We now turn our attention to the problem of finding the basis in which a given linear transformation has the simplest possible representation. Such a representation is frequently called a **canonical form**. Although we would almost always like to find a basis in which the matrix representation of an operator is diagonal, this is in general impossible to do. Basically, in this chapter as well as in Chapters 8 and 10, we will try and find the general conditions that determine exactly what form it is possible for a representation to take.

In the present chapter, we focus our attention on eigenvalues and eigenvectors, which is probably the most important characterization of a linear operator that is available to us. We also treat the triangular form theorem from two distinct viewpoints. Our reason for this is that in this chapter we discuss both quotient spaces and nilpotent transformations, and the triangular form theorem is a good application of these ideas. However, since we also treat this theorem from an entirely different (and much simpler) point of view in the next chapter, the reader should feel free to skip Sections 7.10 to 7.12 if desired. (We also point out that Section 7.9 on quotient spaces is completely independent of the rest of this chapter, and may in fact be read immediately after Chapter 2.)

In Chapter 8 we give a complete discussion of canonical forms of matrices under similarity. All of the results that we prove in the present chapter for canonical forms of operators also follow from the development in Chapter 8. The reason for treating the “operator point of view” as well as the “matrix
point of view” is that the proof techniques and way of thinking can be quite different. The matrix point of view leads to more constructive and insightful proofs, while the operator point of view leads to techniques that are more likely to extend to infinite-dimensional analogs (although there is no complete extension to the infinite-dimensional version).

7.1 MINIMAL POLYNOMIALS

Let \( f = a_0 + a_1x + \cdots + a_nx^n \in \mathcal{F}[x] \) be any polynomial in the indeterminate \( x \). Then, given any linear operator \( T \in L(V) \), we define the linear operator \( f(T) \in L(V) \) as the polynomial in the operator \( T \) defined by substitution as

\[
f(T) = a_0 I + a_1 T + \cdots + a_n T^n
\]

where \( I \) is the identity transformation on \( V \). Similarly, given any matrix \( A \in M_m(\mathcal{F}) \), we define the matrix polynomial \( f(A) \) by

\[
f(A) = a_0 I + a_1 A + \cdots + a_n A^n
\]

where now \( I \) is the \( m \times m \) identity matrix. If \( T \) is such that \( f(T) = 0 \), then we say that \( T \) is a root or zero of the polynomial \( f \). This terminology also applies to a matrix \( A \) such that \( f(A) = 0 \).

If \( A \in M_m(\mathcal{F}) \) is the representation of \( T \in L(V) \) relative to some (ordered) basis for \( V \), then (in view of Theorem 5.13) we expect that \( f(A) \) is the representation of \( f(T) \). This is indeed the case.

**Theorem 7.1** Let \( A \) be the matrix representation of an operator \( T \in L(V) \). Then \( f(A) \) is the representation of \( f(T) \) for any polynomial \( f \in \mathcal{F}[x] \).

**Proof** This is Exercise 7.1.1. \( \blacksquare \)

The basic algebraic properties of polynomials in either operators or matrices are given by the following theorem.

**Theorem 7.2** Suppose \( T \in L(V) \) and let \( f, g \in \mathcal{F}[x] \). Then

(a) \( f(T)T = Tf(T) \).
(b) \( (f \pm g)(T) = f(T) \pm g(T) \).
(c) \( (fg)(T) = f(T)g(T) \).
(d) \( (cf)(T) = cf(T) \) for any \( c \in \mathcal{F} \).
Furthermore, these same results also hold for any matrix representation $A \in M_n(F)$.

**Proof**  In view of Theorem 6.1, we leave this as an easy exercise for the reader (see Exercise 7.1.2). □

From this theorem and the fact that the ring of polynomials is commutative, it should be clear that any two polynomials in the operator $T$ (or matrix $A$) also commute.

This discussion is easily generalized as follows. Let $A$ be any algebra over $F$ with unit element $e$, and let $f = a_0 + a_1 x + \cdots + a_n x^n$ be any polynomial in $F[x]$. Then for any $\alpha \in A$ we define

$$f(\alpha) = a_0 e + a_1 \alpha + \cdots + a_n \alpha^n \in A.$$ 

If $f(\alpha) = 0$, then $\alpha$ is a root of $f$ and we say that $\alpha$ satisfies $f$. We now show that in fact every $\alpha \in A$ satisfies some nontrivial polynomial in $F[x]$. Recall that by definition, an algebra $A$ is automatically a vector space over $F$.

**Theorem 7.3**  Let $A$ be an algebra (with unit element $e$) of dimension $m$ over $F$. Then every element $\alpha \in A$ satisfies some nontrivial polynomial in $F[x]$ of degree at most $m$.

**Proof**  Since $\dim A = m$, it follows that for any $\alpha \in A$, the $m + 1$ elements $e, \alpha, \alpha^2, \ldots, \alpha^m \in A$ must be linearly dependent (Theorem 2.6). This means there exist scalars $a_0, a_1, \ldots, a_m \in F$ not all equal to zero such that

$$a_0 e + a_1 \alpha + \cdots + a_m \alpha^m = 0.$$ 

But then $\alpha$ satisfies the nontrivial polynomial

$$f = a_0 + a_1 x + \cdots + a_m x^m \in F[x]$$ 

which is of degree at most $m$. □

**Corollary**  Let $V$ be a finite-dimensional vector space over $F$, and suppose $\dim V = n$. Then any $T \in L(V)$ satisfies some nontrivial polynomial $g \in F[x]$ of degree at most $n^2$. 


Proof  By Theorem 5.8, \( L(V) \) is an algebra over \( F \), and by Theorem 5.4 we have \( \dim L(V) = \dim L(V, V) = n^2 \). The corollary now follows by direct application of Theorem 7.3.

While this corollary asserts that any \( T \in L(V) \) always satisfies some polynomial \( g \in F[x] \) of degree at most \( n^2 \), we shall see a little later on that \( g \) can be chosen to have degree at most \( n \) (this is the famous Cayley-Hamilton theorem).

Now, for a given \( T \in L(V) \), consider the set of all \( f \in F[x] \) with the property that \( f(T) = 0 \). This set is not empty by virtue of the previous corollary. Hence (by well-ordering) we may choose a polynomial \( p \in F[x] \) of least degree with the property that \( p(T) = 0 \). Such a polynomial is called a **minimal polynomial** for \( T \) over \( F \). (We will present an alternative definition in terms of ideals in Section 7.4.)

**Theorem 7.4**  Let \( V \) be finite-dimensional and suppose \( T \in L(V) \). Then there exists a unique monic polynomial \( m \in F[x] \) such that \( m(T) = 0 \) and, in addition, if \( q \in F[x] \) is any other polynomial such that \( q(T) = 0 \), then \( m \mid q \).

Proof  The existence of a minimal polynomial \( p \in F[x] \) was shown in the previous paragraph, so all that remains is to prove the uniqueness of a particular (i.e., monic) minimal polynomial. Suppose

\[
p = a_0 + a_1 x + \cdots + a_n x^n
\]

so that \( \deg p = n \). Multiplying \( p \) by \( a_n^{-1} \) we obtain a monic polynomial \( m \in F[x] \) with the property that \( m(T) = 0 \). If \( m' \) is another distinct monic polynomial of degree \( n \) with the property that \( m'(T) = 0 \), then \( m - m' \) is a nonzero polynomial of degree less than \( n \) (since the leading terms cancel) that is satisfied by \( T \), thus contradicting the definition of \( n \). This proves the existence of a unique monic minimal polynomial.

Now let \( q \) be another polynomial satisfied by \( T \). Applying the division algorithm we have \( q = mg + r \) where either \( r = 0 \) or \( \deg r < \deg m \). Substituting \( T \) into this equation and using the fact that \( q(T) = 0 \) and \( m(T) = 0 \) we find that \( r(T) = 0 \). But if \( r \neq 0 \), then we would have a polynomial \( r \) with \( \deg r < \deg m \) such that \( r(T) = 0 \), contradicting the definition of \( m \). We must therefore have \( r = 0 \) so that \( q = mg \), and hence \( m \mid q \).

From now on, all minimal polynomials will be assumed to be monic unless otherwise noted. Furthermore, in Section 7.3 we will show (as a consequence of the Cayley-Hamilton theorem) the existence of a minimal polynomial for matrices. It then follows as a consequence of Theorem 7.1 that
any $T \in L(V)$ and its corresponding matrix representation $A$ both have the same minimal polynomial (since $m(T) = 0$ if and only if $m(A) = 0$).

Recall that $T \in L(V)$ is invertible if there exists an element $T^{-1} \in L(V)$ such that $TT^{-1} = T^{-1}T = 1$ (where $1$ is the identity element of $L(V)$). It is interesting to note that for any invertible $T \in L(V)$, its inverse $T^{-1}$ is actually a polynomial in $T$. This fact is essentially shown in the proof of the next theorem.

**Theorem 7.5** Let $V$ be finite-dimensional over $\mathcal{F}$. Then $T \in L(V)$ is invertible if and only if the constant term in the minimal polynomial for $T$ is not equal to zero.

**Proof** Let the minimal polynomial for $T$ over $\mathcal{F}$ be

$$m = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n.$$  

We first assume that $a_0 \neq 0$. Since $m$ is the minimal polynomial for $T$, we have

$$m(T) = a_01 + a_1T + \cdots + a_{n-1}T^{n-1} + T^n = 0$$

and hence multiplying by $a_0^{-1}$ and using Theorem 7.2 yields

$$0 = 1 + a_0^{-1}T(a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1})$$

or

$$1 = T[-a_0^{-1}(a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1})]$$

$$= [-a_0^{-1}(a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1})]T.$$  

This shows that $T^{-1} = -a_0^{-1}(a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1})$, and hence $T$ is invertible.

Now suppose $T$ is invertible, but that $a_0 = 0$. Then we have

$$0 = a_1T + a_2T^2 + \cdots + a_{n-1}T^{n-1} + T^n$$

$$= (a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1})T.$$  

Multiplying from the right by $T^{-1}$ yields

$$0 = a_1 + a_2T + \cdots + a_{n-1}T^{n-2} + T^{n-1}$$

and hence $T$ satisfies the polynomial $p = a_1 + a_2x + \cdots + a_{n-1}x^{n-2} + x^{n-1} \in \mathcal{F}[x]$. But $\deg p = n - 1 < n$ which contradicts the definition of $m$ as the minimal polynomial. Therefore we must have $a_0 \neq 0$.  ■
**Corollary**  Let $V$ be finite-dimensional over $\mathcal{F}$, and assume that $T \in \text{L}(V)$ is invertible. Then $T^{-1}$ is a polynomial in $T$ over $\mathcal{F}$.

**Proof**  If $T$ is invertible, then $m(T) = a_0 I + a_1 T + \cdots + a_{n-1} T^{n-1} + T^n = 0$ with $a_0 \neq 0$. Multiplying by $a_0^{-1}$ then shows that

$$T^{-1} = -a_0^{-1} (a_1 I + a_2 T + \cdots + a_{n-1} T^{n-2} + T^{n-1}) .$$

While we have so far shown the existence of minimal polynomials, most readers would be hard-pressed at this point to actually find one given any particular linear operator. Fortunately, we will discover a fairly general method for finding the minimal polynomial of a matrix in Chapter 8 (see Theorem 8.10).

As we stated earlier, $V$ will always denote a finite-dimensional vector space over a field $\mathcal{F}$. In addition, we will let $I \in \text{L}(V)$ denote the identity transformation on $V$ (i.e., the unit element of $\text{L}(V)$), and we let $I \in M_n(\mathcal{F})$ be the identity matrix.

**Exercises**

1. Prove Theorem 7.1.

2. Prove Theorem 7.2.

3. Let $V$ be finite-dimensional over $\mathcal{F}$, and suppose $T \in \text{L}(V)$ is singular. Prove there exists a nonzero $S \in \text{L}(V)$ such that $ST = TS = 0$.

4. Suppose $V$ has a basis $\{e_1, e_2\}$. If $T \in \text{L}(V)$, then $Te_i = \sum_j a_{ij} e_j$ for some $(a_{ij}) \in M_2(\mathcal{F})$. Find a nonzero polynomial of degree 2 in $\mathcal{F}[x]$ satisfied by $T$.

5. Repeat the previous problem, but let $\dim V = 3$ and find a polynomial of degree 3.

6. Let $\alpha \in \mathcal{F}$ be fixed, and define the linear transformation $T \in \text{L}(V)$ by $T(v) = \alpha v$. This is called the **scalar mapping** belonging to $\alpha$. Show that $T$ is the scalar mapping belonging to $\alpha$ if and only if the minimal polynomial for $T$ is $m(x) = x - \alpha$. 
7.2 EIGENVALUES AND EIGENVECTORS

We now make a very important definition. If $T \in \mathcal{L}(V)$, then an element $\lambda \in \mathcal{F}$ is called an eigenvalue (also called a characteristic value or characteristic root) of $T$ if there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$. In this case, we call the vector $v$ an eigenvector (or characteristic vector) belonging to the eigenvalue $\lambda$. Note that while an eigenvector is nonzero by definition, an eigenvalue may very well be zero.

Throughout the remainder of this chapter we will frequently leave off the parentheses around vector operands. In other words, we sometimes write $Tv$ rather than $T(v)$. This simply serves to keep our notation as uncluttered as possible.

An important criterion for the existence of an eigenvalue of $T$ is the following.

**Theorem 7.6** A linear operator $T \in \mathcal{L}(V)$ has eigenvalue $\lambda \in \mathcal{F}$ if and only if $\lambda 1 - T$ is singular.

**Proof** Suppose $\lambda 1 - T$ is singular. By definition, this means there exists a nonzero $v \in V$ such that $(\lambda 1 - T)v = 0$. But this is just $Tv = \lambda v$. The converse should be quite obvious.

Note, in particular, that $0$ is an eigenvalue of $T$ if and only if $T$ is singular. In an exactly analogous manner, we say that an element $\lambda \in \mathcal{F}$ is an eigenvalue of a matrix $A \in \mathcal{M}_n(\mathcal{F})$ if there exists a nonzero (column) vector $v \in \mathcal{F}^n$ such that $Av = \lambda v$, and we call $v$ an eigenvector of $A$ belonging to the eigenvalue $\lambda$. Given a basis $\{e_i\}$ for $\mathcal{F}^n$, we can write this matrix eigenvalue equation in terms of components as

$$
\sum_{j=1}^{n} a_{ij} v_j = \lambda v_i, \quad i = 1, \ldots, n.
$$

Now suppose $T \in \mathcal{L}(V)$ and $v \in V$. If $\{e_1, \ldots, e_n\}$ is a basis for $V$, then $v = \sum_i v_i e_i$ and hence

$$
T(v) = T(\sum_i v_i e_i) = \sum_i v_i T(e_i) = \sum_{i,j} e_j a_{ij} v_i,
$$

where $A = (a_{ij})$ is the matrix representation of $T$ relative to the basis $\{e_i\}$. Using this result, we see that if $T(v) = \lambda v$, then

$$
\sum_{i,j} e_j a_{ij} v_i = \lambda \sum_i v_j e_j
$$
and hence equating components shows that $\sum a_{ij}v_i = \lambda v_j$. We thus see that (as expected) the isomorphism between $L(V)$ and $M_n(F)$ (see Theorem 5.13) shows that $\lambda$ is an eigenvalue of the linear transformation $T$ if and only if $\lambda$ is also an eigenvalue of the corresponding matrix representation $A$. Using the notation of Chapter 5, we can say that $T(v) = \lambda v$ if and only if $[T]_e[v]_e = \lambda [v]_e$.

**Example 7.1** Let us find all of the eigenvectors and associated eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$  

This means that we must find a vector $v = (x, y)$ such that $Av = \lambda v$. In matrix notation, this equation takes the form

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$  

This is equivalent to the system

$$(1-\lambda)x + 2y = 0$$

$$3x + (2-\lambda)y = 0$$  

(*)

Since this homogeneous system of equations has a nontrivial solution if and only if the determinant of the coefficient matrix is nonzero (Corollary to Theorem 4.13), we must have

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0.$$  

We thus see that the eigenvalues are $\lambda = 4$ and $\lambda = -1$. (The roots of this polynomial are found either by inspection, or by applying the quadratic formula proved following Theorem 6.14.)

Substituting $\lambda = 4$ into (*) yields

$$-3x + 2y = 0$$

$$3x - 2y = 0$$
or \( y = (3/2)x \). This means that every eigenvector corresponding to the eigenvalue \( \lambda = 4 \) has the form \( v = (x, 3x/2) \). In other words, every multiple of the vector \( v = (2, 3) \) is also an eigenvector with eigenvalue equal to 4. If we substitute \( \lambda = -1 \) in (*), then we similarly find \( y = -x \), and hence every multiple of the vector \( v = (1, -1) \) is an eigenvector with eigenvalue equal to -1. //

We will generalize this approach in the next section. However, let us first take a brief look at some of the relationships between the eigenvalues of an operator and the roots of its minimal polynomial.

**Theorem 7.7**  Let \( \lambda \) be an eigenvalue of \( T \in L(V) \). Then \( p(\lambda) \) is an eigenvalue of \( p(T) \) for any \( p \in \mathcal{F}[x] \).

**Proof**  If \( \lambda \) is an eigenvalue of \( T \), then there exists a nonzero \( v \in V \) such that \( Tv = \lambda v \). But then

\[
T^2(v) = T(Tv) = T(\lambda v) = \lambda T(v) = \lambda^2 v
\]

and by induction, it is clear that \( T^k(v) = \lambda^k v \) for any \( k = 1, 2, \ldots \) If we define \( p = a_0 + a_1x + \cdots + a_m x^m \), then we have

\[
p(T) = a_0 I + a_1 T + \cdots + a_m T^m
\]

and hence

\[
p(T)v = a_0 v + a_1 \lambda v + \cdots + a_m \lambda^m v
\]

\[
= (a_0 + a_1 \lambda + \cdots + a_m \lambda^m )v
\]

\[
= p(\lambda)v.
\]

**Corollary**  Let \( \lambda \) be an eigenvalue of \( T \in L(V) \). Then \( \lambda \) is a root of the minimal polynomial for \( T \).

**Proof**  If \( m(x) \) is the minimal polynomial for \( T \), then \( m(T) = 0 \) by definition. From Theorem 7.7, we have \( m(\lambda)v = m(T)v = 0 \) where \( v \neq 0 \) is an eigenvector corresponding to \( \lambda \). But then \( m(\lambda) = 0 \) (see Theorem 2.1(b)) so that \( \lambda \) is a root of \( m(x) \).

Since any eigenvalue of \( T \) is a root of the minimal polynomial for \( T \), it is natural to ask about the number of eigenvalues that exist for a given \( T \in L(V) \). Recall from the corollary to Theorem 6.4 that if \( c \in \mathcal{F} \) is a root of \( f \in \mathcal{F}[x] \), then \( (x - c)|f \). If \( c \) is such that \( (x - c)^m | f \) but no higher power of \( x - c \) divides \( f \), then we say that \( c \) is a root of **multiplicity** \( m \). (The context should make it
clear whether we mean the multiplicity \( m \) or the minimal polynomial \( m(x) \). In counting the number of roots that a polynomial has, we shall always count a root of multiplicity \( m \) as \( m \) roots. A root of multiplicity 1 is frequently called a simple root.

If \( \dim V = n \) then, since the minimal polynomial \( m \) for \( T \in \text{L}(V) \) is of degree at most \( n^2 \) (Corollary to Theorem 7.3), there can be at most \( n^2 \) roots of \( m \) (Theorem 6.14). In particular, we see that any \( T \in \text{L}(V) \) has a finite number of distinct eigenvalues. Moreover, if the field over which \( V \) is defined is algebraically closed, then \( T \) will in fact have at least as many (not necessarily distinct) eigenvalues as is the degree of its minimal polynomial.

**Theorem 7.8** If \( v_1, \ldots, v_r \) are eigenvectors belonging to the distinct eigenvalues \( \lambda_1, \ldots, \lambda_r \) of \( T \in \text{L}(V) \), then the set \( \{v_1, \ldots, v_r\} \) is linearly independent.

**Proof** If \( r = 1 \) there is nothing to prove, so we proceed by induction on \( r \). In other words, we assume that the theorem is valid for sets of less than \( r \) eigenvectors and show that in fact it is valid for sets of size \( r \). Suppose that

\[
a_1 v_1 + \cdots + a_r v_r = 0
\]  

for some set of scalars \( a_i \in F \). We apply \( T \) to this relation to obtain

\[
a_1 T(v_1) + \cdots + a_r T(v_r) = a_1 \lambda_1 v_1 + \cdots + a_r \lambda_r v_r = 0 .
\]  

On the other hand, if we multiply (1) by \( \lambda_r \) and subtract this from (2), we find (since \( Tv_i = \lambda_i v_i \))

\[
a_i (\lambda_i - \lambda_r) v_1 + \cdots + a_{r-1} (\lambda_{r-1} - \lambda_r) v_{r-1} = 0 .
\]

By our induction hypothesis, the set \( \{v_1, \ldots, v_{r-1}\} \) is linearly independent, and hence \( a_i (\lambda_i - \lambda_r) = 0 \) for each \( i = 1, \ldots, r - 1 \). But the \( \lambda_i \) are distinct so that \( \lambda_i - \lambda_r \neq 0 \) for \( i \neq r \), and therefore \( a_i = 0 \) for each \( i = 1, \ldots, r - 1 \). Using this result in (1) shows that \( a_r = 0 \) (since \( v_r \neq 0 \) by definition), and therefore \( a_1 = \cdots = a_r = 0 \). This shows that the entire collection \( \{v_1, \ldots, v_r\} \) is independent. \( \blacksquare \)

**Corollary 1** Suppose \( T \in \text{L}(V) \) and \( \dim V = n \). Then \( T \) can have at most \( n \) distinct eigenvalues in \( F \).
Proof Since \( \dim V = n \), there can be at most \( n \) independent vectors in \( V \). Since \( n \) distinct eigenvalues result in \( n \) independent eigenvectors, this corollary follows directly from Theorem 7.8. ■

Corollary 2 Suppose \( T \in L(V) \) and \( \dim V = n \). If \( T \) has \( n \) distinct eigenvalues, then there exists a basis for \( V \) which consists of eigenvectors of \( T \).

Proof If \( T \) has \( n \) distinct eigenvalues, then (by Theorem 7.8) \( T \) must have \( n \) linearly independent eigenvectors. But \( n \) is the number of elements in any basis for \( V \), and hence these \( n \) linearly independent eigenvectors in fact form a basis for \( V \). ■

It should be remarked that one eigenvalue can belong to more than one linearly independent eigenvector. In fact, if \( T \in L(V) \) and \( \lambda \) is an eigenvalue of \( T \), then the set \( V_\lambda \) of all eigenvectors of \( T \) belonging to \( \lambda \) is a subspace of \( V \) called the eigenspace of \( \lambda \). It is also easy to see that \( V_\lambda = \ker(\lambda I - T) \) (see Exercise 7.2.1).

Exercises

1. (a) If \( T \in L(V) \) and \( \lambda \) is an eigenvalue of \( T \), show that the set \( V_\lambda \) of all eigenvectors of \( T \) belonging to \( \lambda \) is a \( T \)-invariant subspace of \( V \) (i.e., a subspace with the property that \( T(v) \in V_\lambda \) for all \( v \in V_\lambda \)).
   (b) Show that \( V_\lambda = \ker(\lambda I - T) \).

2. An operator \( T \in L(V) \) with the property that \( T^n = 0 \) for some \( n \in \mathbb{Z}^+ \) is said to be nilpotent. Show that the only eigenvalue of a nilpotent operator is 0.

3. If \( S, T \in L(V) \), show that \( ST \) and \( TS \) have the same eigenvalues. [\textit{Hint:} First use Theorems 5.16 and 7.6 to show that 0 is an eigenvalue of \( ST \) if and only if 0 is an eigenvalue of \( TS \). Now assume \( \lambda \neq 0 \), and let \( ST(v) = \lambda v \). Show that \( Tv \) is an eigenvector of \( TS \).]

4. (a) Consider the rotation operator \( R(\alpha) \in L(\mathbb{R}^2) \) defined in Example 1.2. Does \( R(\alpha) \) have any eigenvectors? Explain.
   (b) Repeat part (a) but now consider rotations in \( \mathbb{R}^3 \).

5. For each of the following matrices, find all eigenvalues and linearly independent eigenvectors:
7.2 EIGENVALUES AND EIGENVECTORS

(a) \[
\begin{pmatrix}
2 & 2 \\
1 & 3 \\
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
4 & 2 \\
3 & 3 \\
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
5 & -1 \\
1 & 3 \\
\end{pmatrix}
\]

6. Consider the spaces \( \mathcal{D}[\mathbb{R}] \) and \( \mathcal{F}[\mathbb{R}] \) defined in Exercise 2.1.6, and let \( d: \mathcal{D}[\mathbb{R}] \to \mathcal{F}[\mathbb{R}] \) be the usual derivative operator.
(a) Show that the eigenfunctions (i.e., eigenvectors) of \( d \) are of the form \( \exp(\lambda x) \) where \( \lambda \) is the corresponding eigenvalue.
(b) Suppose \( \lambda_1, \ldots, \lambda_r \in \mathbb{R} \) are distinct. Show that the set
\[
S = \{\exp(\lambda_1 x), \ldots, \exp(\lambda_r x)\}
\]
is linearly independent. [Hint: Consider the linear span of \( S \).]

7. Suppose \( T \in \mathcal{L}(V) \) is invertible. Show that \( \lambda \) is an eigenvalue of \( T \) if and only if \( \lambda \neq 0 \) and \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \).

8. Suppose \( T \in \mathcal{L}(V) \) and \( \dim V = n \). If \( T \) has \( n \) linearly independent eigenvectors, what can you say about the matrix representation of \( T \)?

9. Let \( V \) be a two-dimensional space over \( \mathbb{R} \), and let \( \{e_1, e_2\} \) be a basis for \( V \). Find the eigenvalues and eigenvectors of the operator \( T \in \mathcal{L}(V) \) defined by:
(a) \( Te_1 = e_1 + e_2 \quad Te_2 = e_1 - e_2 \).
(b) \( Te_1 = 5e_1 + 6e_2 \quad Te_2 = -7e_2 \).
(c) \( Te_1 = e_1 + 2e_2 \quad Te_2 = 3e_1 + 6e_2 \).

10. Suppose \( A \in M_n(\mathbb{C}) \) and define \( R_i = \sum_{j=1}^{n} |a_{ij}| \) and \( P_i = R_i - |a_{ii}| \).
(a) Show that if \( Ax = 0 \) for some nonzero \( x = (x_1, \ldots, x_n) \), then for any \( r \) such that \( x_r \neq 0 \) we have
\[
|a_{rr}| x_r = \left| \sum_{j \neq r} a_{jr} x_j \right|.
\]
(b) Show that part (a) implies that for some \( r \) we have \( |a_{rr}| \leq P_r \).
(c) Prove that if \( |a_{ii}| > P_i \) for all \( i = 1, \ldots, n \), then all eigenvalues of \( A \) are nonzero (or, equivalently, that \( \det A \neq 0 \)).

11. (a) Suppose \( A \in M_n(\mathbb{C}) \) and let \( \lambda \) be an eigenvalue of \( A \). Using the previous exercise prove \textbf{Gershgorin's Theorem}: \( |\lambda - a_{rr}| \leq P_r \) for some \( r \) with \( 1 \leq r \leq n \).
(b) Use this result to show that every eigenvalue \( \lambda \) of the matrix
\[
A = \begin{pmatrix}
4 & 1 & 1 & 0 & 1 \\
1 & 3 & 1 & 0 & 0 \\
1 & 2 & 3 & 1 & 0 \\
1 & 1 & -1 & 4 & 0 \\
1 & 1 & 1 & 1 & 5
\end{pmatrix}
\]

satisfies \(1 \leq |\lambda| \leq 9\).

### 7.3 CHARACTERISTIC POLYNOMIALS

So far our discussion has dealt only theoretically with the existence of eigenvalues of an operator \(T \in L(V)\). From a practical standpoint (as we saw in Example 7.1), it is much more convenient to deal with the matrix representation of an operator. Recall that the definition of an eigenvalue \(\lambda \in F\) and eigenvector \(v = \sum v_i e_i\) of a matrix \(A = (a_{ij}) \in M_n(F)\) is given in terms of components by \(\sum a_{ij} v_j = \lambda v_i\) for each \(i = 1, \ldots, n\). This may be written in the form

\[
\sum_{j=1}^{n} a_{ij} v_j = \lambda \sum_{j=1}^{n} \delta_{ij} v_j
\]

or, alternatively, as

\[
\sum_{j=1}^{n} (\lambda \delta_{ij} - a_{ij}) v_j = 0.
\]

In matrix notation, this is

\[
(\lambda I - A)v = 0.
\]

By the corollary to Theorem 4.13, this set of homogeneous equations has a nontrivial solution if and only if \(\det(\lambda I - A) = 0\).

Another way to see this is to note that by Theorem 7.6, \(\lambda\) is an eigenvalue of the operator \(T \in L(V)\) if and only if \(\lambda I - T\) is singular. But according to Theorem 5.16, this means that \(\det(\lambda I - T) = 0\) (recall that the determinant of a linear transformation \(T\) is defined to be the determinant of any matrix representation of \(T\)). In other words, \(\lambda\) is an eigenvalue of \(T\) if and only if \(\det(\lambda I - T) = 0\). This proves the following important result.

**Theorem 7.9** Suppose \(T \in L(V)\) and \(\lambda \in F\). Then \(\lambda\) is an eigenvalue of \(T\) if and only if \(\det(\lambda I - T) = 0\).
Let \([T]\) be a matrix representation of \(T\). The matrix \(xI - [T]\) is called the characteristic matrix of \([T]\), and the expression \(\det(xI - T) = 0\) is called the characteristic (or secular) equation of \(T\). The determinant \(\det(xI - T)\) is frequently denoted by \(\Delta_T(x)\). Writing out the determinant in a particular basis, we see that \(\det(xI - T)\) is of the form

\[
\Delta_T(x) = \begin{vmatrix}
-x + a_{11} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & x - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & x - a_{nn}
\end{vmatrix}
\]

where \(A = (a_{ij})\) is the matrix representation of \(T\) in the chosen basis. Since the expansion of a determinant contains exactly one element from each row and each column, we see that (see Exercise 7.3.1)

\[
\det(xI - T) = (x - a_{11})(x - a_{22})\cdots(x - a_{nn})
\]

+ terms containing \(n - 1\) factors of the form \(x - a_{ii}
\]

+\ldots+ terms with no factors containing \(x\)

\[
= x^n - (\text{Tr} A)x^{n-1} + \text{terms of lower degree in } x + (-1)^n \det A.
\]

This polynomial is called the characteristic polynomial of \(T\).

From the discussion following Theorem 5.18, we see that if \(A' = P^{-1}AP\) is similar to \(A\), then

\[
\det(xI - A') = \det(xI - P^{-1}AP) = \det[P^{-1}(xI - A)P] = \det(xI - A)
\]

(since \(\det P^{-1} = (\det P)^{-1}\)). We thus see that similar matrices have the same characteristic polynomial (the converse of this statement is not true), and hence also the same eigenvalues. Therefore the eigenvalues (not eigenvectors) of an operator \(T \in L(V)\) do not depend on the basis chosen for \(V\). Note also that according to Exercise 4.2.14, we may as well write \(\text{Tr} T\) and \(\det T\) (rather than \(\text{Tr} A\) and \(\det A\)) since these are independent of the particular basis chosen. Using this terminology, we may rephrase Theorem 7.9 as follows.

**Theorem 7.9'**  A scalar \(\lambda \in \mathcal{F}\) is an eigenvalue of \(T \in L(V)\) if and only if \(\lambda\) is a root of the characteristic polynomial \(\Delta_T(x)\).

Since the characteristic polynomial is of degree \(n\) in \(x\), the corollary to Theorem 6.14 tells us that if we are in an algebraically closed field (such as
If \( C \), then there must exist \( n \) roots. In this case, the characteristic polynomial may be factored into the form

\[
det(xI - T) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)
\]

where the eigenvalues \( \lambda_i \) are not necessarily distinct. Expanding this expression we have

\[
det(xI - T) = x^n - (\sum \lambda_i)x^{n-1} + \cdots + (-1)^n \lambda_1\lambda_2 \cdots \lambda_n.
\]

Comparing this with the above general expression for the characteristic polynomial, we see that

\[
\text{Tr} T = \sum_{i=1}^{n} \lambda_i
\]

and

\[
det T = \prod_{i=1}^{n} \lambda_i.
\]

It should be remembered that this result only applies to an algebraically closed field (or to any other field \( F \) as long as all \( n \) roots of the characteristic polynomial lie in \( F \)).

**Example 7.2**  Let us find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.
\]

The characteristic polynomial of \( A \) is given by

\[
\Delta_A(x) = \begin{vmatrix} x - 1 & -4 \\ -2 & x - 3 \end{vmatrix} = x^2 - 4x - 5 = (x - 5)(x + 1)
\]

and hence the eigenvalues of \( A \) are \( \lambda = 5, -1 \). To find the eigenvectors corresponding to each eigenvalue, we must solve \( Av = \lambda v \) or \( (\lambda I - A)v = 0 \). Written out for \( \lambda = 5 \) this is

\[
\begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

We must therefore solve the set of homogeneous linear equations
7.3 CHARACTERISTIC POLYNOMIALS

\[
\begin{align*}
4x - 4y &= 0 \\
-2x + 2y &= 0
\end{align*}
\]

which is clearly equivalent to the single equation \(x - y = 0\), or \(x = y\). This means that every eigenvector corresponding to the eigenvalue \(\lambda = 5\) is a multiple of the vector \((1, 1)\), and thus the corresponding eigenspace is one-dimensional.

For \(\lambda = -1\) we have
\[
\begin{pmatrix}
-2 & -4 \\
-2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

and the equation to be solved is (since both are the same) \(-2x - 4y = 0\). The solution is thus \(-x = 2y\) so that the eigenvector is a multiple of \((2, -1)\).

We now note that
\[
\text{Tr } A = 1 + 3 = 4 = \sum \lambda_i
\]
and
\[
\det A = 3 - 8 = -5 = \prod \lambda_i.
\]

It is also easy to see that these relationships hold for the matrix given in Example 7.1.

It is worth remarking that the existence of eigenvalues of a given operator (or matrix) depends on the particular field we are working with. For example, the matrix
\[
A = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
has characteristic polynomial \(x^2 + 1\) which has no real roots, but does have the complex roots \(\pm i\). In other words, \(A\) has no eigenvalues in \(\mathbb{R}\), but does have the eigenvalues \(\pm i\) in \(\mathbb{C}\) (see Exercise 7.3.6).

Returning to the general case of an arbitrary field, it is clear that letting \(\lambda = 0\) in \(\Delta_T(\lambda) = \det(\lambda I - T) = 0\) shows that the constant term in the characteristic polynomial of \(A\) is given by \(\Delta_T(0) = (-1)^n \det A\). In view of Theorems 4.6, 7.5 and the corollary to Theorem 7.7, we wonder if there are any relationships between the characteristic and minimal polynomials of \(T\). There are indeed, and the first step in showing some of these relationships is to prove that every \(A \in M_n(\mathbb{F})\) satisfies some nontrivial polynomial. This is the essential content of our next result, the famous Cayley-Hamilton theorem.

Before we prove this theorem, we should point out that we will be dealing with matrices that have polynomial entries, rather than entries that are simply elements of the field \(\mathbb{F}\). However, if we regard the polynomials as elements of
the field of quotients (see Section 6.5), then all of our previous results (for example, those dealing with determinants) remain valid. We shall elaborate on this problem in detail in the next chapter. Furthermore, the proof we are about to give is the standard one at this level. We shall find several other methods of proof throughout this text, including a remarkably simple one in the next chapter (see the discussion of matrices over the ring of polynomials).

**Theorem 7.10 (Cayley-Hamilton Theorem)**  Every matrix $A \in M_n(F)$ satisfies its characteristic polynomial.

**Proof**  First recall from Theorem 4.11 that any matrix $A \in M_n(F)$ obeys the relation

$$A(\text{adj } A) = (\det A)I_n$$

where $\text{adj } A$ is the matrix whose elements are the determinants of the minor matrices of $A$. In particular, the characteristic matrix $xI - A$ obeys

$$(xI - A)B(x) = \det(xI - A)I$$

where we let

$$B(x) = \text{adj}(xI - A).$$

Thus the entries of the $n \times n$ matrix $B(x)$ are polynomials in $x$ of degree $\leq n - 1$. For example, if

$$B(x) = \begin{pmatrix} x^2 + 2 & x & 3 \\ -x + 1 & 1 & 0 \\ 0 & 4 & x^2 \end{pmatrix}$$

then

$$B(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & x^2 + 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}x + \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}x^2.$$

Hence in general, we may write $B(x)$ in the form

$$B(x) = B_0 + B_1x + \cdots + B_{n-1}x^{n-1}$$

where each $B_i \in M_n(F)$.

Now write

$$\Delta_A(x) = \det(xI - A) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n.$$
Then Theorem 4.11 becomes

\[(xI - A)(B_0 + B_1x + \cdots + B_{n-1}x^{n-1}) = (a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n)I.\]

Equating powers of \(x\) in this equation yields

\[
\begin{align*}
-AB_0 &= a_0I \\
B_0 - AB_1 &= a_1I \\
B_1 - AB_2 &= a_2I \\
&\vdots \\
B_{n-2} - AB_{n-1} &= a_{n-1}I \\
B_{n-1} &= I
\end{align*}
\]

We now multiply the first of these equations by \(A^0 = I\), the second by \(A^1 = A\), the third by \(A^2\), \ldots, the \(n\)th by \(A^{n-1}\), and the last by \(A^n\) to obtain

\[
\begin{align*}
-AB_0 &= a_0I \\
AB_0 - A^2B_1 &= a_1A \\
A^2B_1 - A^3B_2 &= a_2A^2 \\
&\vdots \\
A^{n-1}B_{n-2} - A^nB_{n-1} &= a_{n-1}A^{n-1} \\
A^nB_{n-1} &= A^n
\end{align*}
\]

Adding this last group of matrix equations, we see that the left side cancels out and we are left with

\[0 = a_0I + a_1A + a_2A^2 + \cdots + a_{n-1}A^{n-1} + A^n.\]

This shows that \(A\) satisfies its characteristic polynomial. \(\blacksquare\)

In view of this theorem, we see that there exists a nonempty set of nonzero polynomials \(p(x) \in \mathcal{F}[x]\) such that \(p(A) = 0\) for any \(A \in M_n(\mathcal{F})\). (Alternatively, Theorem 7.3 and its corollary apply equally well to the algebra of matrices, although the degree is bounded by \(n^2\) rather than by \(n\).) As we did for linear transformations, we may define the \textbf{minimal polynomial} for \(A\) as that polynomial \(p(x)\) of least degree for which \(p(A) = 0\). We also noted following Theorem 7.4 that any \(T \in \text{L}(V)\) and its corresponding representation \(A \in M_n(\mathcal{F})\) satisfy the same minimal polynomial. Theorem 7.4 thus applies equally well to matrices, and hence there exists a unique monic
minimal polynomial \( m(x) \) for \( A \) such that \( m(A) = 0 \). In addition, \( m(x) \) divides every other polynomial which has \( A \) as a zero. In particular, since \( A \) satisfies \( \Delta_A(x) \), we must have \( m(x) | \Delta_A(x) \).

**Theorem 7.11** Suppose \( A \in M_n(F) \) and \( m(x) \) is the minimal polynomial for \( A \). Then \( m(x) | \Delta_A(x) \) and \( \Delta_A(x) | [m(x)]^n \).

**Proof** That \( m(x) | \Delta_A(x) \) was proved in the previous paragraph. Let \( m(x) = x^k + m_1x^{k-1} + \cdots + m_{k-1}x + m_k \). We define the matrices \( B_i \in M_n(F) \) by

\[
B_0 = I \\
B_1 = A + m_1I \\
B_2 = A^2 + m_1A + m_2I \\
\vdots \\
B_{k-1} = A^{k-1} + m_1A^{k-2} + \cdots + m_{k-1}I
\]

where \( I = I_n \). Working our way successively down this set of equations, we may rewrite them in the form

\[
B_0 = I \\
B_1 - AB_0 = m_1I \\
B_2 - AB_1 = m_2I \\
\vdots \\
B_{k-1} - AB_{k-2} = m_{k-1}I
\]

From the previous expression for \( B_{k-1} \), we multiply by \(-A\) and then add and subtract \( m_kI \) to obtain (using \( m(A) = 0 \))

\[
-AB_{k-1} = m_kI - (A^k + m_1A^{k-1} + \cdots + m_{k-1}A + m_kI) \\
= m_kI - m(A) \\
= m_kI
\]

Now define \( B(x) = x^{k-1}B_0 + x^{k-2}B_1 + \cdots + xB_{k-2} + B_{k-1} \) (which may be viewed either as a polynomial with matrix coefficients or as a matrix with polynomial entries). Then, using our previous results, we find that
\( (xI - A)B(x) = xB(x) - AB(x) \)
\[
= (x^k B_0 + x^{k-1}B_1 + \cdots + x^2 B_{k-2} + x B_{k-1}) \\
- (x^{k-1}AB_0 + x^{k-2}AB_1 + \cdots + xAB_{k-2} + AB_{k-1}) \\
= x^k B_0 + x^{k-1}(B_1 - AB_0) + x^{k-2}(B_2 - AB_1) \\
+ \cdots + x(B_{k-1} - AB_{k-2}) - AB_{k-1} \\
= x^k I + m_1 x^{k-1} I + m_2 x^{k-2} I + \cdots + m_{k-1} x I + m_k I \\
= m(x)I. \quad (*)
\]

Since the determinant of a diagonal matrix is the product of its (diagonal) elements (see the corollary to Theorem 4.5), we see that

\[
\det[m(x)I] = [m(x)]^n.
\]

Therefore, taking the determinant of both sides of (*) and using Theorem 4.8 we find that

\[
[\det(xI - A)] \det B(x) = \det[m(x)I] = [m(x)]^n.
\]

But \( \det B(x) \) is just some polynomial in \( x \), so this equation shows that \([m(x)]^n\) is some multiple of \( \Delta_A(x) = \det(xI - A) \). In other words, \( \Delta_A(x)[m(x)]^n \).

**Theorem 7.12** The characteristic polynomial \( \Delta_A(x) \) and minimal polynomial \( m(x) \) of a matrix \( A \in M_n(F) \) have the same prime factors.

**Proof** Let \( m(x) \) have the prime factor \( f(x) \) so that \( f(x)|m(x) \). Since we showed in the above discussion that \( m(x)|\Delta_A(x) \), it follows that \( f(x)|\Delta_A(x) \) and hence \( f(x) \) is a factor of \( \Delta_A(x) \) also. Now suppose that \( f(x)|\Delta_A(x) \). Theorem 7.11 shows that \( \Delta_A(x)[m(x)]^n \), and therefore \( f(x)|[m(x)]^n \). However, since \( f(x) \) is prime, Corollary 2' of Theorem 6.5 tells us that \( f(x)|m(x) \).

It is important to realize that this theorem does not say that \( \Delta_A(x) = m(x) \), but only that \( \Delta_A(x) \) and \( m(x) \) have the same prime factors. However, each factor can be of a different multiplicity in \( m(x) \) from what it is in \( \Delta_A(x) \). In particular, since \( m(x)|\Delta_A(x) \), the multiplicity of any factor in \( \Delta_A(x) \) must be greater than or equal to the multiplicity of the same factor in \( m(x) \). Since a linear factor (i.e., a factor of the form \( x - \lambda \)) is prime, it then follows that \( \Delta_A(x) \) and \( m(x) \) have the same roots (although of different multiplicities).

**Theorem 7.13** Suppose \( A \in M_n(F) \) and \( \lambda \in F \). Then \( \lambda \) is an eigenvalue of \( A \) if and only if \( \lambda \) is a root of the minimal polynomial for \( A \).
Proof By Theorem 7.9, \( \lambda \) is an eigenvalue of \( A \) if and only if \( \Delta_A(\lambda) = 0 \). But from the above remarks, \( \lambda \) is a root of \( \Delta_A(x) \) if and only if \( \lambda \) is a root of the minimal polynomial for \( A \).

An alternative proof of Theorem 7.13 is to note that since \( m(x) \mid \Delta_A(x) \), we may write \( \Delta_A(x) = m(x)p(x) \) for some polynomial \( p(x) \). If \( \lambda \) is a root of \( m(x) \), then \( \Delta_A(\lambda) = m(\lambda)p(\lambda) = 0 \) so that \( \lambda \) is also a root of \( \Delta_A(x) \). In other words, \( \lambda \) is an eigenvalue of \( A \). The converse is just the corollary to Theorem 7.7. If we use this proof, then Theorem 7.12 is essentially just a corollary of Theorem 7.13.

Using Theorem 7.13, we can give another proof of Theorem 7.5 which also applies to the characteristic polynomial of any \( T \in L(V) \). In particular, from Theorem 5.10 we see that \( T \) is invertible if and only if \( T \) is nonsingular if and only if 0 is not an eigenvalue of \( T \) (because this would mean that \( Tv = 0v = 0 \) for some \( v \neq 0 \)). But from Theorem 7.13, this is true if and only if 0 is not a root of the minimal polynomial for \( T \). Writing the minimal polynomial as \( m(x) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + x^k \), we then see that \( a_0 = m(0) \neq 0 \) as claimed.

Example 7.3 Consider the matrix \( A \) given by

\[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 5
\end{pmatrix}.
\]

The characteristic polynomial is given by

\[
\Delta_A(x) = \det(xI - A) = (x - 2)^3(x - 5)
\]

and hence Theorem 7.12 tells us that both \( x - 2 \) and \( x - 5 \) must be factors of \( m(x) \). Furthermore, it follows from Theorems 7.4 and 7.10 that \( m(x) \mid \Delta_A(x) \), and thus the minimal polynomial must be either \( m_3(x) \), \( m_2(x) \) or \( m_3(x) \) where

\[
m_3(x) = (x - 2)^3(x - 5).
\]

From the Cayley-Hamilton theorem we know that \( m_3(A) = \Delta_A(A) = 0 \), and it is easy to show that \( m_2(A) = 0 \) also while \( m_1(A) \neq 0 \). Therefore the minimal polynomial for \( A \) must be \( m_3(x) \).
We now turn our attention to one of the most important aspects of the existence of eigenvalues. Suppose that $T \in L(V)$ with $\dim V = n$. If $V$ has a basis $\{v_1, \ldots, v_n\}$ that consists entirely of eigenvectors of $T$, then the matrix representation of $T$ in this basis is defined by

$$T(v_i) = \sum_{j=1}^{n} v_j a_{ji} = \lambda_i v_i = \sum_{j=1}^{n} \delta_{ji} \lambda_j v_j$$

and therefore $a_{ji} = \delta_{ji} \lambda_j$. In other words, $T$ is represented by a diagonal matrix in a basis of eigenvectors. Conversely, if $T$ is represented by a diagonal matrix $a_{ji} = \delta_{ji} \lambda_j$ relative to some basis $\{v_i\}$, then reversing the argument shows that each $v_i$ is an eigenvector of $T$. This proves the following theorem.

**Theorem 7.14** A linear operator $T \in L(V)$ can be represented by a diagonal matrix if and only if $V$ has a basis consisting of eigenvectors of $T$. If this is the case, then the diagonal elements of the matrix representation are precisely the eigenvalues of $T$. (Note however, that the eigenvalues need not necessarily be distinct.)

If $T \in L(V)$ is represented in some basis $\{e_i\}$ by a matrix $A$, and in the basis of eigenvectors $v_i$ by a diagonal matrix $D$, then Theorem 5.18 tells us that $A$ and $D$ must be similar matrices. This proves the following version of Theorem 7.14, which we state as a corollary.

**Corollary 1** A matrix $A \in M_n(F)$ is similar to a diagonal matrix $D$ if and only if $A$ has $n$ linearly independent eigenvectors.

**Corollary 2** A linear operator $T \in L(V)$ can be represented by a diagonal matrix if $T$ has $n = \dim V$ distinct eigenvalues.

**Proof** This follows from Corollary 2 of Theorem 7.8. ■

Note that the existence of $n = \dim V$ distinct eigenvalues of $T \in L(V)$ is a sufficient but not necessary condition for $T$ to have a diagonal representation. For example, the identity operator has the usual diagonal representation, but its only eigenvalues are $\lambda = 1$. In general, if any eigenvalue has multiplicity greater than 1, then there will be fewer distinct eigenvalues than the dimension of $V$. However, in this case we may be able to choose an appropriate linear combination of eigenvectors in each eigenspace so that the matrix of $T$ will still be diagonal. We shall have more to say about this in Section 7.7.

We say that a matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$. If $P$ is a nonsingular matrix such that $D = P^{-1}AP$, then we say that $P$
diagonalizes $A$. It should be noted that if $\lambda$ is an eigenvalue of a matrix $A$ with eigenvector $v$ (i.e., $Av = \lambda v$), then for any nonsingular matrix $P$ we have

$$(P^{-1}AP)(P^{-1}v) = P^{-1}Av = P^{-1}\lambda v = \lambda(P^{-1}v).$$

In other words, $P^{-1}v$ is an eigenvector of $P^{-1}AP$. Similarly, we say that $T \in \text{L}(V)$ is diagonalizable if there exists a basis for $V$ that consists entirely of eigenvectors of $T$.

While all of this sounds well and good, the reader might wonder exactly how the transition matrix $P$ is to be constructed. Actually, the method has already been given in Section 5.4. If $T \in \text{L}(V)$ and $A$ is the matrix representation of $T$ in a basis $\{e_i\}$, then $P$ is defined to be the transformation that takes the basis $\{e_i\}$ into the basis $\{v_i\}$ of eigenvectors. In other words, $v_i = Pe_i = \sum_j e_j p_{ji}$. This means that the $i$th column of $(p_{ij})$ is just the $i$th eigenvector of $A$. The fact that $P$ must be nonsingular coincides with the requirement that $T$ (or $A$) have $n$ linearly independent eigenvectors $v_i$.

**Example 7.4** Referring to Example 7.2, we found the eigenvectors $v_1 = (1, 1)$ and $v_2 = (2, -1)$ belonging to the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$  

Then $P$ and $P^{-1}$ are given by

$$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

and

$$P^{-1} = \frac{\text{adj } P}{\det P} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$$

and therefore

$$D = P^{-1}AP = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}. $$

We see that $D$ is a diagonal matrix, and that the diagonal elements are just the eigenvalues of $A$. Note also that

$$D(P^{-1}v_1) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
with a similar result holding for $P^{-1}v_2$.

**Example 7.5** Let us show that the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable. The characteristic equation is $(x - 1)^2 = 0$, and hence there are two identical roots $\lambda = 1$. If there existed an eigenvector $v = (x, y)$, it would have to satisfy the equation $(\lambda I - A)v = 0$ or

$$\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since this yields $-2y = 0$, the eigenvectors must be of the form $(x, 0)$, and hence it is impossible to find two linearly independent such eigenvectors.

Note that the minimal polynomial for $A$ is either $x - 1$ or $(x - 1)^2$. But since $A - I \neq 0$, $m(x)$ must be $(x - 1)^2$. We will see later (Theorem 7.24) that a matrix is diagonalizable if and only if its minimal polynomial is a product of distinct linear factors.

**Exercises**

1. Suppose $T \in \text{L}(V)$ has matrix representation $A = (a_{ij})$, and dim $V = n$. Prove that

$$\det(xI - T) = x^n - (\text{Tr } A)x^{n-1} + \text{terms of lower degree in } x + (-1)^n \det A.$$

   [*Hint: Use the definition of determinant.*]

2. Suppose $T \in \text{L}(V)$ is diagonalizable. Show that the minimal polynomial $m(x)$ for $T$ must consist of distinct linear factors. [*Hint: Let $T$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ and consider the polynomial*]
f(x) = (x - \lambda_r) \cdots (x - \lambda_r).

Show that m(x) = f(x).

3. Prove by direct substitution that $\Delta_A(A) = 0$ if $A \in M_n(\mathbb{F})$ is diagonal.

4. Find, in the form $a_0 + a_1x + a_2x^2 + a_3x^3$, the characteristic polynomial of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

Show by direct substitution that A satisfies its characteristic polynomial.

5. If $T \in L(V)$ and $\Delta_T(x)$ is a product of distinct linear factors, prove that T is diagonalizable.

6. Consider the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 \\ 13 & -3 \end{pmatrix}$$

(a) Find all eigenvalues and linearly independent eigenvectors over $\mathbb{R}$.
(b) Find all eigenvalues and linearly independent eigenvectors over $\mathbb{C}$.

7. For each of the following matrices, find all eigenvalues, a basis for each eigenspace, and determine whether or not the matrix is diagonalizable:

$$(a) \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \quad (b) \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

8. Consider the operator $T \in L(\mathbb{R}^3)$ defined by

$$T(x, y, z) = (2x + y, y - z, 2y + 4z).$$

Find all eigenvalues and a basis for each eigenspace.

9. Let $A = (a_{ij})$ be a triangular matrix, and assume that all of the diagonal entries of A are distinct. Is A diagonalizable? Explain.
10. Suppose $A \in M_3(\mathbb{R})$. Show that $A$ can not be a zero of the polynomial $f = x^2 + 1$.

11. If $A \in M_n(\mathcal{F})$, show that $A$ and $A^T$ have the same eigenvalues.

12. Suppose $A$ is a block triangular matrix with square matrices $A_{ii}$ on the diagonal. Show that the characteristic polynomial of $A$ is the product of the characteristic polynomials of the $A_{ii}$.

13. Find the minimal polynomial of

$$
A = \begin{pmatrix}
2 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -2 & 4
\end{pmatrix}.
$$

14. For each of the following matrices $A$, find a nonsingular matrix $P$ (if it exists) such that $P^{-1}AP$ is diagonal:

(a) $A = \begin{pmatrix}
3 & 1 & 1 \\
2 & 4 & 2 \\
1 & 1 & 3
\end{pmatrix}$  
(b) $A = \begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & -1 \\
-1 & 1 & 4
\end{pmatrix}$  
(c) $A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$

15. Consider the following real matrix:

$$
A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
$$

Find necessary and sufficient conditions on $a$, $b$, $c$ and $d$ so that $A$ is diagonalizable.

16. Let $A$ be an idempotent matrix (i.e., $A^2 = A$) of rank $r$. Show that $A$ is similar to the matrix

$$
B = \begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix}.
$$

17. Let $V$ be the space of all real polynomials $f \in \mathbb{R}[x]$ of degree at most 2, and define $T \in L(V)$ by $Tf = f + f' + xf'$ where $f'$ denotes the usual derivative with respect to $x$. 
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(a) Write down the most obvious basis \( \{e_1, e_2, e_3\} \) for \( V \) you can think of, and then write down \([T]_e\).

(b) Find all eigenvalues of \( T \), and then find a nonsingular matrix \( P \) such that \( P^{-1}[T]_e P \) is diagonal.

18. Prove that any real symmetric \( 2 \times 2 \) matrix is diagonalizable.

19. (a) Let \( C \in M_2(\mathbb{C}) \) be such that \( C^2 = 0 \). Prove that either \( C = 0 \), or else \( C \) is similar over \( \mathbb{C} \) to the matrix

\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix}
\]

(b) Prove that \( A \in M_2(\mathbb{C}) \) is similar over \( \mathbb{C} \) to one of the following two types of matrices:

\[
\begin{pmatrix}
a & 0 \\
0 & b \\
\end{pmatrix}
\] or
\[
\begin{pmatrix}
a & 0 \\
1 & a \\
\end{pmatrix}
\]

20. Find a matrix \( A \in M_3(\mathbb{R}) \) that has the eigenvalues 3, 2 and 2 with corresponding eigenvectors \((2, 1, 1), (1, 0, 1)\) and \((0, 0, 4)\).

21. Is it possible for the matrix

\[
A = \begin{pmatrix}
3 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 3 & 7 \\
\end{pmatrix}
\]

to have the eigenvalues \(-1, 2, 3\) and \(5\)?

7.4 ANNihilators

The purpose of this section is to repeat our description of minimal polynomials using the formalism of ideals developed in the previous chapter. Our reasons for this are twofold. First, we will gain additional insight into the meaning and action of minimal polynomials. And second, these results will be of use in the next chapter when we discuss cyclic subspaces.

If \( V \) is a vector space of dimension \( n \) over \( \mathcal{F} \), then for any \( v \in V \) and any \( T \in L(V) \), the \( n + 1 \) vectors \( v, T(v), T^2(v), \ldots, T^n(v) \) must be linearly
dependent. This means there exist scalars \(a_0, \ldots, a_n \in \mathcal{F}\) not all equal to zero, such that

\[
\sum_{i=1}^{n} a_i T^i(v) = 0.
\]

If we define the polynomial \(f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{F}[x]\), we see that our relation may be written as \(f(T)(v) = 0\). In other words, given any (fixed) \(v \in V\) and \(T \in \text{L}(V)\), then there exists a polynomial \(f\) of degree \(\leq n\) such that \(f(T)(v) = 0\).

Now, for any fixed \(v \in V\) and \(T \in \text{L}(V)\), we define the set \(N_T(v)\) by

\[ N_T(v) = \{f(x) \in \mathcal{F}[x]: f(T)(v) = 0\}. \]

This set is called the **T-annihilator** of \(v\). If \(f_1, f_2 \in N_T(v)\), then we have

\[ [f_1(T) \pm f_2(T)](v) = f_1(T)(v) \pm f_2(T)(v) = 0 \]

and for any \(g(x) \in \mathcal{F}[x]\), we also have

\[ [g(T)f_1(T)](v) = g(T)[f_1(T)(v)] = 0. \]

This shows that \(N_T(v)\) is actually an ideal of \(\mathcal{F}[x]\). Moreover, from Theorem 6.8 and its corollary, we see that \(N_T(v)\) is a principal ideal, and hence has a unique monic generator, which we denote by \(m_v(x)\). By definition, this means that \(N_T(v) = m_v(x)\mathcal{F}[x]\), and since we showed above that \(N_T(v)\) contains at least one polynomial of degree less than or equal to \(n\), it follows from Theorem 6.2(b) that \(\deg m_v(x) \leq n\). We call \(m_v(x)\) the **minimal polynomial of the vector** \(v\) corresponding to the given transformation \(T\). (Many authors also refer to \(m_v(x)\) as the **T-annihilator** of \(v\), or the **order** of \(v\).) It is thus the unique monic polynomial of least degree such that \(m(T)(v) = 0\).

**Theorem 7.15** Suppose \(T \in \text{L}(V)\), and let \(v \in V\) have minimal polynomial \(m_v(x)\). Assume \(m_v(x)\) is reducible so that \(m_v(x) = m_1(x)m_2(x)\) where \(m_1(x)\) and \(m_2(x)\) are both monic polynomials of degree \(\geq 1\). Then the vector \(w = m_1(T)(v) \in V\) has minimal polynomial \(m_2(x)\). In other words, every factor of the minimal polynomial of a vector is also the minimal polynomial of some other vector.

**Proof** First note that

\[ m_2(T)(w) = m_2(T)[m_1(T)(v)] = m_v(T)(v) = 0 \]
and thus \( m_\nu(x) \in N_T(w) \). Then for any nonzero polynomial \( f(x) \in \mathcal{F}[x] \) with \( f(T)(w) = 0 \), we have \( f(T)[m_\nu(T)(v)] = 0 \), and hence \( f(x)m_\nu(x) \in N_T(v) = m_\nu(x) \mathcal{F}[x] \). Using this result along with Theorem 6.2(b), we see that

\[
\deg m_\nu \geq \deg m_\nu f = \deg m_\nu + \deg f
\]

and therefore \( \deg m_\nu \leq \deg f \). This shows that \( m_\nu(x) \) is the monic polynomial of least degree such that \( m_\nu(T)(w) = 0 \). □

**Theorem 7.16** Suppose \( T \in L(V) \), and let \( m_\mu(x) \) and \( m_\nu(x) \) be the minimal polynomials of \( u \in V \) and \( v \in V \) respectively. Then the least common multiple \( m(x) \) of \( m_\mu(x) \) and \( m_\nu(x) \) is the minimal polynomial of some vector in \( V \).

**Proof** We first assume that \( m_\mu(x) \) and \( m_\nu(x) \) are relatively prime so that their greatest common divisor is 1. By Theorem 6.9 we then have \( m_\mu(x)m_\nu(x) = m(x) \). From Theorem 6.5 we may write \( m_\mu(x)k(x) + m_\nu(x)h(x) = 1 \) for some polynomials \( h, k \in \mathcal{F}[x] \), and therefore

\[
m_\mu(T)k(T) + m_\nu(T)h(T) = 1
\]

Now define the vector \( w \in V \) by

\[
w = h(T)(u) + k(T)(v) .
\]

Then, using \( m_\mu(T)(u) = 0 = m_\nu(T)(v) \) we have

\[
m_\mu(T)(w) = m_\mu(T)h(T)(u) + m_\mu(T)k(T)(v)
\]

\[
= m_\mu(T)k(T)(v)
\]

\[
= [1 - m_\nu(T)h(T)](v)
\]

\[
= v
\]

and similarly, we find that \( m_\nu(T)(w) = u \). This means \( m_\mu(T)m_\nu(T)(w) = 0 \) so that \( m_\mu(x)m_\nu(x) \in N_T(w) \).

Now observe that \( N_T(w) = m_w(x) \mathcal{F}[x] \) where \( m_w(x) \) is the minimal polynomial of \( w \). Then

\[
m_w(T)(u) = m_w(T)m_\nu(T)(w) = 0
\]

and

\[
m_w(T)(v) = m_w(T)m_\mu(T)(w) = 0
\]
so that \( m_w(x) \in N_T(u) \cap N_T(v) \). Since \( N_T(u) = m_u(x)F[x] \) and \( N_T(v) = m_v(x)F[x] \), we see from Example 6.9 (along with the fact that \( m_u(x) \) and \( m_v(x) \) are relatively prime) that

\[
m_w(x) \in m(x)F[x] = m_u(x)m_v(x)F[x]
\]

and hence \( m_u(x)m_v(x) \mid m_w(x) \). On the other hand, since

\[
m_u(x)m_v(x) \in N_T(w) = m_w(x)F[x]
\]

we have \( m_w(x) \mid m_u(x)m_v(x) \). Since \( m_w(x) \), \( m_u(x) \) and \( m_v(x) \) are monic, it follows that \( m_w(x) = m_u(x)m_v(x) = m(x) \). This shows that in the case where \( m_u(x) \) and \( m_v(x) \) are relatively prime, then \( m(x) = m_u(x)m_v(x) \) is the minimal polynomial of \( w \).

Now let \( d(x) \) be the greatest common divisor of \( m_u(x) \) and \( m_v(x) \), and consider the general case where (see Theorem 6.9) \( m_u(x)m_v(x) = m(x)d(x) \). Using the notation of Theorem 6.10, we write \( m_u = \alpha \beta \) and \( m_v = \gamma \delta \). Since \( m_u \) and \( m_v \) are minimal polynomials by hypothesis, Theorem 7.15 tells us that \( \alpha \) and \( \delta \) are each also the minimal polynomial of some vector. However, by their construction, \( \alpha \) and \( \delta \) are relatively prime since they have no factors in common. This means that we may apply the first part of this proof to conclude that \( \alpha \delta \) is the minimal polynomial of some vector. To finish the proof, we simply note that (according to Theorem 6.10) \( \alpha \delta \) is just \( m(x) \).

A straightforward induction argument gives the following result.

**Corollary** For each \( i = 1, \ldots, k \) let \( m_{v_i}(x) \) be the minimal polynomial of a vector \( v_i \in V \). Then there exists a vector \( w \in V \) whose minimal polynomial \( m(x) \) is the least common multiple of the \( m_{v_i}(x) \).

Now suppose that \( T \in L(V) \) and \( V \) has a basis \( \{v_1, \ldots, v_n\} \). If \( m_{v_i}(x) \) is the minimal polynomial of \( v_i \), then by the corollary to Theorem 7.16, the least common multiple \( m(x) \) of the \( m_{v_i}(x) \) is the minimal polynomial of some vector \( w \in V \), and therefore \( \deg m(x) \leq \dim V = n \). But \( m(x) \) is the least common multiple, so that for each \( i = 1, \ldots, n \) we have \( m(x) = f_i(x)m_{v_i}(x) \) for some \( f_i(x) \in F[x] \). This means that

\[
m(T)(v_i) = [f_i(T)m_{v_i}(T)](v_i) = f_i(T)[m_{v_i}(T)(v_i)] = 0
\]

for each \( v_i \), and hence \( m(T) = 0 \). In other words, every \( T \in L(V) \) satisfies some monic polynomial \( m(x) \) with \( \deg m(x) \leq \dim V = n \).
We now define the (nonempty) set

\[ N_T = \{ f(x) \in \mathcal{F}[x] : f(T) = 0 \} . \]

As was the case with the T-annihilator, it is easy to prove that \( N_T \) is an ideal of \( \mathcal{F}[x] \). Since \( N_T \) consists of those polynomials in \( T \) that annihilate every vector in \( V \), it must be the same as the intersection of all T-annihilators \( N_T(v) \) in \( V \), i.e.,

\[ N_T = \bigcap_{v \in V} N_T(v) . \]

By Theorem 6.8 the ideal \( N_T \) is principal, and we define the **minimal polynomial** for \( T \in L(V) \) to be the unique monic generator of \( N_T \). We claim that the minimal polynomial for \( T \) is precisely the polynomial \( m(x) \) defined in the previous paragraph.

To see this, note first that \( \deg m(x) \leq \dim V = n \), and since \( m(x) \) is the minimal polynomial of some \( w \in V \), it follows directly from the definition of the minimal polynomial of \( w \) as the unique monic generator of \( N_T(w) \) that \( N_T(w) = m(x)\mathcal{F}[x] \). Next, the fact that \( m(T) = 0 \) means that \( m(T)(v) = 0 \) for every \( v \in V \), and therefore \( m(x) \in \bigcap_{v \in V} N_T(v) = N_T \). Since any polynomial in \( N_T(w) \) is a multiple of \( m(x) \) and hence annihilates every \( v \in V \), we see that \( N_T(w) \subseteq N_T \). Conversely, any element of \( N_T \) is automatically an element of \( N_T(w) \), and thus \( N_T = N_T(w) = m(x)\mathcal{F}[x] \). This shows that \( m(x) \) is the minimal polynomial for \( T \), and since \( m(x) \) generates \( N_T \), it is the polynomial of least degree such that \( m(T) = 0 \).

**Example 7.6** Let \( V = \mathbb{R}^4 \) have basis \( \{ e_1, e_2, e_3, e_4 \} \) and define the operator \( T \in L(V) \) by

\[
\begin{align*}
T(e_1) &= e_1 + e_3 \\
T(e_2) &= 3e_2 - e_4 \\
T(e_3) &= 3e_1 - e_3 \\
T(e_4) &= 3e_2 - e_4
\end{align*}
\]

Note that since \( T(e_3) = T(e_4) \), the matrix representation of \( T \) has zero determinant, and hence \( T \) is singular (either by Theorem 5.9 or Theorem 5.16). Alternatively, we have \( T(e_2 - e_4) = 0 \) so that \( T \) must be singular since \( e_2 - e_4 \neq 0 \). In any case, we now have

\[
T^2(e_1) = T(e_1 + e_3) = T(e_1) + T(e_3) = 4e_1
\]

so that

\[
(T^2 - 4)(e_1) = (T - 2)(T + 2) = 0 .
\]

Similarly

\[
T^2(e_2) = T(3e_2 - e_4) = 3T(e_2) - T(e_4) = 6e_2 - 2e_4 = 2T(e_2)
\]
so that
\[ (T^2 - 2T)(e_2) = T(T - 2)(e_2) = 0 \, . \]

Thus the minimal polynomial of \( e_1 \) is given by
\[ m_1(x) = x^2 - 4 = (x - 2)(x + 2) \]
and the minimal polynomial of \( e_2 \) is given by
\[ m_2(x) = x(x - 2) \, . \]

That these are indeed minimal polynomials is clear if we note that in neither case will a linear expression in \( T \) annihilate either \( e_1 \) or \( e_2 \) (just look at the definition of \( T \)).

It should be obvious that the least common multiple of \( m_1 \) and \( m_2 \) is
\[ x(x^2 - 4) \]
and hence (by Theorem 7.16) this is the minimal polynomial of some vector \( w \in \mathbb{R}^4 \) which we now try to find. We know that \( m_1(x) = x^2 - 4 \) is the minimal polynomial of \( e_1 \), but is \( x \) the minimal polynomial of some vector \( u \)? Since \( m_2(x) = x(x - 2) \) is the minimal polynomial of \( e_2 \), we see from Theorem 7.15 that the vector \( u = (T - 2)(e_2) = e_2 - e_4 \) has minimal polynomial \( x \). Now, \( x \) and \( x^2 - 4 \) are relatively prime so, as was done in the first part of the proof of Theorem 7.16, we define the polynomials \( h_1(x) = x/4 \) and \( k_1(x) = -1/4 \) by the requirement that \( xh_1 + (x^2 - 4)k_1 = 1 \). Hence
\[ w = (T/4)(e_1) + (-1/4)(u) = (1/4)(e_1 - e_2 + e_3 + e_4) \]
is the vector with minimal polynomial \( x(x^2 - 4) \).

We leave it to the reader to show that \( T^2(e_3) = 4e_3 \) and \( T^2(e_4) = 2T(e_4) \), and thus \( e_3 \) has minimal polynomial \( m_3(x) = x^2 - 4 \) and \( e_4 \) has minimal polynomial \( m_4(x) = x(x - 2) \). It is now easy to see that \( m(x) = x(x^2 - 4) \) is the minimal polynomial for \( T \) since \( m(x) \) has the property that \( m(T)(e_i) = 0 \) for each \( i = 1, \ldots, 4 \) and it is the least common multiple of the \( m_i(x) \). We also note that the constant term in \( m(x) \) is zero, and hence \( T \) is singular by Theorem 7.5.

\[ \Delta \]

**Example 7.7** Relative to the standard basis for \( \mathbb{R}^3 \), the matrix
represents the operator \( T \in L(\mathbb{R}^3) \) defined by
\[
Te_1 = 2e_1 + e_2, \quad Te_2 = -e_1 + e_3 \quad \text{and} \quad Te_3 = -2e_2 + e_3.
\]
It is easy to see that \( T^2e_1 = 3e_1 + 2e_2 + e_3 \) and \( T^3e_1 = 4e_1 + e_2 + 3e_3 \), and hence the set \( \{e_1, Te_1, T^2e_1\} \) is linearly independent, while the four vectors \( \{e_1, Te_1, T^2e_1, T^3e_1\} \) must be linearly dependent. This means there exists \( a, b, c, d \in \mathbb{R} \) such that
\[
ae_1 + bTe_2 + cT^2e_1 + dT^3e_1 = 0.
\]
This is equivalent to the system
\[
\begin{align*}
    a + 2b + 3c + 4d &= 0 \\
    b + 2c + d &= 0 \\
    c + 3d &= 0
\end{align*}
\]
Since there is one free variable, choosing \( d = 1 \) we find \( a = -5 \), \( b = 5 \) and \( c = -3 \). Therefore the minimal polynomial of \( e_1 \) is \( m_1(x) = x^3 - 3x^2 + 5x - 5 \). Since the lcm of the minimal polynomials of \( e_1, e_2 \) and \( e_3 \) must be of degree \( \leq 3 \), it follows that \( m_1(x) \) is in fact the minimal polynomial \( m(x) \) of \( T \) (and hence also of \( A \)). Note that by Theorem 7.5 we have \( A(A^2 - 3A + 5I) = 5I \), and hence \( A^{-1} = (1/5)(A^2 - 3A + 5I) \) or
\[
A^{-1} = \frac{1}{5} \begin{pmatrix}
2 & 1 & 2 \\
-1 & 2 & 4 \\
1 & -2 & 1
\end{pmatrix}.
\]

**Exercises**

1. Show that \( N_T \) is an ideal of \( \mathcal{J}[x] \).

2. For each of the following linear operators \( T \in L(V) \), find the minimal polynomial of each of the (standard) basis vectors for \( V \), find the minimal polynomial of \( T \), and find a vector whose minimal polynomial is the same as that of \( T \):
   (a) \( V = \mathbb{R}^2 \): \( Te_1 = e_2, \quad Te_2 = e_1 + e_2 \).
3. For each of the following matrices, find its minimal polynomial over \( \mathbb{R} \), and then find its inverse if it exists:

\[
(a) \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \quad (b) \begin{pmatrix} 4 & -1 \\ -4 & 1 \end{pmatrix}
\]

\[
(c) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (d) \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 5 & 2 & 1 \end{pmatrix}
\]

\[
(d) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad (e) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

### 7.5 INVARIANT SUBSPACES

Recall that two matrices \( A, B \in M_n(\mathbb{F}) \) are said to be similar if there exists a nonsingular matrix \( P \in M_n(\mathbb{F}) \) such that \( B = P^{-1}AP \). As was shown in Exercise 5.4.1, this defines an equivalence relation on \( M_n(\mathbb{F}) \). Since \( L(V) \) and \( M_n(\mathbb{F}) \) are isomorphic (Theorem 5.13), this definition applies equally well to linear operators. We call the equivalence class of a matrix (or linear operator) defined by this similarity relation its **similarity class**. While Theorem 7.14 gave us a condition under which a matrix may be diagonalized, this form is not possible to achieve in general. The approach we shall now take is to look for a basis in which the matrix of some linear operator in a given similarity class has a particularly simple standard form. As mentioned at the beginning of this chapter, these representations are called canonical forms. Of the many possible canonical forms, we shall consider only several of the more useful and important forms in this book. We begin with a discussion of some additional types of subspaces. A complete discussion of canonical forms under similarity is given in Chapter 8.
Suppose $T \in \mathcal{L}(V)$ and let $W$ be a subspace of $V$. Then $W$ is said to be **invariant under $T$** (or simply **$T$-invariant**) if $T(w) \in W$ for every $w \in W$. For example, if $V = \mathbb{R}^3$ then the $xy$-plane is invariant under the linear transformation that rotates every vector in $\mathbb{R}^3$ about the $z$-axis. As another example, note that if $v \in V$ is an eigenvector of $T$, then $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$, and hence $v$ generates a one-dimensional subspace of $V$ that is invariant under $T$ (this is not necessarily the same as the eigenspace of $\lambda$).

Another way to describe the invariance of $W$ under $T$ is to say that $T(W) \subseteq W$. Then clearly $T^2(W) = T(T(W)) \subseteq W$, and in general $T^n(W) \subseteq W$ for every $n = 1, 2, \ldots$. Since $W$ is a subspace of $V$, this means that $f(T)(W) \subseteq W$ for any $f(x) \in \mathbb{F}[x]$. In other words, if $W$ is invariant under $T$, then $W$ is also invariant under any polynomial in $T$ (over the same field as $W$).

If $W \subseteq V$ is $T$-invariant, we may define the **restriction** of $T$ to $W$ in the usual way as that operator $T|_W: W \to W$ defined by $(T|_W)(w) = T(w)$ for every $w \in W$. We will frequently write $T|_W$ instead of $T|_W$.

**Theorem 7.17** Suppose $T \in \mathcal{L}(V)$ and let $W$ be a $T$-invariant subspace of $V$. Then

(a) $f(T|_W)(w) = f(T)(w)$ for any $f(x) \in \mathbb{F}[x]$ and $w \in W$.

(b) The minimal polynomial $m'(x)$ for $T|_W$ divides the minimal polynomial $m(x)$ for $T$.

**Proof** This is Exercise 7.5.2. ■

If $T \in \mathcal{L}(V)$ and $f(x) \in \mathbb{F}[x]$, then $f(T)$ is also a linear operator on $V$, and hence we may define the kernel (or null space) of $f(T)$ in the usual way by

$$Ker f(T) = \{v \in V: f(T)(v) = 0\}.$$  

**Theorem 7.18** If $T \in \mathcal{L}(V)$ and $f(x) \in \mathbb{F}[x]$, then $Ker f(T)$ is a $T$-invariant subspace of $V$.

**Proof** Recall from Section 5.2 that $Ker f(T)$ is a subspace of $V$. To show that $Ker f(T)$ is $T$-invariant, we must show that $Tv \in Ker f(T)$ for any $v \in Ker f(T)$, i.e., $f(T)(Tv) = 0$. But using Theorem 7.2(a) we see that

$$f(T)(Tv) = T(f(T)(v)) = T(0) = 0$$

as desired. ■

Now suppose $T \in \mathcal{L}(V)$ and let $W \subseteq V$ be a $T$-invariant subspace. Furthermore let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$, where the first $m < n$ vectors form a basis for $W$. If $A = (a_{ij})$ is the matrix representation of $T$ relative to
this basis for \( V \), then a little thought should convince you that \( A \) must be of the block matrix form

\[
A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}
\]

where \( a_{ij} = 0 \) for \( j \leq m \) and \( i > m \). This is because \( T(w) \in W \) and any \( w \in W \) has components \((w_i, \ldots, w_m, 0, \ldots, 0)\) relative to the above basis for \( V \). The formal proof of this fact is given in the following theorem.

**Theorem 7.19**  Let \( W \) be a subspace of \( V \) and suppose \( T \in L(V) \). Then \( W \) is \( T \)-invariant if and only if \( T \) can be represented in the block matrix form

\[
A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}
\]

where \( B \) is a matrix representation of \( T_W \).

**Proof**  First suppose that \( W \) is \( T \)-invariant. Choose a basis \( \{v_1, \ldots, v_m\} \) for \( W \), and extend this to a basis \( \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\} \) for \( V \) (see Theorem 2.10). Then, since \( T(v_i) \in W \) for each \( i = 1, \ldots, m \), there exist scalars \( b_{ij} \) such that

\[
T_W(v_i) = T(v_i) = v_{i,b_{i1}} + \cdots + v_{m,b_{mi}}
\]

for each \( i = 1, \ldots, m \). In addition, since \( T(v_i) \in V \) for each \( i = m + 1, \ldots, n \), there also exist scalars \( c_{ij} \) and \( d_{ij} \) such that

\[
T(v_i) = v_{i,c_{i1}} + \cdots + v_{m,c_{mi}} + v_{m+1,d_{m+1,i}} + \cdots + v_{n,d_{ni}}
\]

for each \( i = m + 1, \ldots, n \).

From Theorem 5.11, we see that the matrix representation of \( T \) is given by an \( n \times n \) matrix \( A \) of the form

\[
A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}
\]

where \( B \) is an \( m \times m \) matrix that represents \( T_W \), \( C \) is an \( m \times (n - m) \) matrix, and \( D \) is an \( (n - m) \times (n - m) \) matrix.

Conversely, if \( A \) has the stated form and \( \{v_1, \ldots, v_n\} \) is a basis for \( V \), then the subspace \( W \) of \( V \) defined by vectors of the form

\[
w = \sum_{i=1}^{m} \alpha_i v_i
\]
where each $\alpha_i \in \mathcal{F}$ will be invariant under $T$. Indeed, for each $i = 1, \ldots, m$ we have

$$T(v_i) = \sum_{j=1}^{n} v_j a_{ji} = v_1 b_{1i} + \cdots + v_m b_{mi} \in W$$

and hence $T(w) \in W$. ■

**Corollary** Suppose $T \in \text{L}(V)$ and $W$ is a $T$-invariant subspace of $V$. Then the characteristic polynomial of $T_W$ divides the characteristic polynomial of $T$.

**Proof** See Exercise 7.5.3. ■

Recall from Theorem 2.18 that the orthogonal complement $W^\perp$ of a set $W \subset V$ is a subspace of $V$. If $W$ is a subspace of $V$ and both $W$ and $W^\perp$ are $T$-invariant, then since $V = W \oplus W^\perp$ (Theorem 2.22), a little more thought should convince you that the matrix representation of $T$ will now be of the block diagonal form

$$A = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}.$$

We now proceed to discuss a variation of Theorem 7.19 in which we take into account the case where $V$ can be decomposed into a direct sum of subspaces.

Let us assume that $V = W_1 \oplus \cdots \oplus W_r$ where each $W_i$ is a $T$-invariant subspace of $V$. Then we define the restriction of $T$ to $W_i$ to be the operator $T_i = T|_{W_i}$. In other words, $T_i(w_i) = T(w_i) \in W_i$ for any $w_i \in W_i$. Given any $v \in V$ we have $v = v_1 + \cdots + v_r$ where $v_i \in W_i$ for each $i = 1, \ldots, r$, and hence

$$T(v) = \sum_{i=1}^{r} T(v_i) = \sum_{i=1}^{r} T_i(v_i).$$

This shows that $T$ is completely determined by the effect of each $T_i$ on $W_i$. In this case we call $T$ the **direct sum** of the $T_i$ and we write

$$T = T_1 \oplus \cdots \oplus T_r.$$
and \( j \) label the rows and columns respectively of the matrix \( A_i \). Therefore we see that

\[
T(w_{ij}) = T_i(w_{ij}) = \sum_{k=1}^{n_i} w_{ik} a_{i,kj}
\]

where \( i = 1, \ldots, r \) and \( j = 1, \ldots, n_i \). If \( A \) is the matrix representation of \( T \) with respect to the basis \( B = \{ w_{11}, \ldots, w_{1n_1}, \ldots, w_{r1}, \ldots, w_{rn_r} \} \) for \( V \), then since the \( ith \) column of \( A \) is just the image of the \( ith \) basis vector under \( T \), we see that \( A \) must be of the block diagonal form

\[
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_r
\end{pmatrix}
\]

If this is not immediately clear, then a minute’s thought should help, keeping in mind that each \( A_i \) is an \( n_i \times n_i \) matrix, and \( A \) is an \( n \times n \) matrix where \( n = \sum_{i=1}^{r} n_i \). It is also helpful to think of the elements of \( B \) as being numbered from 1 to \( n \) rather than by the confusing double subscripts (also refer to the proof of Theorem 7.19).

The matrix \( A \) is called the direct sum of the matrices \( A_1, \ldots, A_r \) and we write

\[
A = A_1 \oplus \cdots \oplus A_r .
\]

In this case we also say that the matrix \( A \) is reducible. Thus a representation \([T]\) of \( T \) is reducible if there exists a basis for \( V \) in which \([T]\) is block diagonal. (Some authors say that a representation is reducible if there exists a basis for \( V \) in which the matrix of \( T \) is triangular. In this case, if there exists a basis for \( V \) in which the matrix is block diagonal, then the representation is said to be completely reducible. We shall not follow this convention.) This discussion proves the following theorem.

**Theorem 7.20**  Suppose \( T \in L(V) \) and assume that \( V = W_1 \oplus \cdots \oplus W_r \) where each \( W_i \) is \( T \)-invariant. If \( A_i \) is the matrix representation of \( T_i = T|W_i \), then the matrix representation of \( T \) is given by the matrix \( A = A_1 \oplus \cdots \oplus A_r \).

**Corollary**  Suppose \( T \in L(V) \) and \( V = W_1 \oplus \cdots \oplus W_r \) where each \( W_i \) is \( T \)-invariant. If \( \Delta_T(x) \) is the characteristic polynomial for \( T \) and \( \Delta_i(x) \) is the characteristic polynomial for \( T_i = T|W_i \), then \( \Delta_T(x) = \Delta_1(x) \cdots \Delta_r(x) \).

**Proof**  See Exercise 7.5.4.  \( \blacksquare \)
Example 7.8  Referring to Example 2.8, consider the space $V = \mathbb{R}^3$. We write $V = W_1 \oplus W_2$ where $W_1 = \mathbb{R}^2$ (the $xy$-plane) and $W_2 = \mathbb{R}^1$ (the $z$-axis). Note that $W_1$ has basis vectors $w_{11} = (1, 0, 0)$ and $w_{12} = (0, 1, 0)$, and $W_2$ has basis vector $w_{21} = (0, 0, 1)$.

Now let $T \in \text{L}(V)$ be the linear operator that rotates any $v \in V$ counterclockwise by an angle $\theta$ about the $z$-axis. Then clearly both $W_1$ and $W_2$ are $T$-invariant. Letting $\{e_i\}$ be the standard basis for $\mathbb{R}^3$, we have $T_i = T|W_i$ and consequently (see Example 1.2),

\[
T_1(e_1) = T(e_1) = (\cos \theta)e_1 + (\sin \theta)e_2 \\
T_1(e_2) = T(e_2) = (-\sin \theta)e_1 + (\cos \theta)e_2 \\
T_2(e_3) = T(e_3) = e_3
\]

Thus $V = W_1 \oplus W_2$ is a $T$-invariant direct sum decomposition of $V$, and $T$ is the direct sum of $T_1$ and $T_2$. It should be clear that the matrix representation of $T$ is given by

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

which is just the direct sum of the matrix representations of $T_1$ and $T_2$. //

Exercises

1. Suppose $V = W_1 \oplus W_2$ and let $T_1: W_1 \rightarrow V$ and $T_2: W_2 \rightarrow V$ be linear. Show that $T = T_1 \oplus T_2$ is linear.

2. Prove Theorem 7.17.

3. Prove the corollary to Theorem 7.19.

4. Prove the corollary to Theorem 7.20.

5. Let $V$ be a finite-dimensional inner product space over $\mathbb{C}$, and let $G$ be a finite group. If for each $g \in G$ there is a $U(g) \in \text{L}(V)$ such that

\[
U(g_1)U(g_2) = U(g_1g_2)
\]

then the collection $U(G) = \{U(g)\}$ is said to form a representation of $G$. If $W$ is a subspace of $V$ with the property that $U(g)(W) \subseteq W$ for all $g \in \cdots$
G, then we will say that W is U(G)-invariant (or simply invariant). Furthermore, we say that U(G) is irreducible if there is no nontrivial U(G)-invariant subspace (i.e., the only invariant subspaces are \( \{0\} \) and V itself).

(a) Prove Schur’s lemma 1: Let U(G) be an irreducible representation of G on V. If A ∈ L(V) is such that AU(g) = U(g)A for all g ∈ G, then A = λI where λ ∈ C. [Hint: Let λ be an eigenvalue of A with corresponding eigenspace \( V_\lambda \). Show that \( V_\lambda \) is U(G)-invariant.]

(b) If S ∈ L(V) is nonsingular, show that \( U'(G) = SU(G)S^{-1} \) is also a representation of G on V. (Two representations of G related by such a similarity transformation are said to be equivalent.)

(c) Prove Schur’s lemma 2: Let U(G) and U'(G) be two irreducible representations of G on V and V' respectively, and suppose A ∈ L(V', V) is such that AU'(g) = U(g)A for all g ∈ G. Then either A = 0, or else A is an isomorphism of V' onto V so that A^{-1} exists and U(G) is equivalent to U'(G). [Hint: Show that Im A is invariant under U(G), and that Ker A is invariant under U'(G).]

6. Suppose A ∈ M_n(F) has minimal polynomial \( m_A \) and B ∈ M_m(F) has minimal polynomial \( m_B \). Let \( m_{A\oplus B} \) be the minimal polynomial for \( A \oplus B \) and let \( p = \text{lcm}\{m_A, m_B\} \). Prove \( m_{A\oplus B} = p \).

7. Let W be a T-invariant subspace of a finite-dimensional vector space V over F, and suppose v ∈ V. Define the set

\[
N_T(v, W) = \{ f \in F[x] : f(T)v \in W \}.
\]

(a) Show that \( N_T(v, W) \) is an ideal of \( F[x] \). This means that \( N_T(v, W) \) has a unique monic generator \( c_v(x) \) which is called the T-conductor of v into W.

(b) Show that every T-conductor divides the minimal polynomial m(x) for T.

(c) Now suppose the minimal polynomial for T is of the form

\[
m(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_r)^{n_r}
\]

and let W be a proper T-invariant subspace of V. Prove there exists a vector \( v \in V \) with \( v \notin W \) such that \( (T - \lambda I)v \in W \) where \( \lambda \) is an eigenvalue of T. [Hint: Suppose \( v_1 \in V \) with \( v_1 \notin W \). Show that the T-conductor \( c_{v_1} \) of \( v_1 \) into W is of the form \( c_{v_1}(x) = (x - \lambda)d(x) \). Now consider the vector \( v = d(T)v_1 \).]
8. Let $V$ be finite-dimensional over $\mathcal{F}$ and suppose $T \in \text{L}(V)$. Prove there exists a basis for $V$ in which the matrix representation $A$ of $T$ is upper-triangular if and only if the minimal polynomial for $T$ is of the form $m(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_r)^{n_r}$ where each $n_i \in \mathbb{Z}^+$ and the $\lambda_i$ are the eigenvalues of $T$. \textit{[Hint: Apply part (c) of the previous problem to the basis $v_1, \ldots, v_n$ in which $A$ is upper-triangular. Start with $W = \{0\}$ to get $v_1$, then consider the span of $v_1$ to get $v_2$, and continue this process.]}

9. Relative to the standard basis for $\mathbb{R}^2$, let $T \in \text{L}(\mathbb{R}^2)$ be represented by

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$  

(a) Prove that the only $T$-invariant subspaces of $\mathbb{R}^2$ are $\{0\}$ and $\mathbb{R}^2$ itself.
(b) Suppose $U \in \text{L}(\mathbb{C}^2)$ is also represented by $A$. Show that there exist one-dimensional $U$-invariant subspaces.

10. Find all invariant subspaces over $\mathbb{R}$ of the operator represented by

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$  

11. (a) Suppose $T \in \text{L}(V)$, and let $v \in V$ be arbitrary. Define the set of vectors

$$Z(v, T) = \{ f(T)(v) : f \in \mathcal{F}[x] \}.$$  

Show that $Z(v, T)$ is a $T$-invariant subspace of $V$. (This is called the \textit{T-cyclic subspace generated} by $v$.)
(b) Let $v$ have minimal polynomial $m_v(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_1x + a_0$. Prove that $Z(v, T)$ has a basis $\{v, Tv, \ldots, T^{r-1}v\}$, and hence also that $\dim Z(v, T) = \deg m_v(x)$. \textit{[Hint: Show that $T^k v$ is a linear combination of $\{v, Tv, \ldots, T^{r-1}v\}$ for every integer $k \geq r$.]}
(c) Let $T \in \text{L}(\mathbb{R}^3)$ be represented in the standard basis by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix}.$$
7.5 INVARIANT SUBSPACES

If \( v = e_1 - e_2 \), find the minimal polynomial of \( v \) and a basis for \( Z(v, T) \). Extend this to a basis for \( \mathbb{R}^3 \), and show that the matrix of \( T \) relative to this basis is block triangular.

7.6 THE PRIMARY DECOMPOSITION THEOREM

We now proceed to show that there exists an important relationship between the minimal polynomial of a linear operator \( T \in L(V) \) and the conditions under which \( V \) can be written as a direct sum of \( T \)-invariant subspaces of \( V \). In the present context, this theorem (the primary decomposition theorem) is best obtained by first proving two relatively simple preliminary results.

**Theorem 7.21** Suppose \( T \in L(V) \) and assume that \( f \in \mathcal{F}[[x]] \) is a polynomial such that \( f(T) = 0 \) and \( f = h_1 h_2 \) where \( h_1 \) and \( h_2 \) are relatively prime. Define \( W_1 = \text{Ker } h_1(T) \) and \( W_2 = \text{Ker } h_2(T) \). Then \( W_1 \) and \( W_2 \) are \( T \)-invariant subspaces, and \( V = W_1 \oplus W_2 \).

**Proof** We first note that \( W_1 \) and \( W_2 \) are \( T \)-invariant subspaces according to Theorem 7.18. Next, since \( h_1 \) and \( h_2 \) are relatively prime, there exist polynomials \( g_1 \) and \( g_2 \) such that \( g_1 h_1 + g_2 h_2 = 1 \) (Theorem 6.5), and hence

\[
g_1(T)h_1(T) + g_2(T)h_2(T) = 1 \quad (*)
\]

Then for any \( v \in V \) we have

\[
g_1(T)h_1(T)(v) + g_2(T)h_2(T)(v) = v.
\]

However

\[
h_2(T)g_1(T)h_1(T)(v) = g_1(T)h_1(T)h_2(T)(v)
= g_1(T)f(T)(v)
= g_1(T)0(v)
= 0
\]

so that \( g_1(T)h_1(T)(v) \in \text{Ker } h_2(T) = W_2 \). Similarly, it is easy to see that \( g_2(T)h_2(T)(v) \in W_1 \), and hence \( v \in W_1 + W_2 \). This shows that \( V = W_1 + W_2 \), and it remains to be shown that this sum is direct.

To show that this sum is direct we use Theorem 2.14. In other words, if \( v = w_1 + w_2 \) where \( w_i \in W_i \), then we must show that each \( w_i \) is uniquely determined by \( v \). Applying \( g_i(T)h_i(T) \) to \( v = w_1 + w_2 \) and using the fact that \( h_i(T)(w_i) = 0 \) we obtain
\( g_i(T)h_i(T)(v) = g_i(T)h_i(T)(w_2) \).

Next, we apply (*) to \( w_2 \) and use the fact that \( h_2(T)(w_2) = 0 \) to obtain

\( g_i(T)h_i(T)(w_2) = w_2 \).

Combining these last two equations shows that \( w_2 = g_i(T)h_i(T)(v) \), and thus \( w_2 \) is uniquely determined by \( v \) (through the action of \( g_i(T)h_i(T) \)). We leave it to the reader to show in a similar manner that \( w_i = g_i(T)h_i(T)(v) \). Therefore \( v = w_1 + w_2 \) is a unique decomposition, and hence \( V = W_1 \oplus W_2 \). ■

**Theorem 7.22** Suppose \( T \in L(V) \), and let \( f = h_1h_2 \in F[x] \) be the (monic) minimal polynomial for \( T \), where \( h_1 \) and \( h_2 \) are relatively prime. If \( W_i = \ker h_i(T) \), then \( h_i \) is the (monic) minimal polynomial for \( T_i = T|W_i \).

**Proof** For each \( i = 1, 2 \) let \( m_i \) be the (monic) minimal polynomial for \( T_i \). Since \( W_i = \ker h_i(T) \), Theorem 7.17(a) tells us that \( h_i(T_i) = 0 \), and therefore (by Theorem 7.4) we must have \( m_i|h_i \). This means that \( m_i|f \) (since \( f = h_1h_2 \)), and hence \( f \) is a multiple of \( m_1 \) and \( m_2 \). From the definition of least common multiple, it follows that the lcm of \( m_1 \) and \( m_2 \) must divide \( f \). Since \( h_1 \) and \( h_2 \) are relatively prime, \( m_1 \) and \( m_2 \) must also be relatively prime (because if \( m_1 \) and \( m_2 \) had a common factor, then \( h_1 \) and \( h_2 \) would each also have this same common factor since \( m_i|h_i \)). But \( m_1 \) and \( m_2 \) are monic, and hence their greatest common divisor is 1. Therefore the lcm of \( m_1 \) and \( m_2 \) is just \( m_1m_2 \) (Theorem 6.9). This shows that \( m_1m_2|f \).

On the other hand, since \( V = W_1 \oplus W_2 \) (Theorem 7.21), we see that for any \( v \in V \)

\[
[m_1(T)m_2(T)](v) = [m_1(T)m_2(T)](w_1 + w_2) \\
= m_2(T)[m_1(T)(w_1)] + m_1(T)[m_2(T)(w_2)] \\
= m_2(T)[m_1(T_1)(w_1)] + m_1(T)[m_2(T_2)(w_2)] \\
= 0
\]

because \( m_i \) is the minimal polynomial for \( T_i \). Therefore \( (m_1m_2)(T) = 0 \), and hence \( f|m_1m_2 \) (from Theorem 7.4 since, by hypothesis, \( f \) is the minimal polynomial for \( T \)). Combined with the fact that \( m_1m_2|f \), this shows that \( f = m_1m_2 \) (since \( m_1, m_2 \) and \( f \) are monic). This result, along with the definition \( f = h_1h_2 \), the fact that \( h_1 \) and \( h_2 \) are monic, and the fact that \( m_i|h_i \) shows that \( m_i = h_i \). ■

We are now in a position to prove the main result of this section.
Theorem 7.23 (Primary Decomposition Theorem) Suppose $T \in \mathcal{L}(V)$ has minimal polynomial
\[ m(x) = f_1(x)^{n_1} f_2(x)^{n_2} \cdots f_r(x)^{n_r} \]
where each $f_i(x)$ is a distinct monic prime polynomial and each $n_i$ is a positive integer. Let $W_i = \text{Ker} f_i(T)^{n_i}$, and define $T_i = T|W_i$. Then $V$ is the direct sum of the $T$-invariant subspaces $W_i$, and $f_i(x)^{n_i}$ is the minimal polynomial for $T_i$.

Proof If $r = 1$ the theorem is trivial since $W_1 = \text{Ker} f_1(T)^{n_1} = \text{Ker} m(T) = V$. We now assume that the theorem has been proved for some $r - 1 \geq 1$, and proceed by induction to show that it is true for $r$. We first remark that the $W_i$ are $T$-invariant subspaces by Theorem 7.18. Define the $T$-invariant subspace
\[ U = \text{Ker}[f_2(T)^{n_2} \cdots f_r(T)^{n_r}] \]
Because the $f_i(x)^{n_i}$ are relatively prime (by Corollary 2 of Theorem 6.5, since the $f_i(x)$ are all distinct primes), we can apply Theorem 7.21 to write $V = W_1 \oplus U$. In addition, since $m(x)$ is the minimal polynomial for $T$, Theorem 7.22 tells us that $f_1(x)^{n_1}$ is the minimal polynomial for $T_1$, and $[f_2(x)^{n_2} \cdots f_r(x)^{n_r}]$ is the minimal polynomial for $T|U = T|U$.

Applying our induction hypothesis, we find that $U = W_2 \oplus \cdots \oplus W_r$ where for each $i = 2, \ldots, r$ we have $W_i = \text{Ker} f_i(T)^{n_i}$, and $f_i(x)^{n_i}$ is the minimal polynomial for $T_i = T|W_i$. However, it is obvious that $f_i(x)^{n_i}$ divides $[f_2(x)^{n_2} \cdots f_r(x)^{n_r}]$ for each $i = 2, \ldots, r$ and hence $\text{Ker} f_i(T)^{n_i} \subseteq U$. Specifically, this means that the set of all vectors $v \in V$ with the property that $f_i(T)^{n_i}(v) = 0$ are also in $U$, and therefore $\text{Ker} f_i(T)^{n_i} = \text{Ker} f_i(T|U)^{n_i} = W_i$. Furthermore, $T|W_i = T|U|W_i = T_i$ and thus $f_i(x)^{n_i}$ is also the minimal polynomial for $T|W_i$.

Summarizing, we have shown that $V = W_1 \oplus U = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ where $W_i = \text{Ker} f_i(T)^{n_i}$ for each $i = 1, \ldots, r$ and $f_i(x)^{n_i}$ is the minimal polynomial for $T|W_i = T_i$. This completes the induction procedure and proves the theorem.

In order to make this result somewhat more transparent, as well as in aiding actual calculations, we go back and look carefully at what we have done in defining the spaces $W_i = \text{Ker} f_i(T)^{n_i}$. For each $i = 1, \ldots, r$ we define the polynomials $g_i(x)$ by
\[ m(x) = f_i(x)^{n_i} g_i(x) \]
In other words, \( g_a(x) \) is a product of the \( r - 1 \) factors \( f_j(x) \) with \( j \neq i \). We claim that in fact

\[
W_i = g_i(T)(V) .
\]

It is easy to see that \( g_i(T)(V) \subset W_i \) because

\[
f_i(T)^{n_i} [g_i(T)(v)] = [f_i(T)^{n_i} g_i(T)](v) = m(T)(v) = 0
\]

for every \( v \in V \). On the other hand, \( f_i(x)^{n_i} \) and \( g_i(x) \) are monic relative primes, and hence (by Theorem 6.5) there exist polynomials \( a(x), b(x) \in \mathbb{F}[x] \) such that

\[
a(x)f_i(x)^{n_i} + b(x)g_i(x) = 1.
\]

Then for any \( v_i \in W_i = \ker f_i(T)^{n_i} \) we have \( f_i(T)^{n_i}(v_i) = 0 \), and hence

\[
v_i = a(T)[f_i(T)^{n_i}(v_i)] + g_i(T)[b(T)(v_i)] = 0 + g_i(T)[b(T)(v_i)] \in g_i(T)(V) .
\]

Hence \( W_i \subset g_i(T)(V) \), and therefore \( W_i = g_i(T)(V) \) as claimed. This gives us, at least conceptually, a practical method for computing the matrix of a linear transformation \( T \) with respect to the bases of the \( T \)-invariant subspaces.

As a final remark, note that for any \( j \neq i \) we have \( g_j(T)(W_j) = 0 \) because \( g_j(T) \) contains \( f_j(T)^{n_j} \) and \( W_j = \ker f_j(T)^{n_j} \). In addition, since \( W_i \) is \( T \)-invariant we see that \( g_i(T)(W_i) \subset W_i \). But we also have

\[
W_i = a(T)[f_i(T)^{n_i}(W_j)] + g_i(T)[b(T)(W_j)]
\]

\[
= 0 + g_i(T)[b(T)(W_j)]
\]

\[
\subset g_i(T)(W_j)
\]

and hence \( g_i(T)(W_i) = W_i \). This should not be surprising for the following reason. If we write

\[
V_i = W_i \oplus \cdots \oplus W_{i-1} \oplus W_{i+1} \oplus \cdots \oplus W_r
\]

then \( g_i(T)(V_i) = 0 \), and therefore

\[
W_i = g_i(T)(V) = g_i(T)(W_i \oplus V_i) = g_i(T)(W_i) .
\]

**Example 7.9** Consider the space \( V = \mathbb{R}^3 \) with basis \( \{u_1, u_2, u_3\} \) and define the linear transformation \( T \in \text{L}(V) \) by
Then
\[ T^2(u_1) = T(u_2) = u_3 \]
and hence
\[ T^3(u_1) = T(u_3) = -2u_1 + 3u_2. \]

Therefore \( T^3(u_1) - 3T(u_1) + 2u_1 = 0 \) so that the minimal polynomial of \( u_1 \) is given by
\[ m_1(x) = x^3 - 3x + 2 = (x - 1)^2(x + 2). \]

Now recall that the minimal polynomial \( m(x) \) for \( T \) is just the least common multiple of the minimal polynomials \( m_i(x) \) of the basis vectors for \( V \), and \( \deg m(x) \leq \dim V = n \) (see the discussion prior to Example 7.6). Since \( m_1(x) \) is written as a product of prime polynomials with \( \deg m_1 = 3 = \dim V \), it follows that \( m(x) = m_1(x) \). We thus have \( f_1(x)^{n_1} = (x - 1)^2 \) and \( f_2(x)^{n_2} = (x + 2) \).

We now define \( W_1 = g_1(T)(V) = (T + 2)(V) \) and \( W_2 = g_2(T)(V) = (T - 1)^2(V) \).

A simple calculation shows that
\[
\begin{align*}
(T + 2)u_1 &= 2u_1 + u_2 \\
(T + 2)u_2 &= 2u_2 + u_3 \\
(T + 2)u_3 &= -2u_1 + 3u_2 + 2u_3 = (T + 2)(-u_1 + 2u_2) 
\end{align*}
\]

Therefore \( W_1 \) is spanned by the basis vectors \( \{2u_1 + u_2, 2u_2 + u_3\} \). Similarly, it is easy to show that \( (T - 1)^2u_1 = u_1 - 2u_2 + u_3 \) and that both \( (T - 1)^2u_2 \) and \( (T - 1)^2u_3 \) are multiples of this. Hence \( \{u_1 - 2u_2 + u_3\} \) is the basis vector for \( W_2 \).

We now see that \( T_1 = T|W_1 \) and \( T_2 = T|W_2 \) yield the transformations
\[
\begin{align*}
T_1(2u_1 + u_2) &= 2u_2 + u_3 \\
T_1(2u_2 + u_3) &= -2u_1 + 3u_2 + 2u_3 = -(2u_1 + u_2) + 2(2u_2 + u_3) \\
T_2(u_1 - 2u_2 + u_3) &= -2(u_1 - 2u_2 + u_3)
\end{align*}
\]

and hence \( T_1 \) and \( T_2 \) are represented by the matrices \( A_1 \) and \( A_2 \), respectively, given by
Therefore \( T = T_1 \oplus T_2 \) is represented by the matrix \( A = A_1 \oplus A_2 \) where
\[
A = \begin{pmatrix}
0 & -1 & 0 \\
1 & 2 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

From Corollary 1 of Theorem 7.14 we know that a matrix \( A \in M_n(\mathbb{F}) \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors, and from Theorem 7.13, the corresponding \textit{distinct} eigenvalues \( \lambda_1, \ldots, \lambda_r \) (where \( r \leq n \)) must be roots of the minimal polynomial for \( A \). The factor theorem (corollary to Theorem 6.4) then says that \( x - \lambda_i \) is a factor of the minimal polynomial for \( A \), and hence the minimal polynomial for \( A \) must contain at least \( r \) distinct linear factors if \( A \) is to be diagonalizable.

We now show that the minimal polynomial of a diagonalizable linear transformation consists precisely of distinct linear factors.

**Theorem 7.24** A linear transformation \( T \in L(V) \) is diagonalizable if and only if the minimal polynomial \( m(x) \) for \( T \) is of the form
\[ m(x) = (x - \lambda_1) \cdots (x - \lambda_r) \]
where \( \lambda_1, \ldots, \lambda_r \) are the distinct eigenvalues of \( T \).

**Proof** Suppose that \( m(x) = (x - \lambda_i) \cdots (x - \lambda_r) \) where \( \lambda_1, \ldots, \lambda_r \in \mathbb{F} \) are distinct. Then, according to Theorem 7.23, \( V = W_1 \oplus \cdots \oplus W_r \) where \( W_i = \ker(T - \lambda_i I) \). But then for any \( w \in W_i \) we have
\[ 0 = (T - \lambda_i I)(w) = T(w) - \lambda_i w \]
and hence any \( w \in W_i \) is an eigenvector of \( T \) with eigenvalue \( \lambda_i \). It should be clear that any eigenvector of \( T \) with eigenvalue \( \lambda_i \) is also in \( W_i \). In particular, this means that any basis vector of \( W_i \) is also an eigenvector of \( T \). By Theorem 2.15, the union of the bases of all the \( W_i \) is a basis for \( V \), and hence \( V \) has a basis consisting entirely of eigenvectors. This means that \( T \) is diagonalizable (Theorem 7.14).

On the other hand, assume that \( T \) is diagonalizable, and hence \( V \) has a basis \( \{v_1, \ldots, v_n\} \) of eigenvectors of \( T \) that correspond to the (not necessarily
7.6 THE PRIMARY DECOMPOSITION THEOREM

distinct) eigenvalues \( \lambda_1, \ldots, \lambda_m \). If the \( v_i \) are numbered so that \( \lambda_1, \ldots, \lambda_r \) are the *distinct* eigenvalues of \( T \), then the operator

\[
f(T) = (T - \lambda_1) \cdots (T - \lambda_r)\]

has the property that \( f(T)(v_i) = 0 \) for each of the basis eigenvectors \( v_1, \ldots, v_n \). Thus \( f(T) = 0 \) and the minimal polynomial \( m(x) \) for \( T \) must divide \( f(x) \) (Theorem 7.4). While this shows that \( m(x) \) must consist of linear factors, it is in fact also true that \( f(x) = m(x) \). To see this, suppose that we delete any factor \( T - \lambda_\alpha 1 \) from \( f(T) \) to obtain a new linear operator \( f'(T) \). But the \( \lambda_\alpha \) are all distinct so that \( f'(T)(v_\alpha) \neq 0 \), and hence \( f'(T) \neq 0 \). Therefore \( f'(x) \) cannot be the minimal polynomial for \( T \), and we must have \( f(x) = m(x) \).

In a manner similar to that used in Corollary 1 of Theorem 7.14, we can rephrase Theorem 7.24 in terms of matrices as follows.

**Corollary 1** A matrix \( A \in M_n(\mathbb{F}) \) is similar to a diagonal matrix \( D \) if and only if the minimal polynomial for \( A \) has the form

\[
m(x) = (x - \lambda_1) \cdots (x - \lambda_r)
\]

where \( \lambda_1, \ldots, \lambda_r \in \mathbb{F} \) are all distinct. If this is the case, then \( D = P^{-1}AP \) where \( P \) is the invertible matrix whose columns are any set of \( n \) linearly independent eigenvectors \( v_1, \ldots, v_n \) of \( A \) corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_n \). (If \( r < n \), then some of the eigenvalues will be repeated.) In addition, the diagonal elements of \( D \) are just the eigenvalues \( \lambda_1, \ldots, \lambda_n \).

**Corollary 2** A linear transformation \( T \in L(V) \) is diagonalizable if and only if \( V = W_1 \oplus \cdots \oplus W_r \) where \( W_i = \text{Ker}(T - \lambda_i 1) = V_{\lambda_i} \).

*Proof* Recall that \( V_{\lambda_i} \) is the eigenspace corresponding to the eigenvalue \( \lambda_i \), and the fact that \( V_{\lambda_i} = \text{Ker}(T - \lambda_i 1) \) was shown in the proof of Theorem 7.24. If \( T \) is diagonalizable, then the conclusion that \( V = W_1 \oplus \cdots \oplus W_r \) follows directly from Theorems 7.24 and 7.23. On the other hand, if \( V = W_1 \oplus \cdots \oplus W_r \) then each \( W_i \) has a basis of eigenvectors and hence so does \( V \) (Theorem 2.15).

It is important to realize that one eigenvalue can correspond to more than one linearly independent eigenvector (recall the comment following Theorem 7.8). This is why the spaces \( W_i \) in the first part of the proof of Theorem 7.24 can have bases consisting of more than one eigenvector. In particular, any eigenvalue of multiplicity greater than one can result in an eigenspace of
dimension greater than one, a result that we treat in more detail in the next section.

Exercises

1. Write each of the following linear transformations \( T \in L(V) \) as a direct sum of linear transformations whose minimal polynomials are powers of prime polynomials:
   (a) \( V = \mathbb{R}^2 \): \( T_1 = e_2 , T_2 = 3e_1 + 2e_2 \).
   (b) \( V = \mathbb{R}^2 \): \( T_1 = -4e_1 + 4e_2 , T_2 = e_2 \).
   (c) \( V = \mathbb{R}^3 \): \( T_1 = e_2 , T_2 = e_3 , T_3 = 2e_1 - e_2 + 2e_3 \).
   (d) \( V = \mathbb{R}^3 \): \( T_1 = 3e_1 , T_2 = e_2 - e_3 , T_3 = e_1 + 3e_2 \).
   (e) \( V = \mathbb{R}^3 \): \( T_1 = 3e_1 + e_2 , T_2 = e_2 - 5e_3 , T_3 = 2e_1 + 2e_2 + 2e_3 \).

2. Let \( V \) be finite-dimensional and suppose \( T \in L(V) \) has minimal polynomial \( m(x) = f_1(x)^{n_1} \cdots f_r(x)^{n_r} \) where the \( f_i(x) \) are distinct monic primes and each \( n_i \in \mathbb{Z}^+ \). Show that the characteristic polynomial is of the form

\[
    \Delta(x) = f_1(x)^{d_1} \cdots f_r(x)^{d_r}
\]

where

\[
    d_i = \frac{\dim(\ker f_i(T)^{n_i})}{\deg f_i}.
\]

3. Let \( \mathcal{D} = \{T_i\} \) be a collection of mutually commuting (i.e., \( T_iT_j = T_jT_i \) for all \( i, j \)) diagonalizable linear operators on a finite-dimensional space \( V \). Prove that there exists a basis for \( V \) relative to which the matrix representation of each \( T_i \) is diagonal. [Hint: Proceed by induction on \( \dim V \). Let \( T \in \mathcal{D} \) (\( T \neq c1 \)) have distinct eigenvalues \( \lambda_i , \ldots , \lambda_r \) and for each \( i \) define \( W_i = \ker(T - \lambda_i I) \). Show that \( W_i \) is invariant under each operator that commutes with \( T \). Define \( \mathcal{D}_i = \{ T_j | W_i : T_j \in \mathcal{D} \} \) and show that every member of \( \mathcal{D}_i \) is diagonalizable.]

7.7 MORE ON DIAGONALIZATION

If an operator \( T \in L(V) \) is diagonalizable, then in a (suitably numbered) basis of eigenvectors, its matrix representation \( A \) will take the form
where each $\lambda_i$ is repeated $m_i$ times and $I_{m_i}$ is the $m_i \times m_i$ identity matrix. Note that $m_1 + \cdots + m_r$ must be equal to $\dim V$. Thus the characteristic polynomial for $T$ has the form

$$
\Delta_T(x) = \det(xI - A) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r}
$$

which is a product of (possibly repeated) linear factors. That the characteristic polynomial of a diagonalizable operator is of this form also follows directly from Theorems 7.24 and 7.12. However, we stress that just because the characteristic polynomial factors into a product of linear terms does not mean that the operator is diagonalizable. We now investigate the conditions that determine just when an operator will be diagonalizable.

Let us assume that $T$ is diagonalizable, and hence that the characteristic polynomial factors into linear terms. For each distinct eigenvalue $\lambda_i$, we have seen that the corresponding eigenspace $V_{\lambda_i}$ is just $\text{Ker}(T - \lambda_i I)$. Relative to a basis of eigenvectors, the matrix $[T - \lambda_i I]$ is diagonal with precisely $m_i$ zeros along its main diagonal (just look at the matrix $A$ shown above and subtract off $\lambda_i I$). From Theorem 5.15 we know that the rank of a linear transformation is the same as the rank of its matrix representation, and hence $r(T - \lambda_i I)$ is just the number of remaining nonzero rows in $[T - \lambda_i I]$ which is $\dim V - m_i$ (see Theorem 3.9). But from Theorem 5.6 we then see that

$$
\dim V_{\lambda_i} = \dim \text{Ker}(t - \lambda_i I) = \text{nul}(T - \lambda_i I) = \dim V - r(T - \lambda_i I) = m_i.
$$

In other words, if $T$ is diagonalizable, then the dimension of each eigenspace $V_{\lambda_i}$ is just the multiplicity of the eigenvalue $\lambda_i$. Let us clarify this in terms of some common terminology. In so doing, we will also repeat this conclusion from a slightly different viewpoint.

Given a linear operator $T \in \text{L}(V)$, what we have called the multiplicity of an eigenvalue $\lambda$ is the largest positive integer $m$ such that $(x - \lambda)^m$ divides the characteristic polynomial $\Delta_T(x)$. This is properly called the **algebraic multiplicity** of $\lambda$, in contrast to the **geometric multiplicity** which is the number of linearly independent eigenvectors belonging to that eigenvalue. In other words, the geometric multiplicity of $\lambda$ is the dimension of $V_{\lambda}$. In general, we will use the word “multiplicity” to mean the algebraic multiplicity. The set of
all eigenvalues of a linear operator $T \in L(V)$ is called the **spectrum** of $T$. If some eigenvalue in the spectrum of $T$ is of algebraic multiplicity $> 1$, then the spectrum is said to be **degenerate**.

If $T \in L(V)$ has an eigenvalue $\lambda$ of algebraic multiplicity $m$, then it is not hard for us to show that the dimension of the eigenspace $V_\lambda$ must be less than or equal to $m$. Note that since every element of $V_\lambda$ is an eigenvector of $T$ with eigenvalue $\lambda$, the space $V_\lambda$ must be a $T$-invariant subspace of $V$. Furthermore, every basis for $V_\lambda$ will obviously consist of eigenvectors corresponding to $\lambda$.

**Theorem 7.25** Let $T \in L(V)$ have eigenvalue $\lambda$. Then the geometric multiplicity of $\lambda$ is always less than or equal to its algebraic multiplicity. In other words, if $\lambda$ has algebraic multiplicity $m$, then $\dim V_\lambda \leq m$.

**Proof** Suppose $\dim V_\lambda = r$ and let \{\(v_1, \ldots, v_r\)\} be a basis for $V_\lambda$. By Theorem 2.10, we extend this to a basis \{\(v_1, \ldots, v_n\)\} for $V$. Relative to this basis, $T$ must have the matrix representation (see Theorem 7.19)

$$
\begin{pmatrix}
\lambda I_r & C \\
0 & D
\end{pmatrix}.
$$

Applying Theorem 4.14 and the fact that the determinant of a diagonal matrix is just the product of its (diagonal) elements, we see that the characteristic polynomial $\Delta_T(x)$ of $T$ is given by

$$
\Delta_T(x) = \begin{vmatrix}
(x - \lambda)I_r & -C \\
0 & xI_{n-r} - D
\end{vmatrix}
= \det((x - \lambda)I_r)\det(xI_{n-r} - D)
= (x - \lambda)^r \det(xI_{n-r} - D)
$$

which shows that $(x - \lambda)^r$ divides $\Delta_T(x)$. Since by definition $m$ is the largest positive integer such that $(x - \lambda)^m | \Delta_T(x)$, it follows that $r \leq m$. ■

Note that a special case of this theorem arises when an eigenvalue is of (algebraic) multiplicity 1. In this case, it then follows that the geometric and algebraic multiplicities are necessarily equal. We now proceed to show just when this will be true in general. Recall that any polynomial over an algebraically closed field will factor into linear terms (Theorem 6.13).
Theorem 7.26  Assume that $T \in \mathbb{L}(V)$ has a characteristic polynomial that factors into (not necessarily distinct) linear terms. Let $T$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ with (algebraic) multiplicities $m_1, \ldots, m_r$ respectively, and let $\dim V_{\lambda_i} = d_i$. Then $T$ is diagonalizable if and only if $m_i = d_i$ for each $i = 1, \ldots, r$.

Proof  Let $\dim V = n$. We note that since the characteristic polynomial of $T$ is of degree $n$ and factors into linear terms, it follows that $m_1 + \cdots + m_r = n$. We first assume that $T$ is diagonalizable. By definition, this means that $V$ has a basis consisting of $n$ linearly independent eigenvectors of $T$. Since each of these basis eigenvectors must belong to at least one of the eigenspaces $V_{\lambda_i}$, it follows that $V = V_{\lambda_1} + \cdots + V_{\lambda_r}$ and consequently $n \leq d_1 + \cdots + d_r$. From Theorem 7.25 we know that $d_i \leq m_i$ for each $i = 1, \ldots, r$ and hence

$$n \leq d_1 + \cdots + d_r \leq m_1 + \cdots + m_r = n$$

which implies that $d_1 + \cdots + d_r = m_1 + \cdots + m_r$ or

$$(m_1 - d_1) + \cdots + (m_r - d_r) = 0.$$ 

But each term in this equation is nonnegative (by Theorem 7.25), and hence we must have $m_i = d_i$ for each $i$.

Conversely, suppose that $d_i = m_i$ for each $i = 1, \ldots, r$. For each $i$, we know that any basis for $V_{\lambda_i}$ consists of linearly independent eigenvectors corresponding to the eigenvalue $\lambda_i$, while by Theorem 7.8, we know that eigenvectors corresponding to distinct eigenvalues are linearly independent. Therefore the union $B$ of the bases of $\{V_{\lambda_i}\}$ forms a linearly independent set of $d_1 + \cdots + d_r = m_1 + \cdots + m_r$ vectors. But $m_1 + \cdots + m_r = n = \dim V$, and hence $B$ forms a basis for $V$. Since this shows that $V$ has a basis of eigenvectors of $T$, it follows by definition that $T$ must be diagonalizable. □

The following corollary is a repeat of Corollary 2 of Theorem 7.24. Its (very easy) proof may be based entirely on the material of this section.

Corollary 1  An operator $T \in \mathbb{L}(V)$ is diagonalizable if and only if

$$V = W_1 \oplus \cdots \oplus W_r$$

where $W_1, \ldots, W_r$ are the eigenspaces corresponding to the distinct eigenvalues of $T$.

Proof  This is Exercise 7.7.1. □
Using Theorem 5.6, we see that the geometric multiplicity of an eigenvalue \( \lambda \) is given by

\[
\dim V_\lambda = \dim(\ker(T - \lambda I)) = \text{nul}(T - \lambda I) = \dim V - r(T - \lambda I).
\]

This observation together with Theorem 7.26 proves the next corollary.

**Corollary 2** An operator \( T \in L(V) \) whose characteristic polynomial factors into linear terms is diagonalizable if and only if the algebraic multiplicity of \( \lambda \) is equal to \( \dim V - r(T - \lambda I) \) for each eigenvalue \( \lambda \).

**Example 7.10** Consider the operator \( T \in L(\mathbb{R}^3) \) defined by

\[
T(x, y, z) = (9x + y, 9y, 7z).
\]

Relative to the standard basis for \( \mathbb{R}^3 \), the matrix representation of \( T \) is given by

\[
A = \begin{pmatrix}
9 & 1 & 0 \\
0 & 9 & 0 \\
0 & 0 & 7
\end{pmatrix}
\]

and hence the characteristic polynomial is

\[
\Delta_A(x) = \det(A - \lambda I) = (9 - \lambda)^2(7 - \lambda)
\]

which is a product of linear factors. However,

\[
A - 9I = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

which clearly has rank 2, and hence \( \text{nul}(T - 9) = 3 - 2 = 1 \) which is not the same as the algebraic multiplicity of \( \lambda = 9 \). Thus \( T \) is not diagonalizable.  

**Example 7.11** Consider the operator on \( \mathbb{R}^3 \) defined by the following matrix:

\[
A = \begin{pmatrix}
5 & -6 & -6 \\
-1 & 4 & 2 \\
3 & -6 & -4
\end{pmatrix}
\]
In order to avoid factoring a cubic polynomial, we compute the characteristic polynomial \( \Delta_A(x) = \det(xI - A) \) by applying Theorem 4.4 as follows (the reader should be able to see exactly what elementary row operations were performed in each step).

\[
\begin{vmatrix}
  x - 5 & 6 & 6 \\
  1 & x - 4 & -2 \\
  -3 & 6 & x + 4
\end{vmatrix} = \begin{vmatrix}
  x - 2 & 0 & -x + 2 \\
  1 & x - 4 & -2 \\
  -3 & 6 & x + 4
\end{vmatrix}
\]

\[
= (x - 2) \begin{vmatrix}
  1 & 0 & -1 \\
  1 & x - 4 & -2 \\
  -3 & 6 & x + 4
\end{vmatrix}
\]

\[
= (x - 2) \begin{vmatrix}
  0 & x - 1 & -1 \\
  0 & 6 & x + 1
\end{vmatrix}
\]

\[
= (x - 2)^2(x - 1).
\]

We now see that \( A \) has eigenvalue \( \lambda_1 = 1 \) with (algebraic) multiplicity 1, and eigenvalue \( \lambda_2 = 2 \) with (algebraic) multiplicity 2. From Theorem 7.25 we know that the algebraic and geometric multiplicities of \( \lambda_1 \) are necessarily the same and equal to 1, so we need only consider \( \lambda_2 \). Observing that

\[
A - 2I = \begin{pmatrix}
  3 & -6 & -6 \\
  -1 & 2 & 2 \\
  3 & -6 & -6
\end{pmatrix}
\]

it is obvious that \( r(A - 2I) = 1 \), and hence \( \text{nul}(A - 2I) = 3 - 1 = 2 \). This shows that \( A \) is indeed diagonalizable.

Let us now construct bases for the eigenspaces \( W_i = V_{\lambda_i} \). This means that we seek vectors \( v = (x, y, z) \in \mathbb{R}^3 \) such that \( (A - \lambda_i I)v = 0 \). This is easily solved by the usual row reduction techniques as follows. For \( \lambda_1 = 1 \) we have

\[
A - I = \begin{pmatrix}
  4 & -6 & -6 \\
  -1 & 3 & 2 \\
  3 & -6 & -5
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -1 \\
  -1 & 3 & 2 \\
  3 & -6 & -5
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -1 \\
  0 & 3 & 1 \\
  0 & -6 & -2
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -1 \\
  0 & 3 & 1 \\
  0 & 0 & 0
\end{pmatrix}
\]
which has the solutions \( x = z \) and \( y = -z/3 = -x/3 \). Therefore \( W_1 \) is spanned by the single eigenvector \( v_1 = (3, -1, 3) \). As to \( \lambda_2 = 2 \), we proceed in a similar manner to obtain

\[
A - 2I = \begin{pmatrix}
3 & -6 & -6 \\
-1 & 2 & 2 \\
3 & -6 & -6
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & -2 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

which implies that any vector \((x, y, z)\) with \( x = 2y + 2z \) will work. For example, we can let \( x = 0 \) and \( y = 1 \) to obtain \( z = -1 \), and hence one basis vector for \( W_2 \) is given by \( v_2 = (0, 1, -1) \). If we let \( x = 1 \) and \( y = 0 \), then we have \( z = 1/2 \) so that another independent basis vector for \( W_2 \) is given by \( v_3 = (2, 0, 1) \). In terms of these eigenvectors, the transformation matrix \( P \) that diagonalizes \( A \) is given by

\[
P = \begin{pmatrix}
3 & 0 & 2 \\
-1 & 1 & 0 \\
3 & -1 & 1
\end{pmatrix}
\]

and we leave it to the reader to verify that \( AP = PD \) (i.e., \( P^{-1}AP = D \)) where \( D \) is the diagonal matrix with diagonal elements \( d_{11} = 1 \) and \( d_{22} = d_{33} = 2 \).

Finally, we note that since \( A \) is diagonalizable, Theorems 7.12 and 7.24 show that the minimal polynomial for \( A \) must be \( (x - 1)(x - 2) \).

Exercises


2. Show that two similar matrices \( A \) and \( B \) have the same eigenvalues, and these eigenvalues have the same geometric multiplicities.

3. Let \( \lambda_1, \ldots, \lambda_r \in \mathcal{F} \) be distinct, and let \( D \in M_n(\mathcal{F}) \) be diagonal with a characteristic polynomial of the form

\[
\Delta_D(x) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_r)^{d_r}.
\]

Let \( V \) be the space of all \( n \times n \) matrices \( B \) that commute with \( D \), i.e., the set of all \( B \) such that \( BD = DB \). Prove that \( \dim V = d_1^2 + \cdots + d_r^2 \).

4. Relative to the standard basis, let \( T \in L(\mathbb{R}^4) \) be represented by
Find conditions on $a$, $b$ and $c$ such that $T$ is diagonalizable.

5. Determine whether or not each of the following matrices is diagonalizable. If it is, find a nonsingular matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0
\end{pmatrix}.
\]

6. Determine whether or not each of the following operators $T \in \text{L}(\mathbb{R}^3)$ is diagonalizable. If it is, find an eigenvector basis for $\mathbb{R}^3$ such that $[T]$ is diagonal.

(a) $T(x, y, z) = (-y, x, 3z)$.
(b) $T(x, y, z) = (8x + 2y - 2z, 3x + 3y - z, 24x + 8y - 6z)$.
(c) $T(x, y, z) = (4x + z, 2x + 3y + 2z, x + 4z)$.
(d) $T(x, y, z) = (-2y - 3z, x + 3y + 3z, z)$.

7. Suppose a matrix $A$ is diagonalizable. Prove that $A^m$ is diagonalizable for any positive integer $m$.

8. Summarize several of our results by proving the following theorem:
Let $V$ be finite-dimensional, suppose $T \in L(V)$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, and let $W_i = \text{Ker}(T - \lambda_i I)$. Then the following are equivalent:

(a) $T$ is diagonalizable.
(b) $\Delta_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r}$ and $W_i$ is of dimension $m_i$ for each $i = 1, \ldots, r$.
(c) $\dim W_1 + \cdots + \dim W_r = \dim V$.

9. Let $V_3$ be the space of real polynomials of degree at most 3, and let $f'$ and $f''$ denote the first and second derivatives of $f \in V$. Define $T \in L(V_3)$ by $T(f) = f'$ + $f''$. Decide whether or not $T$ is diagonalizable, and if it is, find a basis for $V_3$ such that $[T]$ is diagonal.

10. (a) Let $V_2$ be the space of real polynomials of degree at most 2, and define $T \in L(V_2)$ by $T(ax^2 + bx + c) = cx^2 + bx + a$. Decide whether or not $T$ is diagonalizable, and if it is, find a basis for $V_2$ such that $[T]$ is diagonal.

(b) Repeat part (a) with $T = (x + 1)(d/dx)$. (See Exercise 7.3.17.)

### 7.8 PROJECTIONS

In this section we introduce the concept of projection operators and show how they may be related to direct sum decompositions where each of the subspaces in the direct sum is invariant under some linear operator.

Suppose that $U$ and $W$ are subspaces of a vector space $V$ with the property that $V = U \oplus W$. Then every $v \in V$ has a unique representation of the form $v = u + w$ where $u \in U$ and $w \in W$ (Theorem 2.14). We now define the mapping $E: V \to V$ by $Ev = u$. Note that $E$ is well-defined since the direct sum decomposition is unique. Moreover, given any other $v' \in V = U \oplus W$ with $v' = u' + w'$, we know that $v + v' = (u + u') + (w + w')$ and $kv = ku + kw$, and hence it is easy to see that $E$ is in fact linear because

$$E(v + v') = u + u' = Ev + Ev'$$

and

$$E(kv) = ku = k(Ev).$$

The linear operator $E \in L(V)$ is called the projection of $V$ on $U$ in the direction of $W$. Furthermore, since any $u \in U \subset V$ may be written in the form $u = u + 0$, we also see that $Eu = u$ and therefore

$$E^2 v = E(Ev) = Eu = u = Ev.$$
In other words, a projection operator $E$ has the property that $E^2 = E$. By way of terminology, any operator $T \in L(V)$ with the property that $T^2 = T$ is said to be idempotent.

On the other hand, given a vector space $V$, suppose we have an operator $E \in L(V)$ with the property that $E^2 = E$. We claim that $V = \text{Im } E \oplus \text{Ker } E$. Indeed, first note that if $u \in \text{Im } E$, then by definition this means there exists $v \in V$ with the property that $Ev = u$. It therefore follows that

$$Eu = E(Ev) = E^2v = Ev = u$$

and thus $Eu = u$ for any $u \in \text{Im } E$. Conversely, the equation $Eu = u$ obviously says that $u \in \text{Im } E$, and hence we see that $u \in \text{Im } E$ if and only if $Eu = u$.

Next, note that given any $v \in V$ we may clearly write

$$v = Ev + v - Ev = Ev + (1 - E)v$$

where by definition, $Ev \in \text{Im } E$. Since

$$E[(1 - E)v] = (E - E^2)v = (E - E)v = 0$$

we see that $(1 - E)v \in \text{Ker } E$, and hence $V = \text{Im } E + \text{Ker } E$. We claim that this sum is in fact direct. To see this, let $w \in \text{Im } E \cap \text{Ker } E$. Since $w \in \text{Im } E$ and $E^2 = E$, we have seen that $Ew = w$, while the fact that $w \in \text{Ker } E$ means that $Ew = 0$. Therefore $w = 0$ so that $\text{Im } E \cap \text{Ker } E = \{0\}$, and hence

$$V = \text{Im } E \oplus \text{Ker } E .$$

Since we have now shown that any $v \in V$ may be written in the unique form $v = u + w$ with $u \in \text{Im } E$ and $w \in \text{Ker } E$, it follows that $Ev = Eu + Ew = u + 0 = u$ so that $E$ is the projection of $V$ on $\text{Im } E$ in the direction of $\text{Ker } E$.

It is also of use to note that

$$\text{Ker } E = \text{Im } (1 - E)$$

and

$$\text{Ker } (1 - E) = \text{Im } E .$$

To see this, suppose $w \in \text{Ker } E$. Then

$$w = Ew + (1 - E)w = (1 - E)w$$
which implies that \( w \in \text{Im}(1 - E) \), and hence \( \text{Ker} E \subseteq \text{Im}(1 - E) \). On the other hand, if \( w \in \text{Im}(1 - E) \) then there exists \( w' \in V \) such that \( w = (1 - E)w' \) and hence

\[
Ew = (E - E^2)w' = (E - E)w' = 0
\]

so that \( w \in \text{Ker} E \). This shows that \( \text{Im}(1 - E) \subseteq \text{Ker} E \), and therefore \( \text{Ker} E = \text{Im}(1 - E) \). The similar proof that \( \text{Ker}(1 - E) = \text{Im} E \) is left as an exercise for the reader (Exercise 7.8.1).

**Theorem 7.27** Let \( V \) be a vector space with \( \dim V = n \), and suppose \( E \in \text{L}(V) \) has rank \( k = \dim(\text{Im} E) \). Then \( E \) is idempotent (i.e., \( E^2 = E \)) if and only if any one of the following statements is true:

(a) If \( v \in \text{Im} E \), then \( Ev = v \).

(b) \( V = \text{Im} E \oplus \text{Ker} E \) and \( E \) is the projection of \( V \) on \( \text{Im} E \) in the direction of \( \text{Ker} E \).

(c) \( \text{Im} E = \text{Ker}(1 - E) \) and \( \text{Ker} E = \text{Im}(1 - E) \).

(d) It is possible to choose a basis for \( V \) such that \( [E] = I_k \oplus 0_{n-k} \).

**Proof** Suppose \( E^2 = E \). In view of the above discussion, all that remains is to prove part (d). Applying part (b), we let \( \{e_1, \ldots, e_k\} \) be a basis for \( \text{Im} E \) and \( \{e_{k+1}, \ldots, e_n\} \) be a basis for \( \text{Ker} E \). By part (a), we know that \( Ee_i = e_i \) for \( i = 1, \ldots, k \), and by definition of \( \text{Ker} E \), we have \( Ee_i = 0 \) for \( i = k + 1, \ldots, n \). But then \( [E] \) has the desired form since the \( i \)th column of \( [E] \) is just \( Ee_i \).

Conversely, suppose (a) is true and \( v \in V \) is arbitrary. Then \( E^2v = E(Ev) = Ev \) implies that \( E^2 = E \). Now suppose that (b) is true and \( v \in V \). Then \( v = u + w \) where \( u \in \text{Im} E \) and \( w \in \text{Ker} E \). Therefore \( Ev = Eu + Ew = Eu = u \) (by definition of projection) and \( E^2v = E^2u = Eu = u \) so that \( E^2 = E \) for all \( v \in V \), and hence \( E^2 = E \). If (c) holds and \( v \in V \), then \( Ev \in \text{Im} E = \text{Ker}(1 - E) \) so that \( 0 = (1 - E)Ev = Ev - E^2v \) and hence \( E^2v = Ev \) again. Similarly, \( (1 - E)v \in \text{Im}(1 - E) = \text{Ker} E \) so that \( 0 = E(1 - E)v = Ev - E^2v \) and hence \( E^2v = Ev \). In either case, we have \( E^2 = E \). Finally, from the form of \( [E] \) given in (d), it is obvious that \( E^2 = E \). ■

It is also worth making the following observation. If we are given a vector space \( V \) and a subspace \( W \subseteq V \), then there may be many subspaces \( U \subseteq V \) with the property that \( V = U \oplus W \). For example, the space \( \mathbb{R}^3 \) is not necessarily represented by the usual orthogonal Cartesian coordinate system. Rather, it may be viewed as consisting of a line plus any (oblique) plane not containing the given line. However, in the particular case that \( V = W \oplus W^4 \), then \( W^4 \) is uniquely specified by \( W \) (see Section 2.5). In this case, the projection \( E \in \text{L}(V) \) defined by \( Ev = w \) with \( w \in W \) is called the **orthogonal projection** of \( V \).
on $W$. In other words, $E$ is an orthogonal projection if $(\text{Im } E)^\perp = \text{Ker } E$. By the corollary to Theorem 2.22, this is equivalent to the requirement that $(\text{Ker } E)^\perp = \text{Im } E$.

It is not hard to generalize these results to the direct sum of more than two subspaces. Indeed, suppose that we have a vector space $V$ such that $V = W_1 \oplus \cdots \oplus W_r$. Since any $v \in V$ has the unique representation as $v = w_1 + \cdots + w_r$ with $w_i \in W_i$, we may define for each $j = 1, \ldots, r$ the operator $E_j \in \mathbb{L}(V)$ by $E_j v = w_j$. That each $E_j$ is in fact linear is easily shown exactly as above for the simpler case. It should also be clear that $\text{Im } E_j = W_j$ (see Exercise 7.8.2). If we write

$$w_j = 0 + \cdots + 0 + w_j + 0 + \cdots + 0$$

as the unique expression for $w_j \in W_j \subset V$, then we see that $E_j w_j = w_j$, and hence for any $v \in V$ we have

$$E_j^2 v = E_j(E_j v) = E_j w_j = w_j = E_j v$$

so that $E_j^2 = E_j$.

The representation of each $w_j$ as $E_j v$ is very useful because we may write any $v \in V$ as

$$v = w_1 + \cdots + w_r = E_1 v + \cdots + E_r v = (E_1 + \cdots + E_r) v$$

and thus we see that $E_1 + \cdots + E_r = 1$. Furthermore, since the image of $E_j$ is $W_j$, it follows that if $E_j v = 0$ then $w_j = 0$, and hence

$$\text{Ker } E_j = W_1 \oplus \cdots \oplus W_{j-1} \oplus W_{j+1} \oplus \cdots \oplus W_r.$$ 

We then see that for any $j = 1, \ldots, r$ we have $V = \text{Im } E_j \oplus \text{Ker } E_j$ exactly as before. It is also easy to see that $E_i E_j = 0$ if $i \neq j$ because $\text{Im } E_j = W_j \subset \text{Ker } E_i$.

**Theorem 7.28** Let $V$ be a vector space, and suppose that $V = W_1 \oplus \cdots \oplus W_r$. Then for each $j = 1, \ldots, r$ there exists a linear operator $E_j \in \mathbb{L}(V)$ with the following properties:

(a) $1 = E_1 + \cdots + E_r$.

(b) $E_i E_j = 0$ if $i \neq j$.

(c) $E_j^2 = E_j$.

(d) $\text{Im } E_j = W_j$.

Conversely, if $\{E_1, \ldots, E_r\}$ are linear operators on $V$ that obey properties (a) and (b), then each $E_j$ is idempotent and $V = W_1 \oplus \cdots \oplus W_r$ where $W_j = \text{Im } E_j$. 


Proof In view of the previous discussion, we only need to prove the converse statement. From (a) and (b) we see that

$$E_j = E_j 1 = E_j (E_1 + \cdots + E_r) = E_j^2 + \sum_{i \neq j} E_i E_j = E_j^2$$

which shows that each $E_j$ is idempotent. Next, property (a) shows us that for any $v \in V$ we have

$$v = 1v = E_1 v + \cdots + E_r v$$

and hence $V = W_1 + \cdots + W_r$ where we have defined $W_j = \text{Im } E_j$. Now suppose that $0 = w_1 + \cdots + w_r$ where each $w_j \in W_j$. If we can show that this implies $w_1 = \cdots = w_r = 0$, then any $v \in V$ will have a unique representation $v = v_1 + \cdots + v_r$ with $v_i \in W_i$. This is because if

$$v = v_1 + \cdots + v_r = v_1' + \cdots + v_r'$$

then

$$(v_1 - v_1') + \cdots + (v_r - v_r') = 0$$

would imply that $v_i - v_i' = 0$ for each $i$, and thus $v_i = v_i'$. Hence it will follow that $V = W_1 \oplus \cdots \oplus W_r$ (Theorem 2.14).

Since $w_1 + \cdots + w_r = 0$, it is obvious that $E_j (w_1 + \cdots + w_r) = 0$. However, note that $E_j w_i = 0$ if $i \neq j$ (because $w_i \in \text{Im } E_i$ and $E_j E_i = 0$), while $E_j w_j = w_j$ (since $w_j = E_j w'$ for some $w' \in V$ and hence $E_j w_j = E_j^2 w' = E_j w' = w_j$). This shows that $w_1 = \cdots = w_r = 0$ as desired. ■

We now turn our attention to invariant direct sum decompositions, referring to Section 7.5 for notation. We saw in Corollary 1 of Theorem 7.26 that a diagonalizable operator $T \in L(V)$ leads to a direct sum decomposition of $V$ in terms of the eigenspaces of $T$. However, Theorem 7.28 shows us that such a decomposition should lead to a collection of projections on these eigenspaces. Our next theorem elaborates on this observation in detail. Before stating and proving this result however, let us take another look at a matrix that has been diagonalized. We observe that a diagonal matrix of the form

$$A = \begin{pmatrix}
\lambda_1 I_{m_1} & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 I_{m_2} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \lambda_r I_{m_r}
\end{pmatrix}$$

can also be written as
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If we define $E_i$ to be the matrix obtained from $A$ by setting $\lambda_i = 1$ and $\lambda_j = 0$ for each $j \neq i$ (i.e., the $i$th matrix in the above expression), then this may be written in the simple form

$$A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_r E_r$$

where clearly

$$I = E_1 + E_2 + \cdots + E_r.$$

Furthermore, it is easy to see that the matrices $E_i$ have the property that

$$E_i E_j = 0 \text{ if } i \neq j$$

and

$$E_i^2 = E_i \neq 0.$$

With these observations in mind, we now prove this result in general.

**Theorem 7.29** If $T \in \text{L}(V)$ is a diagonalizable operator with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then there exist linear operators $E_1, \ldots, E_r$ in $\text{L}(V)$ such that:

(a) $I = E_1 + \cdots + E_r$.
(b) $E_i E_j = 0$ if $i \neq j$.
(c) $T = \lambda_1 E_1 + \cdots + \lambda_r E_r$.
(d) $E_j^2 = E_j$.
(e) $\text{Im } E_j = W_j$ where $W_j = \text{Ker}(T - \lambda_j I)$ is the eigenspace corresponding to $\lambda_j$.

Conversely, if there exist distinct scalars $\lambda_1, \ldots, \lambda_r$ and distinct nonzero linear operators $E_1, \ldots, E_r$ satisfying properties (a), (b) and (c), then properties (d) and (e) are also satisfied, and $T$ is diagonalizable with $\lambda_1, \ldots, \lambda_r$ as its distinct eigenvalues.

**Proof** First assume $T$ is diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ and let $W_1, \ldots, W_r$ be the corresponding eigenspaces. By Corollary 1 of Theorem 7.26 we know that $V = W_1 \oplus \cdots \oplus W_r$. (Note that we do not base this result on Theorem 7.24, and hence the present theorem does not depend in any way on the primary decomposition theorem.) Then Theorem 7.28 shows the existence of the projection operators $E_1, \ldots, E_r$ satisfying properties (a),
(b), (d) and (e). As to property (c), we see (by property (a)) that for any \( v \in V \) we have \( v = E_1v + \cdots + E_rv \). Since \( E_jv \in W_j \), we know from the definition of eigenspace that \( T(E_jv) = \lambda_j(E_jv) \), and therefore

\[
Tv = T(E_1v) + \cdots + T(E_rv) \\
= \lambda_1(E_1v) + \cdots + \lambda_r(E_rv) \\
= (\lambda_1E_1 + \cdots + \lambda_rE_r)v
\]

which verifies property (c).

Now suppose that we are given a linear operator \( T \in L(V) \) together with distinct scalars \( \lambda_1, \ldots, \lambda_r \) and (nonzero) linear operators \( E_1, \ldots, E_r \) that obey properties (a), (b) and (c). Multiplying (a) by \( E_i \) and using (b) proves (d). Now multiply (c) from the right by \( E_i \) and use property (b) to obtain \( TE_i = \lambda_i E_i \) or \( (T - \lambda_i 1)E_i = 0 \). If \( w_i \in \text{Im } E_i \) is arbitrary, then \( w_i = E_iw_i' \) for some \( w_i' \in V \) and hence \( (T - \lambda_i 1)w_i = (T - \lambda_i 1)E_iw_i' = 0 \) which shows that \( w_i \in \text{Ker}(T - \lambda_i 1) \). Since \( E_i \neq 0 \), this shows the existence of a nonzero vector \( w_i \in \text{Ker}(T - \lambda_i 1) \) with the property that \( Tw_i = \lambda_i w_i \). This proves that each \( \lambda_i \) is an eigenvalue of \( T \). We claim that there are no other eigenvalues of \( T \) other than \( \{\lambda_i\} \). To see this, let \( \alpha \) be any scalar and assume that \( (T - \alpha 1)v = 0 \) for some nonzero \( v \in V \). Using properties (a) and (c), we see that

\[
T - \alpha 1 = (\lambda_1 - \alpha)E_1 + \cdots + (\lambda_r - \alpha)E_r
\]

and hence letting both sides of this equation act on \( v \) yields

\[
0 = (\lambda_1 - \alpha)E_1v + \cdots + (\lambda_r - \alpha)E_rv.
\]

Multiplying this last equation from the left by \( E_i \) and using properties (b) and (d), we then see that \( (\lambda_i - \alpha)E_i v = 0 \) for every \( i = 1, \ldots, r \). Since \( v \neq 0 \) may be written as \( v = E_1v + \cdots + E_rv \), it must be true that \( E_jv \neq 0 \) for some \( j \), and hence in this case we have \( \lambda_j - \alpha = 0 \) or \( \alpha = \lambda_j \).

We must still show that \( T \) is diagonalizable, and that \( \text{Im } E_i = \text{Ker}(T - \lambda_i 1) \). It was shown in the previous paragraph that any nonzero \( w_i \in \text{Im } E_i \) satisfies \( Tw_i = \lambda_i w_i \), and hence any nonzero vector in the image of any \( E_i \) is an eigenvector of \( E_i \). Note this says that \( \text{Im } E_i \subset \text{Ker}(T - \lambda_i 1) \). Using property (a), we see that any \( w \in V \) may be written as \( w = E_1w + \cdots + E_rw \) which shows that \( V \) is spanned by eigenvectors of \( T \). But this is just what we mean when we say that \( T \) is diagonalizable. Finally, suppose \( w_i \in \text{Ker}(T - \lambda_i 1) \) is arbitrary. Then \( (T - \lambda_i 1)w_i = 0 \) and hence (exactly as we showed above)

\[
0 = (\lambda_1 - \lambda_i)E_iw_i + \cdots + (\lambda_r - \lambda_i)E_rw_i.
\]
Thus for each $j = 1, \ldots, r$ we have

$$0 = (\lambda_j - \lambda_i)E_i w_i$$

which implies $E_i w_i = 0$ for $j \neq i$. Since $w_i = E_1 w_i + \cdots + E_r w_i$ while $E_i w_i = 0$ for $j \neq i$, we conclude that $w_i = E_i w_i$ which shows that $w_i \in \text{Im } E_i$. In other words, we have also shown that $\text{Ker}(T - \lambda_i 1) \subset \text{Im } E_i$. Together with our earlier result, this proves that $\text{Im } E_i = \text{Ker}(T - \lambda_i 1)$. □

**Exercises**

1. (a) Let $E$ be an idempotent linear operator. Show that $\text{Ker}(1 - E) = \text{Im } E$.
   (b) If $E^2 = E$, show that $(1 - E)^2 = 1 - E$.

2. Let $V = W_1 \oplus \cdots \oplus W_r$ and suppose $v = w_1 + \cdots + w_r \in V$. For each $j = 1, \ldots, r$ we define the operator $E_j$ on $V$ by $E_j v = w_j$.
   (a) Show that $E_j \in \text{L}(V)$.
   (b) Show that $\text{Im } E_j = W_j$.

3. Give a completely independent proof of Theorem 7.24 as follows:
   (a) Let $T \in \text{L}(V)$ be diagonalizable with decomposition $T = \lambda_1 E_1 + \cdots + \lambda_r E_r$. Show that $f(T) = f(\lambda_1)E_1 + \cdots + f(\lambda_r)E_r$ for any $f(x) \in F[x]$.
   (b) Use part (a) to conclude that the minimal polynomial for $T$ must be of the form $m(x) = (x - \lambda_1) \cdots (x - \lambda_r)$.
   (c) Now suppose $T \in \text{L}(V)$ has minimal polynomial

$$m(x) = (x - \lambda_1) \cdots (x - \lambda_r)$$

where $\lambda_1, \ldots, \lambda_r \in F$ are distinct. Define the polynomials

$$p_j(x) = \prod_{i \neq j} \left( \frac{x - \lambda_i}{\lambda_j - \lambda_i} \right).$$

Note that $\deg p_j = r - 1 < \deg m$. By Exercise 6.4.2, any polynomial $f$ of degree $\leq r - 1$ can be written as $f = \sum f(\lambda_i)p_i$. Defining $E_j = p_j(T)$, show that $E_j \neq 0$ and that

$$1 = E_1 + \cdots + E_r$$

and

$$T = \lambda_1 E_1 + \cdots + \lambda_r E_r.$$
(d) Show that $m|p_i p_j$ for $i \neq j$, and hence show that $E_i E_j = 0$ for $i \neq j$.
(e) Conclude that $T$ is diagonalizable.

4. Let $E_1, \ldots, E_r$ and $W_1, \ldots, W_r$ be as defined in Theorem 7.28, and suppose $T \in L(V)$.
   (a) If $TE_i = E_i T$ for every $E_i$, prove that every $W_j = \text{Im } E_j$ is $T$-invariant.
   (b) If every $W_j$ is $T$-invariant, prove that $TE_i = E_i T$ for every $E_i$. [Hint: Let $v \in V$ be arbitrary. Show that property (a) of Theorem 7.28 implies $T(E_i v) = w_i$ for some $w_i \in W_i = \text{Im } E_i$. Now show that $E_j (TE_i) v = (E_i w_i) \delta_{ij}$, and hence that $E_j (Tv) = T(E_j v)$]

5. Prove that property (e) in Theorem 7.29 holds for the matrices $E_i$ given prior to the theorem.

6. Let $W$ be a finite-dimensional subspace of an inner product space $V$.
   (a) Show that there exists precisely one orthogonal projection on $W$.
   (b) Let $E$ be the orthogonal projection on $W$. Show that for any $v \in V$ we have $\|v - Ev\| \leq \|v - w\|$ for every $w \in W$. In other words, show that $Ev$ is the unique element of $W$ that is “closest” to $v$.

7.9 QUOTIENT SPACES

Recall that in Section 1.5 we gave a brief description of normal subgroups and quotient groups (see Theorem 1.12). In this section we elaborate on and apply this concept to vector spaces, which are themselves abelian groups. In the next section we will apply this formalism to proving the triangular form theorem for linear operators.

Let $V$ be a vector space and $W$ a subspace of $V$. Since $W$ is an abelian subgroup of $V$, it is easy to see that $W$ is just a normal subgroup since for any $w \in W$ and $v \in V$ we have $v + w + (-v) = w \in W$ (remember that group multiplication in an abelian group is frequently denoted by the usual addition sign, and the inverse of an element $v$ is just $-v$). We may therefore define the quotient group $V/W$ whose elements, $v + W$ for any $v \in V$, are just the cosets of $W$ in $V$. It should be obvious that $V/W$ is also an abelian group. In fact, we will show below that $V/W$ can easily be made into a vector space.

Example 7.12 Let $V = \mathbb{R}^2$ and suppose

$$W = \{(x, y) \in \mathbb{R}^2 : y = mx \text{ for some fixed scalar } m\}.$$
In other words, $W$ is just a line through the origin in the plane $\mathbb{R}^2$. The elements of $V/W$ are the cosets $v + W = \{v + w : w \in W\}$ where $v$ is any vector in $V$.

Therefore, the set $V/W$ consists of all lines in $\mathbb{R}^2$ that are parallel to $W$ (i.e., that are displaced from $W$ by the vector $v$).

While we proved in Section 1.5 that cosets partition a group into disjoint subsets, let us repeat this proof in a different manner that should help familiarize us with some of the properties of $V/W$. We begin with several simple properties that are grouped together as a theorem for the sake of reference.

**Theorem 7.30** Let $W$ be a subspace of a vector space $V$. Then the following properties are equivalent:

(a) $u \in v + W$;
(b) $u - v \in W$;
(c) $v \in u + W$;
(d) $u + W = v + W$.

**Proof**  
(a) $\Rightarrow$ (b): If $u \in v + W$, then there exists $w \in W$ such that $u = v + w$. But then $u - v = w \in W$.
(b) $\Rightarrow$ (c): $u - v \in W$ implies $v - u = -(u - v) \in W$, and hence there exists $w \in W$ such that $v - u = w$. But then $v = u + w \in u + W$.
(c) $\Rightarrow$ (d): If $v \in u + W$, then there exists $w \in W$ such that $v = u + w$. But then $v + W = u + w + W = u + W$.
(d) $\Rightarrow$ (a): $0 \in W$ implies $u = u + 0 \in u + W = v + W$. ☐
**Theorem 7.31** Let $W$ be a subspace of a vector space $V$. Then the cosets of $W$ in $V$ are distinct, and every element of $V$ lies in some coset of $W$.

*Proof* It is easy to see that any $v \in V$ lies in $V/W$ since $v = v + 0 \in v + W$. Now suppose that $v_1 \neq v_2$ and that the cosets $v_1 + W$ and $v_2 + W$ have some element $u$ in common. Then $u \in v_1 + W$ and $u \in v_2 + W$, and hence by Theorem 7.30 we have $v_1 + W = u + W = v_2 + W$. □

Let $V$ be a vector space over $F$, and let $W$ be a subspace of $V$. We propose to make $V/W$ into a vector space. If $\alpha \in F$ and $u + W, v + W \in V/W$, we define

$$(u + W) + (v + W) = (u + v) + W$$

and

$$\alpha(u + W) = \alpha u + W.$$ 

The first thing to do is show that these operations are well-defined. In other words, if we suppose that $u + W = u' + W$ and $v + W = v' + W$, then we must show that $(u + v) + W = (u' + v') + W$ and $\alpha(u + W) = \alpha(u' + W)$. Using $u + W = u' + W$ and $v + W = v' + W$, Theorem 7.30 tells us that $u - u' \in W$ and $v - v' \in W$. But then

$$(u + v) - (u' + v') = (u - u') + (v - v') \in W$$

and hence $(u + v) + W = (u' + v') + W$. Next, we see that $u - u' \in W$ implies $\alpha(u - u') \in W$ since $W$ is a subspace. Then $\alpha u - \alpha u' \in W$ implies $\alpha u + W = \alpha u' + W$, or $\alpha(u + W) = \alpha(u' + W)$.

**Theorem 7.32** Let $V$ be a vector space over $F$ and $W$ a subspace of $V$. For any $u + W, v + W \in V/W$ and $\alpha \in F$, we define the operations

1. $(u + W) + (v + W) = (u + v) + W$
2. $\alpha(u + W) = \alpha u + W.$

Then, with these operations, $V/W$ becomes a vector space over $F$.

*Proof* Since $(0 + W) + (u + W) = (0 + u) + W = u + W$, we see that $W$ is the zero element of $V/W$. Similarly, we see that $-(u + W) = -u + W$. In view of the above discussion, all that remains is to verify axioms (V1) – (V8) for a vector space given at the beginning of Section 2.1. We leave this as an exercise for the reader (see Exercise 7.9.1). □

The vector space $V/W$ defined in this theorem is called the **quotient space** of $V$ by $W$. If $V$ is finite-dimensional, then any subspace $W \subseteq V$ is also finite-
dimensional, where in fact $\dim W \leq \dim V$ (Theorem 2.9). It is then natural to ask about the dimension of $V/W$.

**Theorem 7.33**  Let $V$ be finite-dimensional over $\mathcal{F}$ and $W$ be a subspace of $V$. Then

$$\dim V/W = \dim V - \dim W.$$  

**Proof**  Suppose $\{w_1, \ldots, w_m\}$ is a basis for $W$. By Theorem 2.10 we can extend this to a basis $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ for $V$, where $\dim V = m + r = \dim W + r$. Then any $v \in V$ may be written as

$$v = \alpha_1w_1 + \cdots + \alpha_mw_m + \beta_1v_1 + \cdots + \beta_rv_r$$

where $\{\alpha_i\}, \{\beta_j\} \in \mathcal{F}$. For ease of notation, let us define $\overline{V} = V/W$ and, for any $v \in V$, we let $\overline{v} = v + W \in \overline{V}$. Note that this association is linear because (by Theorem 7.32)

$$\overline{v + v'} = v + v' + W = v + W + v' + W = \overline{v} + \overline{v'}$$

and

$$\overline{kv} = kv + W = k(v + W) = k\overline{v}.$$  

Since $w_i \in W$, we see that $\overline{w_i} = w_i + W = W$, and hence $\overline{v} = \beta_1\overline{v_1} + \cdots + \beta_r\overline{v_r}$. Alternatively, any $\overline{v} \in \overline{V}$ may be written as

$$\overline{v} = v + W = \sum_i\alpha_iw_i + \sum_j\beta_jv_j + W = \sum_j\beta_jv_j + W = \sum_j\beta_j\overline{v_j}$$

since $\sum_i\alpha_iw_i \in W$. In any case, this shows that the $\overline{v}_i$ span $\overline{V}$. Now suppose that $\sum_i\gamma_i\overline{v}_i = 0$ for some scalars $\gamma_i \in \mathcal{F}$. Then

$$\gamma_1\overline{v}_1 + \cdots + \gamma_r\overline{v}_r = 0 = W.$$  

Using $\overline{v}_i = v_i + W$, we then see that $\gamma_1v_1 + \cdots + \gamma_rv_r + W = W$ which implies that $\sum_i\gamma_i\overline{v}_i \in W$. But $\{w_i\}$ forms a basis for $W$, and hence there exist $\delta_1, \ldots, \delta_m \in \mathcal{F}$ such that

$$\gamma_1v_1 + \cdots + \gamma_rv_r = \delta_1w_1 + \cdots + \delta_mw_m.$$  

However, $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ is a basis for $V$ and hence is linearly independent. This means that $\gamma_i = 0$ for each $i = 1, \ldots, r$ and that $\delta_j = 0$ for each $j = 1, \ldots, m$. Thus $\{\overline{v}_i\}$ is linearly independent and forms a basis for $\overline{V} = V/W$, and $\dim V/W = r = \dim V - \dim W$. ✔
There is a slightly different way of looking at this result that will be of use to us later.

**Theorem 7.34** Let $V$ be finite-dimensional over $F$ and $W$ be a subspace of $V$. Suppose that $W$ has basis $w_1, \ldots, w_m$ and $\bar{V} = V/W$ has basis $\bar{v}_1, \ldots, \bar{v}_r$, where $\bar{v}_i = v_i + W$ for some $v_i \in V$. Then $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ is a basis for $V$.

**Proof** Let $u \in V$ be arbitrary. Then $\bar{u} = u + W \in \bar{V}$, and hence there exists $\{\alpha_i\} \in F$ such that

$$u + W = \bar{u} = \alpha_1\bar{v}_1 + \cdots + \alpha_r\bar{v}_r = \alpha_1v_1 + \cdots + \alpha_r v_r + W.$$

By Theorem 7.30 there exists $w = \beta_1w_1 + \cdots + \beta_m w_m \in W$ such that

$$u = \alpha_1v_1 + \cdots + \alpha_r v_r + w = \alpha_1v_1 + \cdots + \alpha_r v_r + \beta_1w_1 + \cdots + \beta_m w_m.$$

This shows that $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ spans $V$.

To show that these vectors are linearly independent, we suppose that

$$\gamma_1w_1 + \cdots + \gamma_m w_m + \delta_1v_1 + \cdots + \delta_r v_r = 0.$$

Since the association between $V$ and $\bar{V}$ is linear (see the proof of Theorem 7.33) and $\bar{w}_i = w_i + W = W$, we see that

$$\delta_1\bar{v}_1 + \cdots + \delta_r \bar{v}_r = \bar{0} = W.$$

But the $\bar{v}_i$ are linearly independent (since they are a basis for $\bar{V}$), and hence $\delta_1 = \cdots = \delta_r = 0$. (This is just the definition of linear independence if we recall that $W$ is the zero vector in $\bar{V} = V/W$.) This leaves us with $\gamma_1 w_1 + \cdots + \gamma_m w_m = 0$. But again, the $w_i$ are linearly independent, and therefore $\gamma_1 = \cdots = \gamma_m = 0$. This shows that $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ is a basis for $V$, and hence $\dim V = \dim W + \dim V/W$. 

**Exercises**

1. Finish the proof of Theorem 7.32.
2. Let $U$ and $V$ be finite-dimensional, and suppose $T$ is a linear transformation of $U$ onto $V$. If $W = \text{Ker } T$, prove that $V$ is isomorphic to $U/W$. \textit{[Hint: See Exercise 1.5.11.]}

3. Let $V$ be a vector space over $\mathcal{F}$, and let $W$ be a subspace of $V$. Define a relation $R$ on the set $V$ by $xRy$ if $x - y \in W$.
   (a) Show that $R$ defines an equivalence relation on $V$.
   (b) Let the equivalence class of $x \in V$ be denoted by $[x]$, and define the quotient set $V/R$ to be the set of all such equivalence classes. For all $x, y \in V$ and $a \in \mathcal{F}$ we define addition and scalar multiplication in $V/R$ by

   \[ [x] + [y] = [x + y] \]

   and

   \[ a[x] = [ax] . \]

   Show that these operation are well-defined, and that $V/R$ is a vector space over $\mathcal{F}$.
   (c) Now assume that $V$ is finite-dimensional, and define the mapping $T$: $V \rightarrow V/R$ by $Tx = [x]$. Show that this defines a linear transformation.
   (d) Using Theorem 5.6, prove that $\dim V/R + \dim W = \dim V$.

7.10 \textbf{THE TRIANGULAR FORM THEOREM *}

Now that we know something about quotient spaces, let us look at the effect of a linear transformation on such a space. Unless otherwise noted, we restrict our discussion to finite-dimensional vector spaces. In particular, suppose that $T \in L(V)$ and $W$ is a $T$-invariant subspace of $V$. We first show that $T$ induces a natural linear transformation on the space $V/W$. (The reader should be careful to note that $\bar{0}$ in Theorem 7.33 is the zero vector in $\bar{V} = V/W$, while in the theorem below, $\bar{0}$ is the zero transformation on $\bar{V}$.)

\textbf{Theorem 7.35} Suppose $T \in L(V)$ and let $W$ be a $T$-invariant subspace of $V$. Then $T$ induces a linear transformation $\bar{T} \in L(V/W)$ defined by

\[ \bar{T}(v + W) = T(v) + W . \]

Furthermore, if $T$ satisfies any polynomial $p(x) \in \mathcal{F}[x]$, then so does $\bar{T}$. In particular, the minimal polynomial $\bar{m}(x)$ for $\bar{T}$ divides the minimal polynomial $m(x)$ for $T$. 

Proof. Our first task is to show that $\overline{T}$ is well-defined and linear. Thus, suppose $v + W = v' + W$. Then $v - v' \in W$ so that $T(v - v') = T(v) - T(v') \in W$ since $W$ is $T$-invariant. Therefore, using Theorem 7.30, we see that

$$
\overline{T}(v + W) = T(v) + W = T(v') + W = \overline{T}(v' + W)
$$

and hence $\overline{T}$ is well-defined. To show that $\overline{T}$ is a linear transformation, we simply calculate

$$
\overline{T}[(v_1 + W) + (v_2 + W)] = \overline{T}(v_1 + v_2 + W) = T(v_1 + v_2) + W
$$

$$
= T(v_1) + T(v_2) + W = T(v_1) + W + T(v_2) + W
$$

and

$$
\overline{T}[\alpha(v + W)] = \overline{T}(\alpha v + W) = T(\alpha v) + W = \alpha T(v) + W
$$

$$
= \alpha[T(v) + W] = \alpha\overline{T}(v + W) .
$$

This proves that $\overline{T}$ is indeed a linear transformation.

Next we observe that for any $T \in L(V)$, $T^2$ is a linear transformation and $W$ is also invariant under $T^2$. This means that we can calculate the effect of $\overline{T^2}$ on any $v + W \in V/W$:

$$
\overline{T^2}(v + W) = T^2(v) + W = T[T(v)] + W = \overline{T}[T(v) + W]
$$

$$
= \overline{T}(v + W) = \overline{T}(v) + W .
$$

This shows that $\overline{T^2} = \overline{T}^2$, and it is easy to see that in general $\overline{T^m} = \overline{T}^m$ for any $m \geq 0$. Then for any $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{F}[x]$, we have

$$
\overline{p(T)}(v + W) = p(T)(v) + W = \sum a_m T^m(v) + W = \sum a_m [T^m(v) + W]
$$

$$
= \sum a_m \overline{T^m}(v + W) = \sum a_m \overline{T^m}(v + W) = p(\overline{T})(v + W)
$$

so that $\overline{p(T)} = p(\overline{T})$.

Now note that for any $v + W \in V/W$ we have $\overline{0}(v + W) = 0(v) + W = W$, and hence $\overline{0}$ is the zero transformation on $V/W$ (since $W$ is the zero vector in $V/W$). Therefore, if $p(T) = 0$ for some $p(x) \in \mathcal{F}[x]$, we see that $\overline{0} = \overline{p(T)} = p(\overline{T})$ and hence $\overline{T}$ satisfies $p(x)$ also.

Finally, let $\overline{m}(x)$ be the minimal polynomial for $\overline{T}$. If $p(x)$ is such that $p(\overline{T}) = \overline{0}$, then we know that $\overline{m} | p$ (Theorem 7.4). If $m(x)$ is the minimal polynomial for $T$, then $m(T) = 0$, and therefore $m(\overline{T}) = \overline{0}$ by what we just proved. Hence $\overline{m} | m$. ■
We now come to the main result of this section. By way of motivation, we saw in Theorem 7.24 that a linear transformation \( T \in \text{L}(V) \) is diagonalizable if and only if the minimal polynomial for \( T \) can be written as the product of distinct linear factors. What we wish to do now is take into account the more general case where all of the distinct factors of the minimal polynomial are linear, but possibly with a multiplicity greater than one. We shall see that this leads to a triangular form for the matrix representation of \( T \).

For definiteness we consider upper-triangular matrices, so what we are looking for is a basis \( \{v_i\} \) for \( V \) in which the action of \( T \) takes the form

\[
T(v_1) = v_1a_{11}
\]
\[
T(v_2) = v_1a_{12} + v_2a_{22}
\]
\[
\vdots
\]
\[
T(v_n) = v_1a_{1n} + \cdots + v_na_{nn}.
\]

We present two versions of this theorem in the present chapter. The first is more intuitive, while the second requires the development of some additional material that is also of use in other applications. Because of this, we postpone the second version until after we have discussed nilpotent transformations in the next section. (Actually, Exercises 7.5.7 and 7.5.8 outlined another way to prove the first version that used the formalism of ideals and annihilators, and made no reference whatsoever to quotient spaces.) Furthermore, in Section 8.1 we give a completely independent proof for the special case of matrices over an algebraically closed field.

Recall from Theorem 7.9 that an element \( \lambda \in \mathcal{F} \) is an eigenvalue of \( T \) if and only if \( \lambda \) is a root of the characteristic polynomial of \( T \). Thus all the eigenvalues of \( T \) lie in \( \mathcal{F} \) if and only if the characteristic polynomial factors into linear terms.

**Theorem 7.36 (Triangular Form Theorem)** Suppose \( T \in \text{L}(V) \) has a characteristic polynomial that factors into (not necessarily distinct) linear terms. Then \( V \) has a basis in which the matrix of \( T \) is triangular.

**Proof** If \( \text{dim } V = 1 \), then \( T \) is represented by a \( 1 \times 1 \) matrix which is certainly triangular. Now suppose that \( \text{dim } V = n > 1 \). We assume the theorem is true for \( \text{dim } V = n - 1 \), and proceed by induction to prove the result for \( \text{dim } V = n \). Since the characteristic polynomial of \( T \) factors into linear polynomials, there exists at least one nonzero eigenvalue \( \lambda_i \) and corresponding eigenvector \( v_i \) such that \( T(v_i) = \lambda_iv_i \). Let \( W \) be the one-dimensional \( T \)-invariant subspace spanned by \( v_i \), and define \( \overline{V} = V/W \) so that (by Theorem 7.33) \( \text{dim } \overline{V} = \text{dim } V - \text{dim } W = n - 1 \).
According to Theorem 7.35, T induces a linear transformation $T\xi$ on $V\xi$ such that the minimal polynomial $m(x)$ for $T\xi$ divides the minimal polynomial $m(x)$ for T. But this means that any root of $m(x)$ is also a root of $m(x)$. Since the characteristic polynomial of T factors into linear polynomials by hypothesis, so does $m(x)$ (see Theorem 7.12). Therefore $m(x)$ must also factor into linear polynomials, and hence so does the characteristic polynomial of T. This shows that $T\xi$ and $V\xi$ satisfy the hypotheses of the theorem, and hence there exists a basis $\{v_2, \ldots , v_n\}$ for $V = V/W$ such that

\[
\begin{align*}
T(v_2) &= v_2 a_{22} \\
T(v_3) &= v_2 a_{23} + v_3 a_{33} \\
& \vdots \\
T(v_n) &= v_2 a_{2n} + \cdots + v_n a_{nn}.
\end{align*}
\]

We now let $v_2, \ldots , v_n$ be elements of $V$ such that $v_i = v_1 + W$ for each $i = 2, \ldots , n$. Since $W$ has basis $\{v_1\}$, Theorem 7.35 tells us that $\{v_1, v_2, \ldots , v_n\}$ is a basis for $V$. According to our above result, we have $T(v_2) = v_2 a_{22}$ which is equivalent to $T(v_2) - v_2 a_{22} = 0$, and hence the definition of $T\xi$ (see Theorem 7.35) along with Theorem 7.30 tells us that $T(v_2) - v_2 a_{22} \in W$. Since $W$ is spanned by $v_i$, this says there exists $a_{12} \in F$ such that $T(v_2) - v_2 a_{22} = v_1 a_{12}$, i.e., $T(v_2) = v_1 a_{12} + v_2 a_{22}$. Clearly, an identical argument holds for any of the $T(v_i)$, and thus for each $i = 2, \ldots , n$ there exists $a_{ii} \in F$ such that $T(v_i) - v_2 a_{2i} - \cdots - v_i a_{ii} \in W$ implies

\[
T(v_i) = v_1 a_{1i} + v_2 a_{2i} + \cdots + v_i a_{ii}.
\]

Written out, this is just

\[
\begin{align*}
T(v_1) &= v_1 a_{11} \\
T(v_2) &= v_1 a_{12} + v_2 a_{22} \\
& \vdots \\
T(v_n) &= v_1 a_{1n} + \cdots + v_n a_{nn}.
\end{align*}
\]

In other words, the elements $v_1, \ldots , v_n \in V$ are a basis for $V$ in which every $T(v_i)$ is a linear combination of $v_j$ for $j \leq i$. This is precisely the definition of triangular form. ■

We now give a restatement of this theorem in terms of matrices. For ease of reference, this version is presented as a corollary.
Corollary Let $A \in M_n(F)$ be a matrix whose characteristic polynomial factors into linear polynomials. Then $A$ is similar to a triangular matrix.

Proof The matrix $A = (a_{ij})$ defines a linear transformation $T$ on the space $F^n$ by $T(v) = \sum_{j=1}^n v_j a_{ij}$ where $\{v_i\}$ is a basis for $F^n$. In particular, relative to the basis $\{v_i\}$, the matrix representation of $T$ is precisely the matrix $A$ (since $T$ takes the $i$th basis vector into the $i$th column of the matrix representing $T$). Since the characteristic polynomial of $T$ is independent of the basis used in the matrix representation of $T$, and the characteristic polynomial of $A$ factors into linear polynomials, we see that Theorem 7.36 applies to $T$. Thus there is a basis for $F^n$ in which the matrix of $T$ (i.e., the matrix $A$) is triangular. By Theorem 5.18 we then see that $A$ must be similar to a triangular matrix. ■

If a linear transformation $T$ can be represented by a triangular matrix, then we say that $T$ can be brought into triangular form. Since $\lambda$ is an eigenvalue of $T$ if and only if $\det(\lambda I - T) = 0$ (Theorem 7.9), Theorem 4.5 tells us that the eigenvalues of a triangular matrix are precisely the diagonal elements of the matrix (this was also discussed in the previous section).

7.11 NILPOTENT TRANSFORMATIONS *

An operator $T \in L(V)$ is said to be nilpotent if $T^n = 0$ for some positive integer $n$. If $T^k = 0$ but $T^{k-1} \neq 0$, then $k$ is called the index of nilpotency of $T$ (note that $T^{k-1} \neq 0$ implies that $T^j \neq 0$ for all $j \leq k - 1$). This same terminology applies to any square matrix $A$ with the property that $A^n = 0$. Some elementary facts about nilpotent transformations are contained in the following theorem. Note Theorem 7.1 implies that if $A$ is the matrix representation of $T$ and $T$ is nilpotent with index $k$, then $A$ is also nilpotent with index $k$.

Theorem 7.37 Suppose $T \in L(V)$, and assume that for some $v \in V$ we have $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$. Define the set

$$S = \{v, T(v), T^2(v), \ldots, T^{k-1}(v)\} .$$

Then $S$ has the following properties:

(a) The elements of $S$ are linearly independent.

(b) The linear span $W$ of $S$ is a $T$-invariant subspace of $V$.

(c) The operator $T_W = T|W$ is nilpotent with index $k$.

(d) With respect to the ordered basis $\{T^{k-1}(v), \ldots, T(v), v\}$ for $W$, the matrix of $T_W$ has the form
Thus the matrix of $T_W$ has all zero entries except on the superdiagonal where they are all equal to one. This shows that the matrix representation of $T_W$ is a nilpotent matrix with index $k$.

**Proof** (a) Suppose that

$$\alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_{k-1} T^{k-1}(v) = 0$$

for some set of scalars $\alpha_i \in F$. Applying $T^{k-1}$ to this equation results in $\alpha_0 T^{k-1}(v) = 0$. Since $T^{k-1}(v) \neq 0$, this implies that $\alpha_0 = 0$. Using this result, apply $T^{k-2}$ to the above equation to obtain $\alpha_1 = 0$. Continuing this procedure, we eventually arrive at $\alpha_i = 0$ for each $i = 0, 1, \ldots, k-1$ and hence the elements of $S$ are linearly independent.

(b) Since any $w \in W$ may be written in the form

$$w = \beta_0 v + \beta_1 T(v) + \cdots + \beta_{k-1} T^{k-1}(v)$$

we see that $T(w) = \beta_0 T(v) + \cdots + \beta_{k-2} T^{k-1}(v) \in W$, and hence $T(W) \subseteq W$.

(c) Using $T^k(v) = 0$, it follows that $T_W^k(T^i(v)) = T^{k+i}(v) = 0$ for each $i = 0, \ldots, k-1$. This shows that $T_W^k$ applied to each element of $S$ (i.e., each of the basis vectors for $W$) is zero, and thus $T_W^k = 0$. In addition, since $v \in W$ we see that $T_W^{k-1}(v) = T^{k-1}(v) \neq 0$, and therefore $T_W$ is nilpotent with index $k$.

(d) Using $T_W(T^i(v)) = T^{i+1}(v)$ along with the fact that the $i$th column of $[T_W]$ is the image of the $i$th basis vector for $W$, it is easy to see that $[T_W]$ has the desired form. 

One must be careful to understand exactly what Theorem 7.37 says and what it does not say. In particular, if $T$ is nilpotent with index $k$, then $T^k(u) = 0$ for all $u \in V$, while $T^{k-1}(v) \neq 0$ for some $v \in V$. This is because if $w = T(v)$, then $T^{k-1}(w) = T^{k-1}(T(v)) = T^k(v) = 0$. Hence it is impossible for $T^{k-1}(v)$ to be nonzero for all $v \in V$. 
It is also interesting to note that according to Theorem 7.37(b), the subspace $W$ is $T$-invariant, and hence by Theorem 7.19, the matrix of $T$ must be of the block form

$$\begin{pmatrix} M_k & B \\ 0 & C \end{pmatrix}$$

where $M_k$ is the $k \times k$ matrix of $T|_W = T|W$ defined in part (d) of Theorem 7.37. If we can find another $T$-invariant subspace $U$ of $V$ such that $V = W \oplus U$, then the matrix representation of $T$ will be in block diagonal form (Theorem 7.20). We now proceed to show that this can in fact be done. Let us first prove two more easy results.

**Theorem 7.38** Let $T \in \mathcal{L}(V)$ be nilpotent, and let $S = \alpha_1 T + \cdots + \alpha_m T^m$ where each $\alpha_i \in \mathcal{F}$. Then $\alpha_0 I + S$ is invertible for any nonzero $\alpha_0 \in \mathcal{F}$.

**Proof** Suppose the index of $T$ is $k$. Then $T^k = 0$, and therefore $S^k = 0$ also since the lowest power of $T$ in the expansion of $S^k$ is $k$. If $\alpha_0 \neq 0$, we leave it as a trivial exercise for the reader to show that

$$(\alpha_0 I + S)(1/\alpha_0 - S/\alpha_0^2 + S^2/\alpha_0^3 + \cdots + (-1)^{k-1} S^{k-1}/\alpha_0^k) = 1.$$ 

This shows that $\alpha_0 I + S$ is invertible, and that its inverse is given by the above polynomial in $S$. ■

**Theorem 7.39** Let $T \in \mathcal{L}(V)$ be nilpotent with index $n$, and let $W$ be the $T$-invariant subspace spanned by $\{T^{n-1}(v), \ldots, T(v), v\}$ where $v \in V$ is such that $T^{n-1}(v) \neq 0$. If $w \in W$ is such that $T^{n-k}(w) = 0$ for some $0 < k \leq n$, then there exists $w_0 \in W$ such that $T^k(w_0) = w$.

**Proof** Since $w \in W$, we have

$$w = \alpha_n T^{n-1}(v) + \cdots + \alpha_{k+1} T^k(v) + \alpha_k T^{n-1}(v) + \cdots + \alpha_2 T(v) + \alpha_1 v$$

and therefore (since $T^n = 0$)

$$0 = T^{n-k}(w) = \alpha_k T^{n-1}(v) + \cdots + \alpha_1 T^{n-k}(v).$$

But $\{T^{n-1}(v), \ldots, T^{n-k}(v)\}$ is linearly independent (Theorem 7.37), and thus $\alpha_k = \cdots = \alpha_1 = 0$. This means that
\[ w = \alpha_n T^{n-1}(v) + \cdots + \alpha_{k+1} T^k(v) = T^k(w_0) \]

where \( w_0 = \alpha_n T^{n-k-1}(v) + \cdots + \alpha_{k+1} v \in W. \]

We are now in a position to prove our above assertion on the decomposition of \( V. \) This (by no means trivial) result will form the basis of the principle theorem dealing with nilpotent transformations (Theorem 7.41 below). It is worth pointing out that while the following theorem will be quite useful to us, its proof is not very constructive.

**Theorem 7.40**  Let \( T \) and \( W \) be as defined in the previous theorem. Then there exists a \( T \)-invariant subspace \( U \) of \( V \) such that \( V = W \oplus U. \)

**Proof**  Let \( U \subset V \) be a \( T \)-invariant subspace of largest dimension with the property that \( W \cap U = \{0\}. \) (Such a space exists since even \( \{0\} \) is \( T \)-invariant, and \( W \cap \{0\} = \{0\} \).) We first show that \( V = W + U. \) If this is not the case, then there exists \( z \in V \) such that \( z \notin W + U. \) Since \( T^0(z) = z \notin W + U \) while \( T^n(z) = 0 \in W + U, \) it follows that there must exist an integer \( k \) with \( 0 < k \leq n \) such that \( T^k(z) \in W + U \) and \( T^j(z) \notin W + U \) for \( j < k. \) We write \( T^k(z) = w + u \) where \( w \in W \) and \( u \in U, \) and therefore

\[ 0 = T^n(z) = T^{n-k}(T^k(z)) = T^{n-k}(w) + T^{n-k}(u). \]

Since both \( W \) and \( U \) are \( T \)-invariant, we have \( T^{n-k}(w) \in W \) and \( T^{n-k}(u) \in U. \)

But \( W \cap U = \{0\} \) so that

\[ T^{n-k}(w) = -T^{n-k}(u) \in W \cap U = 0. \]

(Remember that \( W \) and \( U \) are subspaces so \( x \in U \) implies that \( -x \in U \) also.) We now apply Theorem 7.39 to conclude that there exists \( w_0 \in W \) such that \( T^k(w_0) = w, \) and hence \( T^k(z) = w + u = T^k(w_0) + u. \) Defining \( x = z - w_0, \) we then have

\[ T^k(x) = T^k(z) - T^k(w_0) = u \in U. \]

But \( U \) is \( T \)-invariant, and hence it follows that \( T^m(x) \in U \) for any \( m \geq k. \)

Considering lower powers of \( T, \) let us assume that \( j < k. \) Then the \( T \)-

invariance of \( W \) implies \( T^j(w_0) \in W, \) while we saw above that \( T^j(z) \notin W + U. \) This means that

\[ T^j(x) = T^j(z) - T^j(w_0) \notin U \]
Now let \( U_x \) be that subspace of \( V \) spanned by \( U \) together with the set \( \{ T^{-1}(x), \ldots, T(x), x \} \). Since \( U \) is \( T \)-invariant and \( T(x) \not\in U \), it must be true that \( x \not\in U \). Together with \( U \subseteq U_x \), this means \( \text{dim } U_x > \text{dim } U \). Applying \( T \) to \( \{ T^{-1}(x), \ldots, T(x), x \} \) we obtain the set \( \{ T^k(x), T^{k-1}(x), \ldots, T^2(x), T(x) \} \). Since \( T^k(x) \in U \) and the rest of the vectors in this set are included in the set that spans \( U_x \), it follows that \( U_x \) is also \( T \)-invariant.

By assumption, \( U \) is the subspace of largest dimension that is both \( T \)-invariant and satisfies \( W \cap U = \{0\} \). Since \( \text{dim } U_x > \text{dim } U \) and \( U_x \) is \( T \)-invariant, we must have \( W \cap U_x \neq \{0\} \). Therefore there exists a nonzero element in \( W \cap U_x \) of the form \( u_0 + \alpha_1 T^{-1}(x) + \cdots + \alpha_k T^k(x) + \alpha_1 x \) where \( u_0 \in U \). We can not have \( \alpha_i = 0 \) for every \( i = 1, \ldots, k \) because this would imply that \( 0 \neq u_0 \in W \cap U = \{0\} \), a contradiction. If we let \( \alpha_r \neq 0 \) be the first nonzero \( \alpha_i \), then we have

\[
u_0 + (\alpha_k T^{k-r} + \cdots + \alpha_r T^{r-1})(x) \in W.
\]

From Theorem 7.38 we see that \( \alpha_k T^{k-r} + \cdots + \alpha_r T^{r-1} \) is invertible, and its inverse is given by some polynomial \( p(T) \). Since \( W \) and \( U \) are \( T \)-invariant, they are also invariant under \( p(T) \).

Applying \( p(T) \) to \((*)\), we see that

\[
p(T)(u_0) + T^{r-1}(x) \in p(T)(W) \subseteq W.
\]

This means that \( T^{r-1}(x) \in W + p(T)(U) \subseteq W + U \). But \( r - 1 < r < k \), and hence this result contradicts the earlier conclusion that \( T^j(x) \not\in U \) for \( j < k \). Since this contradiction arose from the assumed existence of an element \( z \in V \) with \( z \not\in W + U \), we conclude that \( V = W + U \). Finally, since \( W \cap U = \{0\} \) by hypothesis, we have \( V = W \oplus U \).

Combining several previous results, the next major theorem follows quite easily.

**Theorem 7.41(a)** Let \( T \in L(V) \) be nilpotent with index of nilpotence \( n \), and let \( M_k \) be the \( k \times k \) matrix containing all 0’s except for 1’s on the superdiagonal (see Theorem 7.37). Then there exists a basis for \( V \) in which the matrix of \( T \) has the block diagonal form

\[
u_0 + (\alpha_k T^{k-r} + \cdots + \alpha_r T^{r-1})(x) \in W.
\]
\[
\begin{pmatrix}
M_{n_1} & 0 & \cdots & 0 \\
0 & M_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_{n_r}
\end{pmatrix}
\]

where \( n_1 \geq \cdots \geq n_r \) and \( n_1 + \cdots + n_r = \dim V \).

**Proof**  
Since \( T^{n_1} = 0 \) but \( T^{n_1-1} \neq 0 \), there exists \( v \in V \) such that \( T^{n_1-1}(v) \neq 0 \). Applying Theorem 7.37, we see that the vectors \( v_1 = T^{n_1-1}(v), \ldots, v_{n_1-1} = T(v), v_{n_1} = v \) are linearly independent and form the basis for a \( T \)-invariant subspace \( W_1 \) of \( V \). Moreover, the matrix of \( T_1 = T|W_1 \) in this basis is just \( M_{n_1} \).

By Theorem 7.40, there exists a \( T \)-invariant subspace \( U \subset V \) such that \( V = W_1 \oplus U \). Define a basis for \( V \) by taking the basis \( \{v_1, \ldots, v_{n_1}\} \) for \( W_1 \) together with any basis for \( U \). Then, according to Theorem 7.20, the matrix of \( T \) with respect to this basis is of the form

\[
\begin{pmatrix}
M_{n_1} & 0 \\
0 & A_2
\end{pmatrix}
\]

where \( A_2 \) is the matrix of \( T_2 = T|U \). For any \( u \in U \) and positive integer \( m \) we have \( T_2^m(u) = T^m(u) \). Since \( T^{n_1} = 0 \), we see that \( T_2^m = 0 \) for all \( m \geq n_1 \), and thus there exists an integer \( n_2 \leq n_1 \) such that \( T_2^{n_2} = 0 \). This shows that \( T_2 \) is nilpotent with index \( n_2 \).

We now repeat the above argument using \( T_2 \) and \( U \) instead of \( T \) and \( V \). This time we will decompose \( A_2 \) into

\[
\begin{pmatrix}
M_{n_2} & 0 \\
0 & A_3
\end{pmatrix}
\]

and therefore the representation of \( T \) becomes

\[
\begin{pmatrix}
M_{n_1} & 0 & 0 \\
0 & M_{n_2} & 0 \\
0 & 0 & A_3
\end{pmatrix}
\]

Continuing this process, it should be clear that we will eventually arrive at a basis for \( V \) in which the matrix of \( T \) has the desired form. It is also obvious that \( \sum_{i=1}^{r} n_i = \dim V = n \) since the matrix of \( T \) must be of size \( n \).
Our next result is a rephrasing of Theorem 7.41(a).

Theorem 7.41(b) Let \( T \in \mathcal{L}(V) \) be nilpotent with index \( k \). Then there exists a basis for \( V \) in which the matrix representation of \( T \) is block diagonal, and where each of these diagonal entries (i.e., square matrices) is of the super-diagonal form \( M \) given in Theorem 7.37. Moreover,

(a) There is at least one \( M \) matrix of size \( k \), and every other \( M \) matrix is of size \( \leq k \).

(b) The total number of \( M \) matrices in the representation of \( T \) (i.e., the total number of blocks in the representation of \( T \)) is just \( \text{nul } T = \dim(\text{Ker } T) \).

Proof See Exercise 7.11.1.

Example 7.13 Consider the matrix

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We leave it to the reader to show that \( A \) is nilpotent with index 3, i.e., \( A^3 = 0 \) but \( A^2 \neq 0 \). We seek the diagonal representation of \( A \) described in Theorem 7.41(b). It is obvious that \( r(A) = 2 \), and therefore (using Theorem 5.6) \( \text{nul } A = 5 - 2 = 3 \). Thus there are three \( M \) matrices in the diagonal representation of \( A \), and one of them must be of size 3. This means that the only possibility for the remaining two matrices is that they both be of size 1. Thus the block diagonal form for \( A \) must be

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]

It is easy to see that this matrix is also nilpotent with index 3.
Exercises

1. Prove Theorem 7.41(b). [Hint: What is the rank of the matrix in Theorem 7.41(a)?]

2. Suppose $S, T \in L(V)$ are nilpotent with the property that $ST = TS$. Show that $S + T$ and $ST$ are also nilpotent.

3. Suppose $A$ is a supertriangular matrix, i.e., all entries of $A$ on or below the main diagonal are zero. Show that $A$ is nilpotent.

4. Let $V_n$ be the vector space of all polynomials of degree $\leq n$, and let $D \in L(V_n)$ be the usual differentiation operator. Show that $D$ is nilpotent with index $n + 1$.

5. Show that the following nilpotent matrices of size $n$ are similar:

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

6. Show that two nilpotent 3 x 3 matrices are similar if and only if they have the same index of nilpotency. Give an example to show that this is not true for nilpotent 4 x 4 matrices.

7.12 THE TRIANGULAR FORM THEOREM AGAIN *

After all the discussion on nilpotent transformations in the previous section, let us return to our second version of the triangular form theorem which, as we shall see in the next chapter, is just the Jordan canonical form. While this theorem applies to a finite-dimensional vector space over an arbitrary field, the minimal polynomial $m(x)$ for $T$ must be factorable into linear polynomials. This means that all the roots of $m(x)$ must lie in $\mathcal{F}$. Clearly, this will always be true if $\mathcal{F}$ is algebraically closed.

**Theorem 7.42 (Jordan Form)** Suppose $T \in L(V)$, and assume that the minimal polynomial $m(x)$ for $T$ can be written in the form
7.12 THE TRIANGULAR FORM THEOREM AGAIN

\[ m(x) = (x - \lambda_i)^{n_i} \cdots (x - \lambda_r)^{n_r} \]

where each \( n_i \) is a positive integer, and the \( \lambda_i \) are distinct elements of \( \mathcal{F} \). Then there exists a basis for \( V \) in which the matrix representation \( A \) of \( T \) has the block diagonal form \( A = A_1 \oplus \cdots \oplus A_r \) where each \( A_i \in \text{M}_{k_i}(\mathcal{F}) \) for some integer \( k_i \geq n_i \), and each \( A_i \) has the upper triangular form

\[
\begin{pmatrix}
\lambda_i & * & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & * & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & * \\
0 & 0 & 0 & \cdots & 0 & \lambda_i \\
\end{pmatrix}
\]

where the *’s may be either 0 or 1.

**Proof**  For each \( i = 1, \ldots, r \) we define \( W_i = \ker(T - \lambda_i)_{n_i} \) and \( k_i = \dim W_i \). By the primary decomposition theorem (Theorem 7.23), we see that \( V = W_1 \oplus \cdots \oplus W_r \) (where each \( W_i \) is \( T \)-invariant), and hence according to Theorem 2.15, \( V \) has a basis which is just the union of bases of the \( W_i \). Letting the basis for \( V \) be the union of the bases for \( W_1, \ldots, W_r \) taken in this order, it follows from Theorem 7.20 that \( A = A_1 \oplus \cdots \oplus A_r \) where each \( A_i \) is the matrix representation of \( T_i = T|W_i \). We must show that each \( A_i \) has the required form, and that \( k_i \geq n_i \) for each \( i = 1, \ldots, r \).

If we define \( N_i = T - \lambda_i 1 \), then \( N_i \in \text{L}(W_i) \) since \( W_i \) is \( T \)-invariant. In other words, \( N_i \) is a linear operator defined on the space \( W_i \), and hence so is \( N_i^{n_i} \). However, since \( W_i = \ker N_i^{n_i} \), it follows from the definition of kernel that \( N_i^{n_i} = 0 \) so that \( N_i \) is nilpotent. The result now follows by applying Theorem 7.41(a) to each \( N_i \) and writing \( T = N_i + \lambda_i 1 \). \( \blacksquare \)

Note that each \( A_i \) in this theorem is a direct sum of matrices of the form

\[
\begin{pmatrix}
\lambda_i & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda_i \\
\end{pmatrix}
\]
which are referred to as **basic Jordan blocks belonging** to $\lambda_i$. This theorem will be discussed in much more detail (and from an entirely different point of view) in Chapter 8.

Many of our earlier results follow in a natural manner as corollaries to Theorem 7.42. In particular, suppose that $V$ is finite-dimensional over an algebraically closed field $F$, and $T \in L(V)$ satisfies the hypotheses of Theorem 7.42. We wish to know the form of the characteristic polynomial $\Delta_T(x)$. Relative to the basis for $V$ given by Theorem 7.42, we see that the characteristic matrix $xI - A$ is given by

\[
\begin{pmatrix}
  x - \lambda_1 & * & & & \\
  & \ddots & & & \downarrow \\
  & & x - \lambda_i & * & \\
  k_1 & & 0 & \ddots & \\
  \downarrow & & & \ddots & \\
  k_2 & & & 0 & \ddots \\
  \vdots & & & & \ddots
\end{pmatrix}
\]

and hence (using Theorem 4.5)

\[
\Delta_T(x) = \det(xI - A) = \prod_{i=1}^{r} (x - \lambda_i)^{k_i}.
\]

On the other hand, since $m(x) = \prod_{i=1}^{r} (x - \lambda_i)^{n_i}$ and $k_i \geq n_i$, properties (a) and (b) in the following corollary should be clear, and property (c) follows from the proof of Theorem 7.42. Note that property (c) is just the Cayley-Hamilton theorem again.

**Corollary** Suppose $T \in L(V)$ where $V$ is finite-dimensional over an algebraically closed field $F$, and let the minimal and characteristic polynomials of $T$ be $m(x)$ and $\Delta_T(x)$ respectively. Then

(a) $m(x)|\Delta_T(x)$.

(b) $m(x)$ and $\Delta_T(x)$ have the same roots (although they are not necessarily of the same algebraic multiplicity).

(c) $\Delta_T(T) = 0$.

**Example 7.14** Let $V = \mathbb{C}^3$ have basis $\{v_1, v_2, v_3\}$ and define $T \in L(V)$ by
Then the matrix representation of \( T \) in this basis is

\[
A = \begin{pmatrix}
-1 & 3 & 0 \\
0 & 2 & 0 \\
2 & 1 & -1
\end{pmatrix}.
\]

We first find the minimal polynomial for \( T \). Note that while we have given many theorems dealing with the minimal polynomial, there has as yet been no general method presented for actually finding it. (We shall see that such a method can be based on Theorem 8.8.) Since the minimal polynomial has the same irreducible factors as does the characteristic polynomial (Theorem 7.12), we begin by finding \( \Delta_T(x) = \det(xI - A) \). A simple calculation yields

\[
\Delta_T(x) = (x + 1)^2(x - 2)
\]

and therefore the minimal polynomial must be either

\[
(x + 1)^2(x - 2)
\]

or

\[
(x + 1)(x - 2).
\]

To decide between these two possibilities, we could simply substitute \( A \) and multiply them out. However, it is worthwhile to instead apply Theorem 7.23. In other words, we find the subspaces \( W_1 = \text{Ker} f_1(x)^{n_1} \) and see which value of \( n_1 \) (i.e., either 1 or 2) results in \( V = W_1 \oplus W_2 \). We must therefore find the kernel (i.e., the null space) of \( (T + 1) \), \( (T + 1)^2 \) and \( (T - 2) \). Applying the operator \( T + 1 \) to each of the basis vectors yields

\[
(T + 1)(v_1) = 2v_3 \\
(T + 1)(v_2) = 3v_1 + 3v_2 + v_3 \\
(T + 1)(v_3) = 0.
\]

Since \( \text{Im}(T + 1) \) is spanned by these three vectors, only two of which are obviously independent, we see that \( r(T + 1) = 2 \). Therefore, applying Theorem 5.6, we find that \( \text{nul}(T + 1) = \dim V - r(T + 1) = 1 \). Similarly, we have
and so $r(T + 1)^2 = 1$ implies $\text{nul}(T + 1)^2 = 2$. It should also be clear that the space $\text{Ker}(T + 1)^2$ is spanned by the set $\{v_1, v_3\}$. Finally,

\[
(T + 1)^2(v_1) = 0 \\
(T + 1)^2(v_2) = 9(v_1 + v_2 + v_3) \\
(T + 1)^2(v_3) = 0
\]

so that $r(T - 2) = 2$ and hence $\text{nul}(T - 2) = 1$. We also note that since

\[
(T - 2)(v_1 + v_2 + v_3) = 0
\]

and $\text{nul}(T - 2) = 1$, it follows that the space $W_2 = \text{Ker}(T - 2)$ must be spanned by the vector $\{v_1 + v_2 + v_3\}$. Alternatively, we could assume that $W_2$ is spanned by some vector $u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ and proceed to find $\alpha_1$, $\alpha_2$ and $\alpha_3$ by requiring that $(T - 2)(u) = 0$. This results in

\[
(T - 2)(u) = \alpha_1(-3v_1 + 2v_3) + \alpha_2(3v_1 + v_3) + \alpha_3(-3v_3) = 0
\]

so that we have the simultaneous set of equations

\[
-3\alpha_1 + 3\alpha_2 = 0 \\
2\alpha_1 + \alpha_2 - 3\alpha_3 = 0
\]

This yields $\alpha_1 = \alpha_2 = \alpha_3$ so that $u = v_1 + v_2 + v_3$ will span $W_2$ as we found by inspection.

From the corollary to Theorem 2.15 we have $\dim V = \dim W_1 + \dim W_2$, and since $\dim W_2 = \text{nul}(T - 2) = 1$, it follows that we must have $\dim W_1 = 2$, and hence $W_1 = \text{Ker}(T + 1)^2$. Thus the minimal polynomial for $T$ must be given by

\[
m(x) = (x + 1)^2(x - 2) .
\]

Note that because of this form for $m(x)$, Theorem 7.24 tells us that $T$ is not diagonalizable.

According to Theorem 7.42, the matrix $A_1$ corresponding to $\lambda = -1$ must be at least a $2 \times 2$ matrix, and the matrix $A_2$ corresponding to $\lambda = 2$ must be at least a $1 \times 1$ matrix. However, since $\dim V = 3$, these are in fact the actual
squares required. While $A_2$ is unambiguous, the matrix $A_1$ could be either a single $2 \times 2$ matrix, or it could be a direct sum of two $1 \times 1$ matrices. To resolve this we use Theorem 7.41(b) which tells us that the number of blocks in the representation of the nilpotent operator $T + 1$ is $\dim(\ker(T + 1)) = 1$. This means that the Jordan form of $T$ must be

$$
\begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{pmatrix}
$$

Exercises

1. Let $V = \mathbb{C}^4$ and suppose $T \in \text{L}(V)$. If $T$ has only one eigenvalue $\lambda$ of multiplicity 4, describe all possible Jordan forms of $T$.

2. Let $V = \mathbb{C}^4$ and suppose $m(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_r)^{n_r}$ is the minimal polynomial for $T \in \text{L}(V)$. Let $A = A_1 \oplus \cdots \oplus A_r$ be the Jordan form of $T$. Prove directly from the structure of the $A_i$ that the largest Jordan block belonging to $\lambda_i$ has size $n_i \times n_i$.

3. Let $V = \mathbb{C}^n$. If the Jordan form of $T \in \text{L}(V)$ consists of just one Jordan block (counting $1 \times 1$ blocks), what is the Jordan form of $T^2$? Explain.

4. Let $V = \mathbb{C}^n$, suppose $T \in \text{L}(V)$, and let $\lambda$ be an eigenvalue of $T$. What is the relationship between the number of Jordan blocks belonging to $\lambda$ and the rank of $T - \lambda I$? Explain.

5. Let $V = \mathbb{C}^n$, and suppose that each matrix below represents $T \in \text{L}(V)$ relative to the standard basis. Determine the Jordan form of $T$. [Hint: Use the previous exercise.]

(a) $\begin{pmatrix}
1 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 2
\end{pmatrix}$

(b) $\begin{pmatrix}
1 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}$

(c) $\begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 2
\end{pmatrix}$

(d) $\begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}$