\theta,pq}, \]  

(40)
where the summation in (40) is over all \( p \in R(D_{\alpha,\sigma}) \), \( q \in C(D_{\alpha,\sigma}) \).

Now \( \delta \) either has precisely one representation of the form \( pq \). or none, and hence (40) has the value \( \varepsilon(q) \) or 0 according as \( \delta \) is of the indicated form \( pq \) or not. But this is the coefficient on \( \delta \) in \( E(D_{\alpha,\sigma}) \).

The preceding result has the following purely combinatorial consequence.

**Theorem 3.3** Let \( D_{\alpha,\sigma} \) and \( D_{\beta,\varphi} \) be two diagrams with the following property: there exists a pair of integers \( s \) and \( t \) that appear in the same row of \( D_{\alpha,\sigma} \) and in the same column of \( D_{\beta,\varphi} \). Then

\[
c(D_{\beta,\varphi})r(D_{\alpha,\sigma}) = 0
\]

(41)

and thus

\[
E(D_{\beta,\varphi})E(D_{\alpha,\sigma}) = 0.
\]

(42)

**Proof:** Consider the transposition \((s \ t)\), which is simultaneously in \( R(D_{\alpha,\sigma}) \) and \( C(D_{\beta,\varphi}) \) by hypothesis. Then by (37) and (38)

\[
(s \ t)r(D_{\alpha,\sigma}) = r(D_{\alpha,\sigma})(s \ t)
\]

\[
= r(D_{\alpha,\sigma})
\]

and

\[
(s \ t)c(D_{\beta,\varphi}) = c(D_{\beta,\varphi})(s \ t)
\]

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Thus,
\[
\text{c}(D_{\beta,\varphi}) \tau(D_{\alpha,\sigma}) = \text{c}(D_{\beta,\varphi})(s\ t)\tau(D_{\alpha,\sigma})
\]
\[
= -\text{c}(D_{\beta,\varphi}) \tau(D_{\alpha,\sigma})
\]
and (41) is proved. Then (42) follows immediately. 

**Theorem 3.4** Let \( D_{\alpha,\sigma} \) and \( D_{\beta,\varphi} \) be two diagrams:

(a) if \( \beta > \alpha \), then \( \varepsilon(D_{\alpha,\sigma}) \varepsilon(D_{\beta,\varphi}) = 0 \);

(b) if \( (\varphi(1), \ldots, \varphi(n)) > (\sigma(1), \ldots, \sigma(n)) \) and \( D_{\alpha,\varphi} \) and \( D_{\alpha,\sigma} \) are standard diagrams then \( \varepsilon(D_{\alpha,\varphi}) \varepsilon(D_{\alpha,\sigma}) = 0 \).

**Proof:** In view of Theorem 3.3 (with the roles of \( \alpha \) and \( \beta \) reversed), it suffices to prove that under the hypothesis (a) there exists a pair of integers appearing in the same column of \( D_{\alpha,\sigma} \) and in the same row of \( D_{\beta,\varphi} \), and under the hypothesis (b) there exists a pair of integers appearing in the same column of \( D_{\alpha,\varphi} \) and the same row of \( D_{\alpha,\sigma} \).

(a) Let \( \beta = (\beta_1, \ldots, \beta_k) \), \( \alpha = (\alpha_1, \ldots, \alpha_m) \). Consider the first row of \( D_{\beta,\varphi} \). There are \( \beta_1 \) integers occurring there; and unless some two of these appear in the same column of \( D_{\alpha,\sigma} \), they must occur in \( \beta_1 \) different columns of \( D_{\alpha,\sigma} \). Now \( \beta > \alpha \) so that \( \beta_1 \geq \alpha_1 \). But to accommodate the \( \beta_1 \) different integers we must have \( \alpha_1 \geq \beta_1 \). Hence \( \alpha_1 = \beta_1 \). The situation looks like this:

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in which \( j_1, \ldots, j_{\alpha_1} \) appear in \( \alpha_1 \) different columns of \( D_{\alpha,\sigma} \).

Now let \( \theta \in C(D_{\alpha,\sigma}) \) be a permutation that brings \( j_1, \ldots, j_{\alpha_1} \) into the first row, and consider the diagrams \( D_{\beta,\varphi} \) and \( \theta D_{\alpha,\sigma} \).

They have precisely the same integers in the first row, so consider the \( \beta_2 \) integers appearing in row 2 of \( D_{\beta,\varphi} \). These must lie below row 1 in \( \theta D_{\alpha,\sigma} \) (since the first rows of \( D_{\beta,\varphi} \) and \( \theta D_{\alpha,\sigma} \) both contain the same integers \( j_1, \ldots, j_{\alpha_1} \)), and unless some two of them are in the same column of \( \theta D_{\alpha,\sigma} \), they must be in different columns of \( \theta D_{\alpha,\sigma} \). Hence by the same argument as above, \( \alpha_2 \geq \beta_2 \), and since \( \beta > \alpha \), we must have \( \alpha_2 = \beta_2 \). We now apply a permutation \( \theta' \in C(D_{\alpha,\sigma}) \) that brings these \( \beta_2 = \alpha_2 \) integers into the second row and leaves the first row unaltered. Now \( \theta D_{\beta,\varphi} \) and \( \theta' \theta D_{\alpha,\sigma} \) have the same integers appearing in the first two rows (except for order, of course). This process is continued, and since \( \beta > \alpha \), we must reach a stage at which some two integers, say \( s \) and \( t \), appearing in a row of \( D_{\beta,\varphi} \) appear in the same column of

\[
\theta'' \cdots \theta' \theta D_{\alpha,\sigma}.
\] (43)
But $\emptyset'' \cdots \emptyset', \emptyset \in \mathcal{C}(D_{\alpha, \sigma})$, and hence two integers appearing in the same column of (43) must appear in the same column of $D_{\alpha, \sigma}$. This establishes (a).

(b) We are given that $(\varphi(1), \ldots, \varphi(n)) > (\sigma(1), \ldots, \sigma(n))$. Then let $i$ be the first integer for which these two sequences disagree [notice that $i > 1$ because a standard diagram must have 1 in the $(1,1)$ position; otherwise it would be preceded in a row or column by a larger integer]. Thus

$$\varphi(1) = \sigma(1), \ldots, \varphi(i-1) = \sigma(i-1), \quad \varphi(i) > \sigma(i). \quad (44)$$

Let $s = \varphi(i)$, $t = \sigma(i)$, so $s > t$. We have the following picture:

Suppose $s$ (and $t$) appear in row $k$ and column $\ell$. Clearly $\ell > 1$ because $\varphi$ and $\sigma$ agree on $1, \ldots, i-1$, and the first column position in any row of a standard diagram is determined completely by the preceding rows. That is, the integers must appear in increasing order in both rows and columns of a standard diagram, and this implies that the beginning of each row must be occupied by the least integer not appearing in previous rows. The
integer $t$ occurs in $D_{\sigma, \phi}$, say at position $(k_1, \ell_1)$. Clearly $t = \sigma(i)$ cannot be any of $\phi(1) = \sigma(1), \ldots, \phi(i-1) = \sigma(i-1)$ because of (44). Thus the possibility

$$k_1 < k$$  \hspace{1cm} (45)

is excluded. Next consider the possibility $k_1 = k$. There are two cases:

$$k_1 = k, \quad \ell_1 \leq \ell,$$  \hspace{1cm} (46)

$$k_1 = k, \quad \ell_1 > \ell.$$  \hspace{1cm} (47)

The possibility (46) is excluded for the same reason as (45) was. If (47) were to hold, then $t < s$ would be in the same row as $s$ in $D_{\sigma, \phi}$ and to the right of $s$. This is impossible because the integers in a standard diagram increase from left to right across a row. Finally we have the possibilities when $k_1 > k$:

$$k_1 > k, \quad \ell_1 \leq \ell,$$  \hspace{1cm} (48)

$$k_1 > k, \quad \ell_1 > \ell.$$  \hspace{1cm} (49)

Now (49) is not possible; otherwise $t < s$ would appear to the right and below $s$ in $D_{\sigma, \phi}$, which is impossible for a standard diagram. For the same reason $k_1 > k$ and $\ell_1 = \ell$ is also excluded. The only possibility left then is

$$k_1 > k, \quad \ell_1 < \ell;$$  \hspace{1cm} (50)
Let $j$ be the integer in position $(k,i_1)$; then in view of (44), $j$ is also in the $(k,i_1)$ position of $D_{\alpha,\sigma}$. Thus $j$ and $t$ lies in row $k$ of $D_{\alpha,\sigma}$ and column $i_1$ of $D_{\alpha,\varphi}$, and (b) is proved.

Our next result is fundamental in the analysis of the representations of $S_n$.

**Theorem 3.5** Let $\alpha$ be a partition of $n$ and let $\sigma \in S_n$. Then

$$\varepsilon(D_{\alpha,\sigma})^2 = \gamma \varepsilon(D_{\alpha,\sigma})$$

where $\gamma$ is a positive integral divisor of $n!$.

**Proof:**

Part I. We show first that if $f \in \mathfrak{U}(S_n)$ and

$$pfq = e(q)f$$  \hspace{1cm} (51)

holds for every $p \in R(D_{\alpha,\sigma})$, $q \in C(D_{\alpha,\sigma})$, then $f$ must be a multiple of $\varepsilon(D_{\alpha,\sigma})$. For, write

$$f = \sum_{\varphi \in S_n} f(\varphi)\varphi,$$

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and then

\[ pfq = \sum_{\varphi \in S_n} f(\varphi) p q \varphi \]

\[ = \sum_{\theta \in S_n} f(\theta^{-1} q \theta^{-1}) \theta. \]

Thus by (51) we have

\[ f(p^{-1} q^{-1}) = \varepsilon(q) f(\varphi), \quad \varphi \in S_n. \quad (52) \]

If we replace the dummy variable \( p^{-1} q^{-1} \) by \( \varphi \), then (52) becomes

\[ f(p q) = \varepsilon(q) f(\varphi), \quad \varphi \in S_n. \quad (53) \]

In particular, if \( \varphi = e \) then

\[ f(p q) = \varepsilon(q) f(e). \]

Suppose next that \( \varphi \) is not of the form \( p q \), \( p \not\in R(D_{\alpha,\sigma}) \), \( q \in C(D_{\alpha,\sigma}) \). We will show now that there must exist two integers that are in the same row of \( D_{\alpha,\sigma} \) and in the same column of \( \varphi D_{\alpha,\sigma} = D_{\alpha,\sigma} \varphi \). If this were not the case, then all the integers in column 1 of \( D_{\alpha,\sigma} \varphi \) must lie in different rows of \( D_{\alpha,\sigma} \). Apply a row permutation \( p_1 \in R(D_{\alpha,\sigma}) \) to \( D_{\alpha,\sigma} \) so that these integers appear in column 1 of \( p_1 D_{\alpha,\sigma} \). Observe that by (29),

\[ R(p_1 D_{\alpha,\sigma}) = p_1 R(D_{\alpha,\sigma}) p_1^{-1} = R(D_{\alpha,\sigma}) \].

Now \( D_{\alpha,\sigma} \varphi \) and \( p_1 D_{\alpha,\sigma} \) have the same integers appearing in column 1. Consider the
integers appearing in column 2 of $D_{\alpha,\omega}$, They appear in different rows of $D_{\alpha,\sigma}$ and hence in different rows of $p_1 D_{\alpha,\sigma}$. Thus by a row permutation $p_2 \in R(p_1 D_{\alpha,\sigma}) = R(D_{\alpha,\sigma})$, we can bring these integers into column 2 of $p_2 p_1 D_{\alpha,\sigma}$ without affecting column 1 of $p_1 D_{\alpha,\sigma}$. Continuing we obtain $p_1, \ldots, p_m$ all in $R(D_{\alpha,\sigma})$ such that $D_{\alpha,\varphi\omega}$ and $p_m \cdots p_1 D_{\alpha,\sigma}$ have the same integers in each column. Let $p = p_m \cdots p_1 \in R(D_{\alpha,\sigma})$. We have just proved that $D_{\alpha,\varphi\omega}$ and $p D_{\alpha,\sigma}$ involve the same integers in each of their columns. Then it follows that there exists $q_1 \in C(p D_{\alpha,\sigma})$ such that

$$D_{\alpha,\varphi\omega} = q_1 p D_{\alpha,\sigma}. \quad (54)$$

According to (30), $C(p D_{\alpha,\sigma}) = p C(D_{\alpha,\sigma}) p^{-1}$ and hence

$$q_1 = p q p^{-1} \quad (55)$$

for $q \in C(D_{\alpha,\sigma})$. Now (54) becomes

$$D_{\alpha,\varphi\omega} = D_{\alpha,\varphi q \sigma}. \quad (56)$$

From (56) we have

$$\omega = p q. \quad (57)$$

But we are assuming that $\omega$ is not of the form (57). Thus there must exist two integers, $s$ and $t$, in the same row of $D_{\alpha,\sigma}$ and in the same column of $D_{\alpha,\varphi\omega}$. Hence $(s, t) \in R(D_{\alpha,\sigma})$ and $(s, t) \in C(D_{\alpha,\varphi\omega})$. Since $C(D_{\alpha,\varphi\omega}) = \varphi C(D_{\alpha,\sigma}) \varphi^{-1}$, we conclude that
\[(s \ t) = q_0 q^{-1}, \ q \in C(D_{\alpha, \sigma}); \quad (58)\]

moreover (58) implies that \(q\) is a transposition so that \(\epsilon(q) = -1\) and \(q^2 = e\). Then from (53) we have, by taking \(p = (s \ t),\)

\[f(q_0) = \epsilon(q)f(p_0 q_0)\]
\[= \epsilon(q)f(q_0 q^{-1} q_0)\]
\[= \epsilon(q)f(q q^{-2})\]
\[= -f(q_0).\]

Hence \(f(q_0) = 0\). We have proved that for any \(p \in R(D_{\alpha, \sigma})\), \(q \in C(D_{\alpha, \sigma})\),

\[f(p q) = \epsilon(q)f(e); \quad (59)\]

and if \(q_0\) is not of the form \(pq\), then

\[f(q_0) = 0. \quad (60)\]

In other words,

\[f = f(e)C(D_{\alpha, \sigma}). \quad (61)\]

**Part II.** We assert that if \(f = \epsilon(D_{\alpha, \sigma})^2\) then

\[pfq = \epsilon(q)f\]

for any \(p \in R(D_{\alpha, \sigma})\), \(q \in C(D_{\alpha, \sigma})\). For, by Theorem 3.2,
\[ p\mathcal{E}(D_{\alpha,\sigma})^2 q = p\mathcal{E}(D_{\alpha,\sigma})\mathcal{E}(D_{\alpha,\sigma}) q \]
\[ = \mathcal{E}(D_{\alpha,\sigma})\mathcal{E}(q)\mathcal{E}(D_{\alpha,\sigma}) \]
\[ = \mathcal{E}(q)\mathcal{E}(D_{\alpha,\sigma})^2. \]

It now follows from Part I, in particular (61), that if
\[ f = \mathcal{E}(D_{\alpha,\sigma})^2 \]
then
\[ f = f(e)\mathcal{E}(D_{\alpha,\sigma}), \quad (63) \]

Part III. Let \( \mathcal{E} \in L(\mathfrak{U}(S_n), \mathfrak{U}(S_n)) \) be defined by
\[ \mathcal{E}x = x\mathcal{E}(D_{\alpha,\sigma}), \quad x \in \mathfrak{U}(S_n). \]

In other words \( \mathcal{E} = \rho_\gamma(\mathcal{E}(D_{\alpha,\sigma})) \). Then we assert that both
\[ \operatorname{tr}(\mathcal{E}) = n! \quad (64) \]
and
\[ \operatorname{tr}(\mathcal{E}) = \gamma k \quad (65) \]

where \( \gamma \) is the number \( f(e) \) appearing in (63) and \( k \) is the rank of \( \mathcal{E} \). [Note that from the definition of multiplication in \( \mathfrak{U}(S_n), f(e) = \gamma \) is an integer so that once (64) and (65) are established it will also follow that \( \gamma \) is a positive integral divisor of \( n! \).] First observe that
\[ \mathcal{E}^2 x = \mathcal{E}(x\mathcal{E}(D_{\alpha,\sigma})) \]
\[ = x\mathcal{E}(D_{\alpha,\sigma})\mathcal{E}(D_{\alpha,\sigma}) \]
\[ \mathcal{L}^2 = f(e) \mathcal{L}. \]  

(66)

Now suppose \( f(e) = 0 \). Then \( \mathcal{L}^2 = 0 \) from (66), and hence \( \text{tr}(\mathcal{L}) = 0 \). On the other hand, consider the following matrix representation of \( \mathcal{L} \). Choose the elements of \( S_n \) as a basis of \( \mathfrak{S}(S_n) \) with \( e \) as the first element of the basis. If \( \theta \in S_n \), then

\[ \mathcal{L} \theta = \theta \mathcal{E}(D_{\alpha, \sigma}) \]

\[ = \sum_{p \in \mathfrak{R}(D_{\alpha, \sigma}), q \in \mathfrak{C}(D_{\alpha, \sigma})} \epsilon(q) pq. \]  

(67)

We want to know the coefficient of \( \theta \) in (67). The permutations \( pq \) appearing in (67) are all distinct as we have seen before, and hence in order that

\[ \theta \epsilon(q) pq = c \theta \]

we must have \( \theta pq = \theta \). But then \( pq = e \) so that \( p = q = e \), \( \epsilon(q) = 1 \). Thus \( c = 1 \). In other words, \( \mathcal{L} \theta \) has a coefficient of 1 on \( \theta \), \( \theta \in S_n \), so that

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\[ \text{tr}(\mathcal{L}) = n!. \]

It follows that \( \text{tr}(\mathcal{L}) = 0 \) is impossible and thus that \( f(e) \neq 0 \).

Now consider the transformation \( \frac{1}{f(e)} \mathcal{L} \), and compute from (66) that

\[
\left( \frac{1}{f(e)} \mathcal{L} \right)^2 = \frac{1}{f(e)} \frac{1}{f(e)} \mathcal{L}^2 = \frac{1}{f(e)} \frac{1}{f(e)} f(e) \mathcal{L} = \mathcal{L}.
\]

Thus \( \frac{1}{f(e)} \mathcal{L} \) is idempotent and hence

\[
\text{tr}\left( \frac{\mathcal{L}}{f(e)} \right) = k. \quad (68)
\]

The statement (68) is precisely (65).

From (64) and (65) we see that

\[ n! = \gamma k \]

and this completes the proof. \( \Box \)

Theorem 3.5 tells us that for any diagram \( D_{\alpha, \sigma} \)

\[
\left( \frac{\mathcal{L}(D_{\alpha, \sigma})}{\gamma} \right)^2 = \frac{\mathcal{L}(D_{\alpha, \sigma})}{\gamma},
\]

i.e., to within a positive integral divisor of \( n! \) the Young
symmetrizers are idempotents. It is not difficult to show that if
\[ (D_{\sigma'\psi})^2 = \lambda E(D_{\alpha'\psi}) \]

then \( \lambda = \gamma \). In other words, it is the frame and not the arrangement of integers in it that determines the coefficient \( \gamma \) in Theorem 3.5. To see this let

\[ L_{\sigma',\sigma} = \rho_x(E(D_{\sigma',\sigma})), \]
\[ L_{\psi,\psi} = \rho_x(E(D_{\psi,\psi})), \]

as in the proof of Part III of Theorem 3.5. Then as we saw in (64) and (65)

\[ n! = \text{tr}(L_{\sigma',\sigma}) \]
\[ = \gamma k, \]
\[ n! = \text{tr}(L_{\psi,\psi}) \]
\[ = \lambda k', \]

where \( k \) and \( k' \) are the ranks of \( L_{\sigma',\sigma} \) and \( L_{\psi,\psi} \) respectively. Thus our assertion will be confirmed once we show that \( k = k' \).

We compute that

\[ L_{\sigma',\psi} \times = \times E(D_{\alpha'\psi}) \]
\[ = \times E(D_{\alpha,\psi^{-1}\sigma}) \]
\[ = \times E(\psi^{-1}D_{\alpha,\sigma}) \quad [\text{see (28)}] \]
\[ = \times (\psi^{-1})E(D_{\alpha,\sigma} \sigma^{-1})^{-1} (\text{Exercise 3}) \]
Let \( \theta = \varphi \sigma^{-1} \) and set \( T_{\theta} = \rho_{\sigma}(\theta) \). Then the preceding calculation shows that

\[
L_{\alpha, \varphi} x = T_{\theta}^{-1} L_{\alpha, \sigma} T_{\theta} x, \quad x \in \mathfrak{u}(S_n)
\]

so that

\[
k = \text{tr}(L_{\alpha, \varphi})
\]

\[
= \text{tr}(L_{\alpha, \sigma})
\]

\[
= k'.
\]

For diagrams with the same table we can now prove the following interesting result.

**Theorem 3.6** For a fixed partition \( \alpha \), consider each of the \( n! \) left ideals in \( \mathfrak{u}(S_n) \):

\[
A_\varphi = \mathfrak{u}(S_n) \mathcal{e}(D_{\alpha, \varphi}), \quad \varphi \in S_n.
\]

Then

(a) the representations \( \rho|_{A_\varphi} \) are equivalent for all \( \varphi \),

where \( \rho : S_n \to L(\mathfrak{u}(S_n), \mathfrak{u}(S_n)) \) is the left regular representation;

(b) if \( \varphi_1, \ldots, \varphi_s \) in \( S_n \) are those permutations for which \( D_{\alpha, \varphi_j} \), \( j = 1, \ldots, s \), are all the standard diagrams for fixed \( \alpha \), then the sum

\[
A_{\varphi_1} + \cdots + A_{\varphi_s}
\]

is direct, that is, there is no nontrivial linear relation.
\[ x_1 + \cdots + x_s = 0 \quad (69) \]

where \( x_i \in A_\varphi, i = 1, \ldots, s \).

**Proof:** (a) Consider the following calculation using Exercise 3:

\[
A_\varphi = \mathfrak{U}(S_n) \mathcal{E}(D_{\alpha, \varphi}) \\
= \mathfrak{U}(S_n) \mathcal{E}(D_{\alpha, (\varphi \sigma^{-1})\sigma}) \\
= \mathfrak{U}(S_n) \mathcal{E}((\varphi \sigma^{-1}) D_{\alpha, \sigma}) \\
= \mathfrak{U}(S_n) \varphi \sigma^{-1} \mathcal{E}(D_{\alpha, \sigma}) \sigma \varphi^{-1} \\
= \mathfrak{U}(S_n) \mathcal{E}(D_{\alpha, \sigma}) \sigma \varphi^{-1} \\
= A_\sigma(\sigma \varphi^{-1}),
\]

where the penultimate equality follows from \( \mathfrak{U}(S_n) \varphi \sigma^{-1} = \mathfrak{U}(S_n) \).

Now set \( \Theta = \sigma \varphi^{-1} \in S_n \), so that

\[
A_\varphi = A_\Theta, \quad (70)
\]

and define the linear transformation \( T : A_\sigma \to A_\varphi \) by

\[
Tf = f\Theta, \quad f \in A_\sigma. \quad (71)
\]

Then by (70), \( T \) is onto \( A_\varphi \), and obviously if \( x \in S_n \) and \( f \in A_\sigma \), then

\[
(\rho(x)|_{A_\varphi})Tf = (\rho(x)|_{A_\varphi})f\Theta.
\]

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= x f \Theta ,

whereas

\[ T(p(x)|A_\Phi) f = Tf \]

\[ = (xf) \Theta . \]

Thus

\[(p(x)|A_\Phi) T = T(p(x)|A_\sigma). \tag{72} \]

Also, if \( T f_1 = T f_2 \), i.e., \( f_1 \Theta = f_2 \Theta \), then since \( \Theta \in S_n \), we can multiply by \( \Theta^{-1} \) to obtain \( f_1 = f_2 \). Thus we know that

\( T \) is injective and (a) is proved.

(b) An element \( x_i \in A_{\omega_1} \) is of the form

\[ f_i \mathcal{E}(D_{\alpha, \Phi_1}) , \quad i = 1, \ldots, s. \tag{73} \]

In (69), assume without loss of generality that as \( i \) increases, the sequences \( (\omega_1(1), \ldots, \omega_1(n)) \) decrease in lexicographic order. Suppose

\[ \sum_{i=1}^{s} f_i \mathcal{E}(D_{\alpha, \Phi_1}) = 0. \tag{74} \]

Now multiply (74) on the right by \( \mathcal{E}(D_{\alpha, \Phi_2}) \):

\[ \sum_{i=1}^{s} f_i \mathcal{E}(D_{\alpha, \Phi_1}) \mathcal{E}(D_{\alpha, \Phi_2}) = 0. \tag{75} \]
By Theorem 3.4(b), $\rho(\mathcal{D}_{\alpha,\Phi_i})\mathcal{E}(\mathcal{D}_{\alpha,\Phi_s}) = 0$ for $i < s$, and by Theorem 3.5, $\rho(\mathcal{D}_{\alpha,\Phi_s})^2 = \gamma_s \mathcal{E}(\mathcal{D}_{\alpha,\Phi_s})$, $\gamma_s \neq 0$. Hence (75) becomes

$$\gamma_s \mathcal{E}(\mathcal{D}_{\alpha,\Phi_s}) = 0$$

or

$$\gamma_s \chi_s = 0.$$ 

Thus $\chi_s = 0$. Now repeat this argument with $\mathcal{E}(\mathcal{D}_{\alpha,\Phi_{s-1}})$, using the fact already established that $\chi_{s-1} = 0$, to obtain $\chi_{s-1} = 0$.

Continuing, we obviously get $\chi_1 = \cdots = \chi_s = 0$, and the proof is complete. \[ \square \]

The next result is preliminary to showing that the left ideals

$$\mathbb{U}(\mathcal{S}_n)\mathcal{E}(\mathcal{D}_{\alpha,\Phi})$$

are minimal in $\mathcal{S}_n$. This crucial fact will then enable us to exhibit a complete list of absolutely irreducible pairwise inequivalent representations of $\mathcal{S}_n$. In fact, as we shall see in Theorem 3.10, a complete list of pairwise inequivalent absolutely irreducible representations of $\mathcal{S}_n$ can be obtained rationally, i.e., over $\mathbb{R}$ itself.

**Theorem 3.7** Let $\mathbb{S}$ be any finite group and let $e$ be a nonzero idempotent in the group algebra $\mathbb{U}(\mathbb{S})$. If for each $a \in \mathbb{U}(\mathbb{S})$ there is an $r \in \mathbb{R}$ such that
\[ eae = re, \]

then \( A = \mathbb{H}(S)e \) is a minimal left ideal in \( \mathbb{H}(S) \).

**Proof:** Suppose \( A \) were not minimal and \( A_1 \subset A \). Since \( \rho \) is fully reducible there exists a left ideal \( A_2 \) such that

\[ A = A_1 + A_2. \]

Since \( e \in A \) write

\[ e = e_1 + e_2, \quad e_i \in A_i, \quad i = 1, 2. \]

As we have computed a number of times (see Exercise 10)

\[ e_i^2 = e_i, \quad i = 1, 2, \]

\[ e_1 e_2 = e_2 e_1 = 0. \]

However by assumption

\[ ee_1 e = re, \]

\[ ee_2 e = se, \quad r, s \in \mathbb{R}, \]

and we compute that

\[ re = ee_1 e \]

\[ = (e_1 + e_2)e_1(e_1 + e_2) \]

\[ = e_1 \]

and similarly \( se = e_2 \). But...
\[ re = e_1 = e_1^2 = r^2 e_2^2 \]

so that
\[ r(r-1)e = 0, \]

and similarly
\[ s(s-1)e = 0. \]

Obviously neither \( r = 0 \) nor \( s = 0 \), otherwise \( e_1 \) or \( e_2 \) would be 0. Hence \( r = s = 1 \) and \( e_1 = e_2 \). This contradiction completes the proof. \( \square \)

The next result allows us to apply the results of Section 6.2.

**Theorem 3.8** If \( \alpha \) is a partition of \( n \) and \( \sigma \in S_n \), then the left ideal

\[ A_{\alpha, \sigma} = \mathbb{U}(S_n) \mathbb{C}(D_{\alpha, \sigma}) \]

is minimal.

**Proof:** Let \( e = y^{-1} \mathbb{C}(D_{\alpha, \sigma}) \) (see Theorem 3.5) and for \( a \in \mathbb{U}(S_n) \) consider the element

\[ f = eae. \]

If \( p \in R(D_{\alpha, \sigma}) \) and \( q \in C(D_{\alpha, \sigma}) \), then by Theorem 3.2,
pfq = peaq

= ε(q)eaq

= ε(q)f.

Thus by Part I of the proof of Theorem 3.5, we conclude that $f$ is a scalar multiple of $\mathcal{E}(\mathcal{D}_{\alpha, \sigma})$:

$$f = r\mathcal{E}(\mathcal{D}_{\alpha, \sigma}).$$

But then Theorem 3.7 immediately implies that

$$A_{\alpha, \sigma} = \mathcal{U}(S_n)\mathcal{E}(\mathcal{D}_{\alpha, \sigma})$$

is a minimal left ideal.

Theorem 3.9 Let $\alpha$ and $\beta$ be partitions of $n$, $\sigma$ and $\omega$ in $S_n$, and define two minimal left ideals by

$$A = A_{\alpha, \sigma} = \mathcal{U}(S_n)\mathcal{E}(\mathcal{D}_{\alpha, \sigma}),$$

$$B = A_{\beta, \omega} = \mathcal{U}(S_n)\mathcal{E}(\mathcal{D}_{\beta, \omega}).$$

Then $\rho|A$ is equivalent to $\rho|B$ if and only if $\alpha = \beta$.

Proof: If $\alpha = \beta$ then Theorem 3.6(a) states that $\rho|A$ and $\rho|B$ are equivalent. Conversely, suppose that $\alpha \neq \beta$, say $\alpha < \beta$. By Theorem 2.14(b), Section 6.2, $AB = 0$ iff $\rho|A$ and $\rho|B$ are equivalent. Thus if we assume that $\rho|A$ and $\rho|B$ are inequivalent, $AB = B \neq 0$, and there exists an element $b \in B$ such that $Ab \neq 0$. But $Ab$ is a left ideal in $B$, $Ab = B$. We can conclude that
\[ \text{From (76) we see that for an appropriate } a \in \mathbb{H}(S_n), \]

\[ \mathcal{E}(D_{\beta,\varphi}) = a \mathcal{E}(D_{\alpha,\sigma}) b. \]

Now \( \mathcal{E}(D_{\beta,\varphi})^2 = \gamma \mathcal{E}(D_{\beta,\varphi}), \) \( 0 \neq \gamma \in \mathbb{R}, \) so that

\[ \gamma \mathcal{E}(D_{\beta,\varphi}) = \mathcal{E}(D_{\beta,\varphi})^2 \]

\[ = (a \mathcal{E}(D_{\alpha,\sigma}) b) \mathcal{E}(D_{\beta,\varphi}) \]

\[ = a(\mathcal{E}(D_{\alpha,\sigma}) b \mathcal{E}(D_{\beta,\varphi})). \] (77)

Let \( \theta \in S_n. \) Then by Exercise 3,

\[ \mathcal{E}(D_{\alpha,\sigma}) \mathcal{E}(D_{\beta,\varphi}) = \mathcal{E}(D_{\alpha,\sigma})(\theta \mathcal{E}(D_{\beta,\varphi}) \theta^{-1}) \theta \]

\[ = \mathcal{E}(D_{\alpha,\sigma}) \mathcal{E}(D_{\beta,\varphi}) \theta \]

\[ = \mathcal{E}(D_{\alpha,\sigma}) \mathcal{E}(D_{\beta,\theta \varphi}) \theta. \] (78)

However, \( \alpha < \beta, \) and hence by Theorem 3.4(a), (78) implies

\[ \mathcal{E}(D_{\alpha,\sigma}) \mathcal{E}(D_{\beta,\varphi}) = 0 \]

for any \( \theta \in S_n. \) But then, using (78) again,

\[ \mathcal{E}(D_{\alpha,\sigma}) b \mathcal{E}(D_{\alpha,\varphi}) = 0 \]

for any \( b \in \mathbb{H}(S_n). \) We conclude from (77) that

\[ \gamma \mathcal{E}(D_{\beta,\varphi}) = 0, \]
a contradiction. Thus $\rho|A$ and $\rho|B$ are inequivalent and the proof is complete.

We now have the following important result.

**Theorem 3.10** A complete list of pairwise inequivalent absolutely irreducible representations of $\mathfrak{S}_n$ is obtained as follows: For each partition $\alpha$ consider the diagram $D_{\alpha,e}$ (e is the identity permutation). Let

$$A_{\alpha} \simeq A_{\alpha,e} = \mathfrak{U}(\mathfrak{S}_n)\mathfrak{E}(D_{\alpha,e}).$$

(79)

Then the $A_{\alpha}$ are minimal left ideals in $\mathfrak{U}(\mathfrak{S}_n)$, and the representations $\rho|A_{\alpha}$, as $\alpha$ varies over all partitions of $n$, constitute the complete list. Moreover, the group algebra $\mathfrak{U}(\mathfrak{S}_n)$ is a direct sum of minimal left ideals:

$$\mathfrak{U}(\mathfrak{S}_n) = \sum_{\alpha} \sum_{s=1}^{n} \mathfrak{U}(\mathfrak{S}_n)\mathfrak{E}(D_{\alpha,\varphi_s}).$$

(80)

In (80) $\alpha$ runs over all partitions of $n$ and for each fixed $\alpha$, $D_{\alpha,\varphi_1}, \ldots, D_{\alpha,\varphi_n}$ are all the standard diagrams with frame $F_{\alpha}$.

**Proof:** Let $\pi_n$ denote the number of partitions of $n$. To say that the $\rho|A_{\alpha}$ constitute a complete list means that any absolutely irreducible representation $L$ must be equivalent to one of them. We know from Theorem 3.9 that the $\rho|A_{\alpha}$ are pairwise inequivalent as $\alpha$ varies over all the distinct partitions of $n$. Moreover, there cannot exist a minimal left ideal $A \subset \mathfrak{U}(\mathfrak{S}_n)$ for which $\rho|A$ is not equivalent to some $\rho|A_{\alpha}$. For obviously there would be at least $\pi_n + 1$ pairwise inequivalent
absolutely irreducible representations of $S_n$, and this contradicts Theorem 3.1. Thus every minimal left ideal $A \subset M(S_n)$ has the property that $\rho|A$ is equivalent to some $\rho|A_\alpha$. On the other hand, the representation $L$ is equivalent to $\rho|A$ where $A$ is some minimal left ideal in $M(S_n)$ [Theorem 2.21(a), Section 6.2] and thus $L$ is equivalent to some $\rho|A_\alpha$.

To prove that the sum on the right in (80) is direct it suffices to show that if

$$\sum_{\alpha_1, \ldots, \alpha_n} \sum_{s=1}^{n_\alpha} a_{\alpha_1, \ldots, \alpha_n, s} \mathcal{E}(D_{\alpha_1}, \ldots, D_{\alpha_n}, \rho_s) = 0$$  \hspace{1cm} (81)

then every $a_{\alpha_1, \ldots, \alpha_n, s} = 0$. To do this multiply (81) on the right by the $\mathcal{E}(D_{\alpha_1}, \ldots, D_{\alpha_n}, \rho_s)$ in the following order: First in order of decreasing $\alpha$ and then for a fixed $\alpha$ in order of increasing $\rho_s^{\alpha}$. Then repeated applications of Theorem 3.4(a), (b) shows that the $a_{\alpha_1, \ldots, \alpha_n, s}$ are indeed 0. The fact that the right side of (80) is in fact equal to $M(S_n)$ depends on a rather intricate purely combinatorial argument. The argument actually hinges on two facts:

First, the number $n_\alpha$, i.e., the number of standard diagrams with frame $\alpha = (\alpha_1, \ldots, \alpha_r)$, is

$$n_\alpha = \frac{n! \prod_{i<k} (\ell_i - \ell_k)}{\ell_1 ! \cdots \ell_r !}$$

where $\ell_1 = \alpha_1 + (r-1), \ldots, \ell_r = \alpha_r$; by using this formula one then proves directly that

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\[ \sum_{\alpha} n_{\alpha}^2 = n! . \]

Once this latter equation is established the equality (80) follows by general considerations. For, the number of times a given component occurs in the regular representation is precisely its degree and hence each \( \Upsilon(S_n) \mathcal{E}(D_{\alpha,s}) \) has dimension \( n_{\alpha,s} \), \( s = 1, \ldots, n_{\alpha} \). But then the preceding equation asserts the equality of the dimensions of the two sides of (80). We omit the details. (See Notes, Section 6.3).

Actually we shall not have occasion to use (80) but only the fact that the ideals (79) provide a complete set of pairwise inequivalent absolutely irreducible representations. Observe that since the coefficients of the idempotents \( \mathcal{E}(D_{\alpha,s}) \) are in \( \mathbb{R} \) we can conclude that all the absolutely irreducible components (79) can be taken over \( \mathbb{R} \), i.e., the reduction (80) can be done rationally.

At this point we shall switch the notation to \( S_m \) (rather than \( S_n \)) and resume our study of the components of \( \Pi' \). Moreover, we take the dimension of \( V \) to be \( n \) as in Section 6.2. If we look back at Theorem 2.16 in Section 6.2 we see that the components

\[ \Pi'(T) \big| P(\cup(\mathcal{E}(D_{\alpha,s}))) \big\otimes V \]

for which \( P(\mathcal{E}(D_{\alpha,s})) \neq 0 \) (see the remark preceding Theorem 2.17 in Section 6.2) are pairwise inequivalent and absolutely irreducible.
Also note that the preceding remarks concerning $\mathcal{E}(D_{\alpha, e})$ imply that the absolutely irreducible components (82) can be taken over $\mathbb{R}$ as well.

Let the symmetry class of tensors
\[
P(\nu(\mathcal{E}(D_{\alpha, e}))) \bigotimes_{1}^{m} V_{\alpha}
\]
be denoted simply by
\[
V_{\alpha}^{\otimes}
\]
and set $\mathcal{E}_{\alpha} = \mathcal{E}(D_{\alpha, e})$. Since $\mathcal{E}(D_{\alpha, e})^2 = \gamma \mathcal{E}(D_{\alpha, e})$ it follows that
\[
P(\nu^{-1}(\mathcal{E}_{\alpha}))) = P_{\alpha}
\]
is a projection and of course
\[
V_{\alpha}^{\otimes} = \text{Im} \ P_{\alpha}.
\]
(The constant $\gamma$ depends only on the partition $\alpha$ and not on the arrangement of integers in the diagram.) We define the linear map
\[
K_{\alpha}(T) : V_{\alpha}^{\otimes} \to V_{\alpha}^{\otimes}
\]
by the formula
\[
K_{\alpha}(T) = \Pi(T) | V_{\alpha}^{\otimes}.
\]
In other words, $K_{\alpha}$ is simply a component of $\Pi$ acting on the absolutely irreducible invariant subspace $V_{\alpha}^{\otimes}$. We have
\[ K_{\alpha}(T)P_{\alpha}x_1 \otimes \cdots \otimes x_m = P_{\alpha'}(T)P_{\alpha'}x_1 \otimes \cdots \otimes x_m \]
\[
= P_{\alpha'}(T)x_1 \otimes \cdots \otimes x_m \\
= P_{\alpha'}x_1 \otimes \cdots \otimes x_m .
\]  

(84)

If, in conformity with an earlier notation, [see Section 2.3, in particular remark (c) following Theorem 3.1] we set

\[ P_{\alpha}x_1 \otimes \cdots \otimes x_m = x_1 * \cdots * x_m , \]  

(85)

then formula (84) can be written very simply in terms of this *star product:*

\[ K_{\alpha}(T)x_1 * \cdots * x_m = Tx_1 * \cdots * Tx_m . \]  

(86)

Note that

\[ K_{\alpha}(T, T_2) = K_{\alpha}(T_1)K_{\alpha}(T_2) . \]  

(87)

Since \( K_{\alpha}(T) \) is simply a restriction of \( \Pi'(T) \) to an invariant subspace it follows that if \( T \) is nonsingular, then \( K_{\alpha}(T) \) is nonsingular. We shall call these components of \( \Pi' \) the Young induced transformations (see Section 2.4, Definition 4.1). Note that \( K_{\alpha}(T) \) is an absolutely irreducible homogeneous polynomial representation of \( GL_n(V) \). We express the fact that the components of \( \Pi'(T) \) have this property by saying that the representation \( \Pi' \) is reductive.

**Example 3.4** If \( \alpha \) is the partition \( 1 + 1 + \cdots + 1 \) then
\[ \epsilon_\alpha = \sum_{\sigma \in S_m} \epsilon(\sigma)\sigma , \]

\[ P_\alpha = \frac{1}{m!} \sum_{\sigma \in S_m} \epsilon(\sigma)P(\sigma) \]

\[ = S_\epsilon , \]

\[ \nu^{\otimes}_\alpha = \bigwedge^m \nu , \]

\[ x_1 \ast \cdots \ast x_m = x_1 \wedge \cdots \wedge x_m , \]

and

\[ K_\alpha(T) = C_m(T) . \]

Similarly, if \( \alpha \) is the partition consisting of the single integer \( m \), then

\[ \epsilon_\alpha = \sum_{\sigma \in S_m} \sigma , \]

\[ P_\alpha = \frac{1}{m!} \sum_{\sigma \in S_m} P(\sigma) \]

\[ = S_1 , \]

\[ \nu^{\otimes}_\alpha = \nu^{(m)} , \]

\[ x_1 \ast \cdots \ast x_m = x_1 \cdot \cdots \cdot x_m , \]
and

\[ K(\alpha)(T) = P_m(T) . \]

It is interesting to know the precise conditions for \( \varepsilon_\alpha \) to be in \( C \), or equivalently, the conditions for \( V^\otimes_\alpha \) to be nonzero.

**Theorem 3.11** The symmetry class \( V^\otimes_\alpha \) is 0 if and only if the first column of the diagram corresponding to \( \alpha \) has length exceeding \( n = \dim V \).

**Proof:** We have \( \varepsilon_\alpha = \varepsilon(D_{\alpha,e}) = r(D_{\alpha,e})c(D_{\alpha,e}) \) and moreover \( \varepsilon_\alpha^\otimes = 0 \) iff \( P(\varepsilon_\alpha^\otimes) = 0 \). As we saw in the discussion immediately following the proof of Theorem 2.16, Section 6.2, \( P(\varepsilon_\alpha^\otimes) = 0 \) iff \( P(\varepsilon_\alpha) = 0 \). Also note that if \( \varphi \in S_m \) then

\[
P(\varepsilon(D_{\alpha,\psi})) = P(\varphi(D_{\alpha,\psi})) = P(\varepsilon(D_{\alpha,\psi})) [\text{see (28)}]
\]

\[
= P(\varphi(D_{\alpha,\psi})\varphi^{-1}) \quad \text{(see Exercise 3)}
\]

\[
= P(\varphi)P(\varepsilon_\alpha)P(\varphi^{-1}) ,
\]

so that \( P(\varepsilon_\alpha) = 0 \) iff \( P(\varepsilon(D_{\alpha,\psi})) = 0 \) for any \( \varphi \in S_m \). Assume that the first column of the diagram \( D_{\alpha,e} \) has length \( N \) exceeding \( n \). Choose \( \varphi \in S_m \) so that the integers appearing in the first column of \( D_{\alpha,\psi} \) are \( 1, \ldots, N \) in order. Let \( x_1, \ldots, x_N, x_{N+1}, \ldots, x_m \) be any \( m \) vectors in \( V \). Then

\[
P(\varepsilon(D_{\alpha,\psi}))x^\otimes = P(r(D_{\alpha,\psi})c(D_{\alpha,\psi}))x^\otimes
\]

\[
= P(r(D_{\alpha,\psi}))P(c(D_{\alpha,\psi}))x^\otimes
\]
\[ = \mathcal{P}(r(D_{\alpha,\varphi})) \sum_{\sigma \in \mathcal{C}(D_{\alpha,\varphi})} \varepsilon(\alpha) \mathcal{P}(\sigma) x^\otimes \]

\[ = \mathcal{P}(r(D_{\alpha,\varphi})) \left( \sum_{q \in S_N} \varepsilon(q) x_q(1) \otimes \cdots \otimes x_q(N) \right) \otimes z \quad (88) \]

where \( z \) is some appropriate tensor. Continuing, we have

\[ \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi})) x^\otimes = N! \mathcal{P}(r(D_{\alpha,\varphi})) x_1 \wedge \cdots \wedge x_N \otimes z. \quad (89) \]

Since \( \dim V < N \) it follows that (89) must be 0 and hence

\[ \mathcal{P}(e_\varphi) = 0. \]

To prove the converse it suffices to exhibit \( \varphi \in S_m \) and \( w \in \otimes V \) such that

\[ \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi})) w \neq 0. \]

For then

\[ 0 \neq \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi}) \varphi^{-1}) \]

\[ = \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi}) \varphi^{-1}) \mathcal{P}(\varphi) \]

\[ = \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi}) \varphi^{-1}) \mathcal{P}(\varphi^{-1}) \]

so that \( \mathcal{P}(e_\varphi) \neq 0 \). Clearly

\[ \mathcal{P}(\mathcal{E}(D_{\alpha,\varphi})) = \mathcal{P}(e(D_{\alpha,\varphi})) \mathcal{P}(r(D_{\alpha,\varphi})) \]

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\[ x = c(D_{\alpha,\varphi}) r(D_{\alpha,\varphi}) . \]

We illustrate with a special choice of \( \varphi \) and \( w \) that makes the general situation clear. Thus suppose \( m = 10, \alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 2, \alpha_4 = 1 \) and \( \varphi \) is chosen so that \( D_{\alpha,\varphi} \) has the following structure:

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 5 & 8 & 10 \\
\hline
2 & 6 & 9 &   \\
\hline
3 & 7 &   &   \\
\hline
4 &   &   &   \\
\hline
\end{array}
\]

Since the first column has length not exceeding \( \text{dim} \ V \) it follows that there are (at least) four i.i. vectors \( e_1, e_2, e_3, e_4 \) in \( V \). Define \( w \) as follows:

\[
w = \frac{f_1}{e_1 \otimes e_2 \otimes e_3 \otimes e_4} \otimes \frac{f_2}{e_1 \otimes e_2 \otimes e_3} \otimes \frac{f_3}{e_1 \otimes e_2} \otimes \frac{f_4}{e_1} .
\]

where \( f_1 = 4, f_2 = 3, f_3 = 2, f_4 = 1 \) are the column lengths.

Clearly if \( p \in R(D_{\alpha,\varphi}) \), then \( P(p)w = w \) and hence

\[
P(\tau(D_{\alpha,\varphi}))w = \alpha_1! \alpha_2! \alpha_3! \alpha_4! w .
\]

Thus

\[
P(\gamma(C(D_{\alpha,\varphi})))w = P(c(D_{\alpha,\varphi}))P(\tau(D_{\alpha,\varphi}))w
\]

\[= P(c(D_{\alpha,\varphi}))\alpha_1! \alpha_2! \alpha_3! \alpha_4! w .\]
It is also clear that

\[ P(c(D_{\alpha,e})) = f_1 f_2 f_3 f_4 (e_1 \wedge e_2 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_2 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes e_1. \]

(90)

None of the indicated exterior products in (90) is 0 and since the tensor algebra over \( V \) contains no zero-divisors [see Example 1.3(d), Section 3.1] it follows that (90) is not 0.

Theorem 3.11 tells us that the symmetry class of tensors

\[ V^\otimes_{\alpha} = P(v(e_\alpha))^m \otimes V \]

is trivial, i.e., 0, iff the diagram \( D_{\alpha,e} \) has a column whose length exceeds \( n = \text{dim} V \). It is interesting to see when

\[ \text{dim} V^\otimes_{\alpha} = 1. \]

(91)

For, the corresponding associated transformation

\[ K_{\alpha} : \text{GL}_n(V) \rightarrow \text{GL}_{n}^\otimes_{\alpha} \]

is then nothing more or less than an abelian character of \( \text{GL}_n(V) \).

Suppose that \( m = dn \) and every column of \( D_{\alpha,e} \) has length \( n \), i.e., the frame has the form

\[ \begin{array}{cccc}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 \\
\end{array} \]

(92)

As we saw in the discussion following Theorem 2.16 in Section 6.2,
\( P(\nu(f)) \) and \( P(f) \) always have the same rank so that \( V^\otimes \) has the same dimension as \( P(\varepsilon_\alpha) \otimes V \). Moreover
\[
P(\varepsilon(D_{\alpha,\varphi})) = P(\varepsilon_\alpha \otimes \varphi^{-1})
\]
\[
= P(\varphi)P(\varepsilon_\alpha)P(\varphi^{-1})
\]
so that \( \dim V^\otimes \) is the rank of \( P(\varepsilon(D_{\alpha,\varphi})) \). Now choose \( \varphi \) so that \( D_{\alpha,\varphi} \) has the following form

\[
D_{\alpha,\varphi}:
\begin{array}{cccc}
1 & n & \ldots & dn-n+1 \\
2 & n+2 & \ldots & \\
\vdots & \ddots & \ddots & \ddots \\
n & 2n & \ldots & dn \\
\end{array}
\tag{93}
\]

Observe that if \( S_t^n \) is the subgroup of \( S_m \) consisting of those permutations that hold fixed all symbols except possibly those in the \( i^{th} \) column of (93), then
\[
P(\varepsilon(D_{\alpha,\varphi})) = \sum_{q \in C(D_{\alpha,\varphi})} \varepsilon(q)P(q)
\]
\[
= \sum_{q_t \in S_t^n} \varepsilon(q_1) \cdots \varepsilon(q_d)P(q_1) \cdots P(q_d), \tag{94}
\]
where the summation is over all choices \( q_1 \in S_t^n, \ldots, q_d \in S_d^n \).

Let \( x^\otimes = (x_1 \otimes \cdots \otimes x_n) \otimes (x_{n+1} \otimes \cdots \otimes x_{2n}) \otimes \cdots \otimes (x_{dn-n+1} \otimes \cdots \otimes x_{dn}) \).
where \( x_i \in V \), \( i = 1, \ldots, dn = m \). Then from (94)

\[
P(c(D_{\alpha,\phi}^\alpha))x^\otimes = \left( \sum_{q_1 \in S_n^1} e(q_1)x_{q_1(1)} \otimes \cdots \otimes x_{q_1(n)} \right) \otimes \left( \sum_{q_2 \in S_n^2} e(q_2)x_{q_2(n+1)} \otimes \cdots \otimes x_{q_2(2n)} \right) \otimes \cdots \\
\otimes \left( \sum_{q_d \in S_n^d} e(q_d)x_{q_d(dn-n+1)} \otimes \cdots \otimes x_{q_d(dn)} \right)
\]

\[
= (n!)^d (x_1 \wedge \cdots \wedge x_n) \otimes (x_{n+1} \wedge \cdots \wedge x_{2n}) \otimes \cdots \otimes (x_{dn-n+1} \wedge \cdots \wedge x_{dn}). \tag{95}
\]

Since \( \dim V = n \), \( \dim \wedge V = 1 \). If \( e_1, \ldots, e_n \) is a basis of \( V \) and \( e^\wedge = e_1 \wedge \cdots \wedge e_n \), then (95) shows that every element in \( \text{Im} P(c(D_{\alpha,\phi}^\alpha)) \) has the form

\[
aP(r(D_{\alpha,\phi}^\alpha)) e^\wedge \otimes \cdots \otimes e^\wedge, \quad a \in \mathbb{R}. \tag{96}
\]

Thus \( V^\otimes_{\alpha} \) is at most one dimensional. On the other hand let

\[
z^\otimes = (x_1 \otimes \cdots \otimes x_n) \otimes (x_1 \otimes \cdots \otimes x_n) \otimes \cdots \otimes (x_1 \otimes \cdots \otimes x_n)
\]

and compute easily that

\[
P(r(D_{\alpha,\phi}^\alpha))z^\otimes = (n!)^d z^\otimes. \tag{97}
\]

But then it follows from (95) and (97) that
\[ P(\gamma^{-1}E(D_{\alpha,\varphi})))z^\otimes = \gamma^{-1}P(\nu(E(D_{\alpha,\varphi})))z^\otimes \]
\[ = \gamma^{-1}P(c(D_{\alpha,\varphi}))P(r(D_{\alpha,\varphi}))z^\otimes \]
\[ = \gamma^{-1}(n!)dP(c(D_{\alpha,\varphi}))z^\otimes \]
\[ = \gamma^{-1}(n!)2d(x_1 \wedge \cdots \wedge x_n) \otimes \cdots \otimes (x_1 \wedge \cdots \wedge x_n). \quad (98) \]

It follows from (98) that if \( x_1, \ldots, x_n \) are i.i. then \( P_{\alpha}z^\otimes \neq 0 \)
and this fact combined with (96) shows that
\[ \dim V_{\alpha}^\otimes = 1 \]
whenever \( m = nd \), \( \dim V = n \), and every column of \( D_{\alpha,e} \) has
length \( n \). The converse of this result is contained in Exercise 9.

Exercises

1. Show that \( R(D_{\alpha,\sigma}) \cap C(D_{\alpha,\sigma}) = \{e\} \).

   Hint: Suppose \( p \in R(D_{\alpha,\sigma}) \cap C(D_{\alpha,\sigma}) \) and \( p \) is not the
identity. Then there exists some row of \( D_{\alpha,\sigma} \), say row \( i \),
in which the element in position \( (i,j) \) is moved to position \( (i,k) \) by \( p \), \( j \neq k \). But then \( p \) would not be in \( C(D_{\alpha,\sigma}) \),
i.e., it moves an element from column \( j \) to column \( k \) in
the diagram.

2. Prove formula (30).

3. Prove that if \( \varphi \in S_n \) then
\[ E(\varphi D_{\alpha,\sigma}) = \varphi E(D_{\alpha,\sigma}) \varphi^{-1}. \]
Hint: From (29) and (30) we have
\[ R(\varphi D_{\alpha,\sigma}) = \varphi R(D_{\alpha,\sigma}) \varphi^{-1}, \]
\[ C(\varphi D_{\alpha,\sigma}) = \varphi C(D_{\alpha,\sigma}) \varphi^{-1} \] so that
\[ \varepsilon(\varphi D_{\alpha,\sigma}) = \sum_{p \in R(\varphi D_{\alpha,\sigma})} \sum_{q \in C(\varphi D_{\alpha,\sigma})} \varepsilon(q)pq \]
\[ = \sum_{\varphi^{-1}p \in R(D_{\alpha,\sigma})} \sum_{\varphi^{-1}q \in C(D_{\alpha,\sigma})} \varepsilon(q)pq \]
\[ = \sum_{p' \in R(D_{\alpha,\sigma})} \sum_{q' \in C(D_{\alpha,\sigma})} \varepsilon(p'\varphi^{-1})p'\varphi^{-1}pq'\varphi^{-1} \]
\[ = \varphi \varepsilon(D_{\alpha,\sigma}) \varphi^{-1}. \]

In Exercises 4-9 the following notation will be used.

Let \( \varphi \in S_m \) and define
\[ V_{\alpha,\varphi} = P(\nu(\varepsilon(D_{\alpha,\varphi})))^m \]
\[ = \bigotimes_{\lambda=1}^{m} V_{\lambda}. \] \hfill (99)

Set
\[ P_{\alpha,\varphi} = P(\gamma^{-1} \nu(\varepsilon(D_{\alpha,\varphi}))) \] \hfill (100)

and define
\[ K_{\alpha,\varphi}(T) : \bigotimes_{\alpha,\varphi} \rightarrow \bigotimes_{\alpha,\varphi} \] \hfill (101)
by the formula
\[ K_{\alpha, \phi}(T) = \prod_{i=1}^{n} |V_{\alpha_i, \phi_i}^{\otimes}. \] (102)

Define the \( \phi \)-star product of vectors \( x_1, \ldots, x_m \) in \( V \) to be
\[ x_1 \ast \cdots \ast x_m = P_{\alpha, \phi} x_1 \otimes \cdots \otimes x_m \] (103)
precisely as in (85) (we do not subscript the \( \ast \) with \( \phi \) in order to avoid clumsy notation).

4. Prove that \( P_{\alpha, \phi} = P(\phi) P_{\alpha} P(\phi)^{-1} \) (i.e., \( P_{\alpha} = P_{\alpha, e} \)).
   Hint: \( E(\alpha, \phi) = \phi E_{\alpha, e} \phi^{-1} \) so that \( P_{\alpha, \phi} = P(\phi) P_{\alpha} P(\phi)^{-1} \).

5. Prove that \( P_{\alpha, \phi}^2 = P_{\alpha, \phi} \).
   Hint: Apply \( P_{\alpha}^2 = P_{\alpha} \) and Exercise 4.

6. Prove that for any \( \phi \), \( \dim V_{\alpha, \phi}^{\otimes} = \dim V_{\alpha, e}^{\otimes} = \dim V_{\alpha}^{\otimes} \).
   Hint: From Exercise 4, \( P_{\alpha, \phi} \) and \( P_{\alpha} \) have the same rank.

7. Assume that \( V_{\alpha, \phi}^{\otimes} \neq 0 \). Prove that there exists a symmetry operator \( P(x) \) such that \( P(x) \) maps \( V_{\alpha, \phi}^{\otimes} \) onto \( V_{\alpha}^{\otimes} \) and the mapping
\[ S_{\alpha, \phi} = P(x) |V_{\alpha, \phi}^{\otimes} \]
is a bijection satisfying
\[ S_{\alpha, \phi} K_{\alpha, \phi}(T) S_{\alpha, \phi}^{-1} = K_{\alpha}(T). \]
   Hint: From Theorem 3.9 we know that the two minimal left ideals in \( \mathfrak{g}(S) \) determined by \( e_1 = E(\alpha, e) \) and \( e_2 = E(\alpha, \phi) \) are
equivalent. Thus by Theorem 2.15, Section 6.2, these minimal left ideals must lie in the same two-sided ideal \( \mathcal{R} \) that appears in the unique decomposition of \( U(S_m) \) into minimal two-sided ideals. Of course, since we are assuming \( V_{\alpha} \neq 0 \) it follows that \( V_{\alpha, \varphi} \neq 0 \) (see Exercise 4). Moreover \( \mathcal{R} \) [and \( \mathcal{W}(R) \)] must be contained in \( \mathcal{O} \), for as we have seen many times, \( \mathcal{O} \) is a direct sum of some of these unique two-sided ideals and \( \ker P \) is the direct sum of the remaining ones so that if \( V_{\alpha} \neq 0 \) it means that \( \psi(\varepsilon_1) \notin \ker P \) and hence \( \mathcal{R} \subseteq \mathcal{O} \).

By Example 2.2 in Section 6.2 we can obtain \( x \neq 0 \) such that

\[
x = \varepsilon_1 x \varepsilon_2, \quad \psi(x) = \psi(\varepsilon_2) \psi(x) \psi(\varepsilon_1) \in \psi(\varepsilon_2) U(S_m) \subseteq \psi(R) \subseteq \mathcal{O}
\]

and \( P(\psi(x)) \neq 0 \). We have

\[
P(\psi(x)) \psi_{\alpha} = P(\psi(x)) \psi_{\alpha} \otimes V
\]

\[
= P(\psi(\varepsilon_2)) P(\psi(x)) P(\psi(\varepsilon_1)) \otimes V
\]

\[
\subseteq \psi_{\alpha, \varphi}.
\]

Thus \( P(\psi(x)) : \psi_{\alpha} \twoheadrightarrow \psi_{\alpha, \varphi} \) is a linear map, it is nonzero, it is a symmetry operator, and finally, it commutes with every \( \Pi'(T) \). But then by Theorem 1.8(b), Section 6.1,

\[
\Pi'(T) |_{\psi_{\alpha}} \sim \Pi'(T) |_{\psi_{\alpha, \varphi}}
\]

and in fact the linking transformation is precisely \( P(\psi(x))|_{\psi_{\alpha}} \).

This completes the argument, but the reader should realize that this proof is actually a recapitulation of the proof of
Theorem 2.16, Section 6.2.

8. Prove that $K_{\alpha, \varphi}(T)x_1 \otimes \cdots \otimes x_m = Tx_1 \otimes \cdots \otimes Tx_m$ ($\varphi$-star product).

Hint:

$$K_{\alpha, \varphi}(T)P_{\alpha, \varphi}x_1 \otimes \cdots \otimes x_m = \Pi(T)P_{\alpha, \varphi}x_1 \otimes \cdots \otimes x_m = P_{\alpha, \varphi}\Pi(T)x_1 \otimes \cdots \otimes x_m = Tx_1 \otimes \cdots \otimes Tx_m.$$

9. Assume that every column in the frame corresponding to $\alpha$ has length at most $n = \dim V$ and that not all columns have the same length. Prove that $\dim V_{\alpha}^{\otimes} > 1$. Also prove that $\dim V_{\alpha}^{\otimes} > 1$ when every column in the frame corresponding to $\alpha$ has length $k < n$.

Hint: We use Exercise 6 and prove that $\dim V_{\alpha, \varphi}^{\otimes} > 1$ for an appropriate choice of $\varphi$. Suppose that the first $r$ columns of the frame corresponding to $\alpha$ have length $k$ ($\leq n$) and the $(r+1)^{st}$ column has length $h$, $h < k$. Then obtain $\varphi$ so that

<table>
<thead>
<tr>
<th>col 1</th>
<th>col 2</th>
<th>\cdots</th>
<th>col r</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$k+1$</td>
<td>\cdots</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>\cdots</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\ddots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$k$</td>
<td>$2k$</td>
<td>\cdots</td>
<td>$rk$</td>
</tr>
</tbody>
</table>

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It suffices to prove that there are at least two l.i. tensors in $\text{Im} \ P(\nu(D_{\alpha,\varphi}))$. First let $e_1, \ldots, e_n$ be a basis of $V$, $e_k = e_1 \otimes \cdots \otimes e_k$, and set

$$z = e_k \otimes \cdots \otimes e_k \otimes (e_1 \otimes \cdots \otimes e_h) \otimes (e_1 \otimes e_2 \otimes \cdots \otimes e_{p_1}) \otimes \cdots \otimes (e_1 \otimes \cdots \otimes e_{p_L})$$

where $L$ is the number of columns after column $r+1$ and $p_1, p_2, \ldots, p_L$ are the lengths of these columns. Then clearly

$$P(\nu(D_{\alpha,\varphi}))z = \mu z$$

where $\mu$ is a product of factorials depending on the row lengths. Also

$$P(c(D_{\alpha,\varphi}))z = \tau e_k^\wedge \otimes \cdots \otimes e_k^\wedge \otimes e_h^\wedge \otimes e_{p_1}^\wedge \otimes \cdots \otimes e_{p_L}^\wedge$$

where

$$e_k^\wedge = e_1 \wedge \cdots \wedge e_k, \quad e_h^\wedge = e_1 \wedge \cdots \wedge e_h, \quad e_{p_i}^\wedge = e_1 \wedge \cdots \wedge e_{p_i}, \quad i = 1, \ldots, L$$

and $\tau$ is a positive integer. Thus

$$P(\varepsilon(D_{\alpha,\varphi}))z = \mu \tau e_k^\wedge \otimes \cdots \otimes e_k^\wedge \otimes e_h^\wedge \otimes e_{p_1}^\wedge \otimes \cdots \otimes e_{p_L}^\wedge (104)$$

There is a convenient way of visualizing the action of $P(\varepsilon(D_{\alpha,\varphi}))$ on $z$. In a frame corresponding to $\alpha$ write
arbitrary vectors $x_1, \ldots, x_k$ down the columns as follows:

\[
\begin{array}{cccccc}
 x_1 & x_1 & x_1 & x_1 & x_1 \\
 x_2 & x_2 & x_2 & x_2 & \ddots \\
 \vdots & \vdots & \vdots & \vdots & \ddots & x_h \\
 x_k & x_k & x_k & x_k & \ddots & \ddots \\
\end{array}
\]

(105)

If

\[
w^\otimes = \left( x_1 \otimes \cdots \otimes x_k \right) \otimes \left( x_1 \otimes \cdots \otimes x_k \right) \otimes \cdots \otimes \left( x_1 \otimes \cdots \otimes x_k \right)
\]

\[
\otimes \left( x_1 \otimes \cdots \otimes x_h \right) \otimes \left( x_1 \otimes \cdots \otimes x_{p_1} \right) \otimes \cdots \otimes \left( x_1 \otimes \cdots \otimes x_{p_{\ell}} \right),
\]

then

\[
P(\mathcal{E}(D_{\alpha,\psi}))w^\otimes = \delta \left( x_1 \wedge \cdots \wedge x_k \right) \otimes \cdots \otimes \left( x_1 \wedge \cdots \wedge x_k \right) \otimes \left( x_1 \wedge \cdots \wedge x_h \right)
\]

\[
\otimes \left( x_1 \wedge \cdots \wedge x_{p_1} \right) \otimes \cdots \otimes \left( x_1 \wedge \cdots \wedge x_{p_{\ell}} \right)
\]

(106)

where $\delta$ is a positive integer. Thus the exterior product of the vectors in each column of (105) is computed, and the tensor product of these exterior products is formed in the formula for $P(\mathcal{E}(D_{\alpha,\psi}))w^\otimes$. Now construct a new arrangement of the vectors $x_1, \ldots, x_k$ in which $x_1$ and $x_k$ are interchanged but the rest of the vectors remain in their original positions in (105). Thus the new arrangement is
If

\[ u^\otimes = \left( x_k \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_1 \right) \otimes \cdots \otimes \left( x_k \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_1 \right) \]

then by (106)

\[ P(\mathcal{E}(D_{\alpha}, \Phi))u^\otimes = \pm \delta(x_1 \wedge x_2 \wedge \cdots \wedge x_k) \otimes \cdots \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_k) \]

\[ \otimes (x_k \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_h) \otimes (x_k \wedge x_2 \wedge \cdots \wedge x_p) \]

\[ \otimes \cdots \otimes (x_k \wedge x_2 \wedge \cdots \wedge x_p) \].

(108)

In (108) specialize \( x_i \) to \( e_i \), \( i = 1, \ldots, k \) so that

\[ P(\mathcal{E}(D_{\alpha}, \Phi))u^\otimes = \pm \delta e_1^\wedge \otimes \cdots \otimes e_k^\wedge (e_k \wedge e_2 \wedge \cdots \wedge e_h) \]

\[ \otimes (e_k \wedge e_2 \wedge \cdots \wedge e_p) \otimes \cdots \otimes (e_k \wedge \cdots \wedge e_p) \].

(109)

Now let \( f_1, \ldots, f_n \) be a basis of \( V^\ast \) dual to \( e_1, \ldots, e_n \), let
\( \psi = (f_1 \ldots f_k) \ldots (f_1 \ldots f_k)(f_1 \ldots f_h)(f_1 \ldots f_{p_1}) \ldots (f_1 \ldots f_{p_k}) \),

(110)

and factor \( \psi \) with a linear \( H_1 : \otimes V \rightarrow R \). Then a simple computation using (104) shows that

\[
H_1(P(\mathcal{E}(D_{\alpha}, \varphi))^\otimes) = \mu \tau \neq 0.
\]

(111)

On the other hand

\[
H_1(P(\mathcal{E}(D_{\alpha}, \varphi))^\otimes) = \pm \delta \det \begin{bmatrix}
  f_1(e_k) & f_2(e_k) & \cdots & f_h(e_k) \\
  f_1(e_2) & f_2(e_2) & \cdots & f_h(e_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(e_h) & f_2(e_h) & \cdots & f_h(e_h)
\end{bmatrix}^\otimes
\]

(112)

where the dots in (112) indicate products of appropriate determinants. But since \( k > h \), it follows that the first row of the determinant exhibited in (112) is 0 and hence the value of (112) is 0. The tensor (109) is not 0 and thus we have exhibited a linear functional \( H_1 \) that annihilates the tensor (109) and is nonzero on the tensor (104). Thus \( \dim \psi^\otimes > 1 \).

We remark that in fact by interchanging \( x_1 \) with \( x_{k-1} \), \( x_1 \) with \( x_{k-2} \), \ldots, \( x_1 \) with \( x_{h+1} \), each time repeating the preceding argument, it is easy to show that \( \dim \psi^\otimes > k - h \). We leave the proof of the second assertion to the reader.
10. Prove that $e_1^2 = e_1$ and $e_1 e_2 = e_2 e_1 = 0$ as asserted in the proof of Theorem 3.7.

Hint: Since $e_1 \in \mathbb{H}(S) e$, write $e_1 = ae$, $a \in \mathbb{H}(S)$. Then $e_1 = ae = (ae)e = ae(e_1 + e_2) = (ae)e_1 + (ae)e_2$. However, such a representation of an element in $A = A_1 + A_2$ is unique so $e_1 = (ae)e_1 = e_1^2$. Similarly, write $e_2 = be$ so that $e_2 = be = (be)e = be(e_1 + e_2) = (be)e_1 + (be)e_2$. Thus $e_2 = (be)e_2 = e_2^2$, $e_2 e_1 = (be)e_1 = 0$.

11. Let $\dim V = n$, $e_1, \ldots, e_n$ be a basis of $V$ and let $v_k = \sum_{j=1}^{n} a_{kj} e_j$, $k = 1, \ldots, n$. The following bracket notation is classical for denoting the determinant of the coefficients:

$$\det(a_{kj}) = [v_1 \cdots v_n].$$

Let $m = dn$ and assume each column of $D_{\alpha, \phi}$ has length $n$, i.e., $\dim v_{\alpha, \phi}^\otimes = 1$ by Exercise 9. Let $x_1, \ldots, x_{nd}$ be arbitrary vectors in $V$ and let $F$ be a nonzero linear functional, $F: v_{\alpha, \phi}^\otimes \to \mathbb{R}$ (all linear functionals on $v_{\alpha, \phi}^\otimes$ are multiples of $F$ because $\dim v_{\alpha, \phi}^\otimes = 1$). Prove that

$$F(D_{\alpha, \phi} x_{\sigma}^\otimes) = \sum_{\sigma \in S_m} b_{\sigma} [x_{\sigma(1)} \cdots x_{\sigma(n)}] [x_{\sigma(n+1)} \cdots x_{\sigma(2n)}]$$

$$\cdots [x_{\sigma((d-1)n+1)} \cdots x_{\sigma(dn)}],$$

(115)

where each coefficient $b_{\sigma} \in \mathbb{R}$ in (113) depends only on $\alpha, \phi$.  

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and $F$ and not on the vectors $x_1, \ldots, x_{nd}$.

Hint: For a given $p \in \mathcal{R}(D_{\alpha, \phi})$ let $z^p_{k_1}, \ldots, z^p_{k_n}$ denote the vectors that appear in $x^p_{(1)} \otimes \cdots \otimes x^p_{(m)}$ in positions numbered $\phi(k), \ldots, \phi((n-1)d+k)$, i.e., these are the positions that correspond to the integers appearing in the $k^{th}$ column of $D_{\alpha, \phi}$. Then

$$p_{\alpha, \phi}^x = \gamma^{-1} p(e(D_{\alpha, \phi})) \sum_{p \in \mathcal{R}(D_{\alpha, \phi})} \cdots \otimes \cdots \otimes z^p_{k_1}$$

$$\phi((n-1)d+k)$$

$$\cdots \otimes z^p_{k_n} \otimes \cdots \otimes$$

$$= \gamma^{-1} \sum_{p \in \mathcal{R}(D_{\alpha, \phi})} \sum_{q_1, \ldots, q_d \in S_n} \varepsilon(q_1) \cdots \varepsilon(q_d) \otimes$$

$$\phi(k)$$

$$\phi((n-1)d+k)$$

$$\cdots \otimes z^p_{kq_k(1)} \otimes \cdots \otimes z^p_{kq_k(n)} \otimes \cdots \otimes$$

(114)

Next let $f_1, \ldots, f_n$ be dual to $e_1, \ldots, e_n$ and let $\psi$ be the multilinear functional obtained by forming a product of the $f'$s in which $f_1, \ldots, f_n$ appear in positions $\phi(k), \ldots, \phi((n-1)d+k)$ respectively, for each $k = 1, \ldots, d$.

Then let $F_1$ factor $\psi$, i.e., $\psi = F_1 \otimes$ and denote by $F_2$ the restriction of $F_1$ to $\psi_{\alpha, \phi}^x$. We compute from (114) that

$$F_2(p_{\alpha, \phi}^x) = \gamma^{-1} \prod_{p \in \mathcal{R}(D_{\alpha, \phi})} \sum_{k=1}^d \sum_{q_k \in S_n} \varepsilon(q_k) f_1(z^p_{kq_k(1)}) \cdots f_n(z^p_{kq_k(n)})$$

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\[ \gamma^{-1} \sum_{p} \prod_{k=1}^{d} [z_{k1}^p \cdots z_{kn}^p]. \]  \hspace{1cm} (115)

Observe that if we choose the vectors \( x_1, \ldots, x_m \) so that in positions \( \varphi(1), \ldots, \varphi(d) \) in \( x^\otimes \), \( e_1 \) occurs, in positions \( \varphi(d+1), \ldots, \varphi(2d) \) in \( x^\otimes \), \( e_2 \) occurs, \ldots, in positions \( \varphi((n-1)d+1), \ldots, \varphi(nd) \) in \( x^\otimes \), \( e_n \) occurs, then clearly \( z_{k1}^p = e_1, \ldots, z_{kn}^p = e_n \) independently of \( p \in R(D_{\alpha, \varphi}) \) and the value of each of the brackets in (115) is 1. Thus \( F_2 \) is not 0 on \( V_{\alpha, \varphi}^\otimes \) and since \( \dim V_{\alpha, \varphi}^\otimes = 1 \) we conclude that \( F = aF_2 \), \( a \in R \). Now for any fixed \( p \in R(D_{\alpha, \varphi}), \ z_{11}^p, \ldots, z_{1n}^p, z_{21}^p, \ldots, z_{2n}^p, \ldots, z_{dn}^p \) is simply a permutation of the vectors \( x_1, \ldots, x_n \) and we have established (113).

12. **(First main theorem of invariant theory)** If \( \psi : \chi V \to R \) is a nonzero multilinear functional then it is called a **simultaneous multilinear relative invariant of** \( \text{GL}_n(V) \) if for any \( x_1, \ldots, x_m \) in \( V \) and any \( T \in \text{GL}_n(V) \)

\[ \psi(Tx_1, \ldots, Tx_m) = \chi(T) \psi(x_1, \ldots, x_m) \] \hspace{1cm} (116)

where \( \chi : \text{GL}_n(V) \to R \) is an abelian character. Prove that if such a \( \psi \) exists then \( m = nd \) and \( \psi \) has precisely the form given in (113).

**Hint:** Let \( f : \otimes^l V \to R \) factor \( \psi \) so that
\[ f(x^\otimes) = \psi(x_1, \ldots, x_m). \]  

(117)

Then

\[ f(\Pi'(T)x^\otimes) = f(Tx_1 \otimes \cdots \otimes Tx_m) \]

\[ = \psi(Tx_1, \ldots, Tx_m) \]

\[ = \chi(T)\psi(x_1, \ldots, x_m) \]

\[ = \chi(T)f(x^\otimes), \]  

(118)

for arbitrary \( x_1, \ldots, x_m \) in \( V \). Both sides of (118) are linear in \( x^\otimes \) so that (118) holds with \( x^\otimes \) replaced by an arbitrary \( z \in \bigotimes V \). But then for any \( \alpha \) and \( \varphi \),

\[ f(\Pi'(T)P^\alpha_\varphi x^\otimes) = \chi(T)f(P^\alpha_\varphi x^\otimes), \]

or in view of (102)

\[ f^\alpha_\varphi K^\alpha_\varphi (T) = \chi(T)f^\alpha_\varphi \]  

(119).

where \( f^\alpha_\varphi = f|_{V^\otimes} \). The formula (119) states that \( f^\alpha_\varphi \)

links \( K^\alpha_\varphi \) and \( \chi \). Now we know that

\[ \bigotimes^m \frac{V}{\varphi_1} = \sum_{\alpha_\varphi} \frac{V^\otimes}{\alpha_\varphi} \]  

(120)

is a decomposition of \( \bigotimes V \) into absolutely irreducible invariant subspaces of \( \Pi' \). (The dash on the summation indicates that the sum is over only those partitions \( \alpha \) and permutations
φ for which the corresponding symmetry classes \( V^\otimes_{\alpha, \phi} \) are nonzero. An equivalent statement of (120) is

\[
x^\otimes = \sum_{\alpha, \phi} ^{'} P_{\alpha, \phi} x^\otimes .
\] (121)

Corresponding to (120) we also have

\[
\Pi'(T) = \sum_{\alpha, \phi} ^{'} K_{\alpha, \phi}(T) .
\] (122)

Then from (118), (119) and (121) we have

\[
\chi(T) \psi(x_1, \ldots, x_m) = \epsilon \Pi'(T) \sum_{\alpha, \phi} ^{'} P_{\alpha, \phi} x^\otimes
\]

\[
= \sum_{\alpha, \phi} ^{'} K_{\alpha, \phi}(T) P_{\alpha, \phi} x^\otimes
\]

\[
= \sum_{\alpha, \phi} ^{'} \epsilon K_{\alpha, \phi}(T) P_{\alpha, \phi} x^\otimes
\]

\[
= \sum_{\alpha, \phi} ^{'} \epsilon K_{\alpha, \phi}(T) P_{\alpha, \phi} x^\otimes
\]

\[
= \sum_{\alpha, \phi} ^{'} \chi(T) \epsilon P_{\alpha, \phi} x^\otimes .
\] (123)

Now, each nonzero \( K_{\alpha, \phi} \) is absolutely irreducible as is
\( \chi(T) \) so that since \( f_{\alpha, \phi} \) links the two, it follows from Schur's Lemma (Section 6.1, Theorem 1.8) that either \( f_{\alpha, \phi} = 0 \) or \( f_{\alpha, \phi} \) is a bijection. We see from (123) that since \( \psi \neq 0 \) [and \( \chi(T) \neq 0 \)] it must be the case that \( f_{\alpha, \phi} \neq 0 \) for some \( \alpha \) and \( \phi \) so that \( \dim V^{\otimes}_{\alpha, \phi} = 1 \) (i.e., the degree of \( \chi \) is 1 and \( f_{\alpha, \phi} \) links \( \chi \) and \( K_{\alpha, \phi} \)). As we know (see Exercise 9) \( m = nd \) and every column of \( D_{\alpha, \phi} \) has length \( n \). Thus we can rewrite (123) [after cancelling \( \chi(T) \)] as

\[
\psi(x_1, \ldots, x_m) = \sum_{\alpha, \phi} f_{\alpha, \phi} x^{\otimes}_{\alpha, \phi}
\]  

(124)

in which the only terms in (124) are those for which \( m = nd \), each column of \( D_{\alpha, \phi} \) has length \( n \), and finally \( f_{\alpha, \phi} \) is a bijection between the field \( R \) [on which \( \chi(T) \) acts] and the 1-dimensional symmetry class \( V^{\otimes}_{\alpha, \phi} \). But then we immediately apply (113) to each term \( f_{\alpha, \phi} x^{\otimes}_{\alpha, \phi} \) in (124) and the argument is complete.

13. Show that the standard diagram

\[
D_{\alpha, e} : \begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\]

gives rise to an absolutely irreducible representation of degree 2 according to Theorem 3.10.

Hint: The corresponding \( \mathcal{E}(D_{\alpha, e}) \) will simply be denoted by \( e \) so that
\[ \varepsilon = [e + (1 2) + (3 4) + (1 2)(3 4)][e - (1 3) - (2 4) + (1 3)(2 4)]. \]

The minimal left ideal \( A_\alpha = \mathbb{H}(S_4)\varepsilon \) is clearly the image under the mapping \( \rho_\varepsilon : \mathbb{H}(S_4) \to A_\alpha \), \( \rho_\varepsilon(x) = x\varepsilon \). Now let \( T = \rho_\varepsilon(\varepsilon) \) so that \( T^2\varepsilon = x\varepsilon^2 = \gamma x\varepsilon = \gamma T \varepsilon \) and hence \( T \gamma \) is an idempotent linear transformation. Obviously \( \dim A_\alpha = \rho_{\gamma}(T) = \frac{1}{\gamma} \text{tr}(T) \) (here \( \rho \) denotes rank). The problem then is to compute both \( \text{tr}(T) \) and \( \gamma \). We use \( S_4 \) itself as the basis for \( \mathbb{H}(S_4) \) in order to obtain the matrix representation of \( T \). The main diagonal entry in the \( (\sigma, \sigma) \) position in \( [T]^S_{S_4} \) is the coefficient on \( \sigma \) in \( T\sigma = \sigma\varepsilon = \sigma \sum e(q)pq = \sum e(q)pq \). Now \( \sigma pq = \sigma p_1 q_1 \) iff \( p = p_1, \ p_2 q = q_2 \). Thus each element \( \sigma pq = \sigma \) iff \( p = q = e \) so the coefficient on \( \sigma \) in \( T\sigma = \sigma\varepsilon \) is precisely 1. Hence \( \text{tr}(T) = 24 \). Next we multiply to obtain

\[ \varepsilon = e - (1 3) - (2 4) + (1 3)(2 4) + (1 2) - (1 3 2) - (1 2 4) + (1 3 2 4) + (3 4) - (1 4 3) - (2 3 4) + (1 4 2 3) + (1 2)(3 4) - (1 4 3 2) - (1 2 3 4) + (1 4)(2 3). \]

Since \( \varepsilon^2 = \gamma \varepsilon \) it suffices to compute the coefficient of \( e \) in \( \varepsilon^2 \). Now observe that in computing \( \varepsilon^2 \) we have:

(a) The square of a 2-cycle, or product of disjoint 2-cycles, contributes 1 to \( e \) in \( \varepsilon^2 \). The total contribution obtained in this way is then 8.

(b) No square of a 3-cycle or a 4-cycle makes a
contribution.

(c) No product of a 2-cycle and a 3-cycle, a 3-cycle and a 4-cycle, a product of disjoint two cycles and either a 2-cycle, a 3-cycle or a 4-cycle can make a contribution.

(d) \((1\ 3\ 2\ 4)(1\ 4\ 2\ 3) = e\) and the product in the other order each contribute 1 to the coefficient of \(e\).

Similarly for \((1\ 4\ 3\ 2)(1\ 2\ 3\ 4) = e\) since both these permutations occur with a coefficient of -1 in \(\sigma\).

Thus from (a) and (d) we see that \(\gamma = 12\), \(\frac{\text{tr}(T)}{\gamma} = \frac{24}{12} = 2\), and \(\rho|A_\alpha\) is a representation of degree 2.

14. Determine the degrees of the other absolutely irreducible representations of \(S_4\) using the result of Exercise 13.

Hint: See Section 6.2, Theorem 2.21(c).
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