Document Clustering Using Word Clusters via the Information Bottleneck Method

Noam Slonim and Naftali Tishby

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Document clustering is closely related to text classification.
Traditional Clustering Methods

- Represent a document as a vector of weights for the terms that occur in the document.

\[
\begin{align*}
&\text{w}_1 \quad \text{w}_2 \quad \text{w}_3 \quad \ldots \quad \text{w}_{124080} \quad \text{w}_{124081} \quad \ldots \quad \text{w}_{\text{word}_n} \\
\text{doc}_1: & \quad 0.0 \quad 0.75 \quad 0.0 \quad \ldots \quad 0.0 \quad 0.13 \quad \ldots \quad 0.0 \\
\text{doc}_2: & \quad 0.6 \quad 0.21 \quad 0.0 \quad \ldots \quad 0.36 \quad 0.0 \quad \ldots \quad 0.0
\end{align*}
\]

- This representation has many disadvantages:
  - High dimensionality
  - Sparseness
  - Loss of word ordering information

- Clustering documents using the distances between pairs of vectors is troublesome.
  - The Information Bottleneck is an alternative method that does not rely on vector distances.
Dimensionality Reduction

• Dimensionality reduction is beneficial for improved accuracy and efficiency when clustering documents.
  – Latent semantic indexing (LSI)
  – Information Gain and Mutual Information Measures
  – Chi-Squared Statistic
  – Term Strength Algorithm
  – Distributional Clustering
    • Cluster words based on their distribution across documents
    • The Information Bottleneck is a distributional clustering method
The Information Bottleneck

- A distributional clustering method
  - Used to cluster words, reducing the dimensionality of document representations.
  - Used to cluster documents.

- The agglomerative algorithm presented in the paper is a special case of a general approach:
The Information Bottleneck: $X \rightarrow \tilde{X}$

- Find a mapping between $x \in X$ and $\tilde{x} \in \tilde{X}$, characterized by a conditional probability distribution $p(\tilde{x} | x)$.
  - For example, if $X$ is the set of words, $\tilde{X}$ is a new representation of words where $|\tilde{X}| < |X|$.

- This mapping induces a soft partitioning of $X$: each $x \in X$ maps to $\tilde{x} \in \tilde{X}$ with probability $p(\tilde{x} | x)$.

```
\begin{array}{|c|c|c|}
  \hline
  x & p(\tilde{x}_1 | x) & p(\tilde{x}_2 | x) \\
  \hline
  x_1 & 0.8 & 0.2 \\
  x_2 & 0.0 & 1.0 \\
  x_3 & 0.6 & 0.4 \\
  \hline
\end{array}
```
The Information Bottleneck: \( Y \rightarrow X \rightarrow \tilde{X} \)

- Suppose the variable \( X \) is an observation of \( Y \), where \( Y \) is the variable of interest.
  - \( x \in X \) is evidence concerning the outcome \( y \in Y \)
  - For example, \( x \in X \) is a word and \( y \in Y \) is a document

- We want the mapping from \( x \in X \) to \( \tilde{x} \in \tilde{X} \) to preserve as much information about \( Y \) as possible.
Entropy

- **Entropy** measures the uncertainty about a discrete random variable $X$:
  \[ H(X) = - \sum_{x \in X} p(X = x) \log_2 p(X = x) \]

- Entropy defines the lower bound on the number of bits needed to accurately encode $X$.

- **Conditional entropy** of $X$ given $Y$ describes the amount of remaining uncertainty about $X$ after an observation of $Y$:
  \[ H(X \mid Y) = E[H(X \mid y)] = - \sum_{x} \sum_{y} p(x, y) \log p(x \mid y) \]

- **Relative entropy**, or *Kullback-Leibler (KL) distance*, measures the distance between two distributions:
  \[ D_{KL}(p, q) = \sum_{x} p(x) \log \frac{p(x)}{q(x)} \]
The **Mutual Information** of $X$ and $Y$ measures the amount of uncertainty about $X$ that is resolved by observing $Y$:

\[
I(X, Y) = H(X) - H(X \mid Y)
\]

\[
I(X, Y) = \sum_x \sum_y p(x, y) \log \left( \frac{p(x, y)}{p(x) p(y)} \right)
\]

This is also the relative entropy between the joint distribution of $X$ and $Y$ and the product of the distributions of $X$ and $Y$.

Note that $I(X, Y) = I(Y, X)$.
Information Theory Examples

Examples using two different joint probability distributions

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

$H(X) = -0.5 \log 0.5 - 0.5 \log 0.5 = 1.0$

$H(Y) = 1.0$

$H(X|Y) = -0.25 \log 0.5 - 0.25 \log 0.5 - 0.25 \log 0.5 - 0.25 \log 0.5 = 1.0$

$I(X,Y) = H(X) - H(X|Y) = 1.0 - 1.0 = 0$

$H(X) = -0.8 \log 0.8 - 0.2 \log 0.2 = 0.72$

$H(Y) = -0.76 \log 0.76 - 0.24 \log 0.24 = 0.79$

$H(X|Y) = -0.75 \log 0.98 - 0.01 \log 0.01 - 0.05 \log 0.21 - 0.19 \log 0.79 = 0.25$

$I(X,Y) = H(X) - H(X|Y) = 0.72 - 0.25 = 0.47$

$H(Y) = -0.75 \log 0.94 - 0.01 \log 0.05 - 0.05 \log 0.06 - 0.19 \log 0.95 = 0.32$

$I(X,Y) = H(Y) - H(Y|X) = 0.79 - 0.32 = 0.47$
Lossy Compression

• We want to transmit signal $X$ using a compressed version $\tilde{X}$.
  – Lower the bit-rate $R$ (compression) by sending $\tilde{X}$ with some acceptable distortion $D$.

• Two compression problems:
  – Given a maximum rate $R$, minimize distortion $D$
  – Given an acceptable amount of distortion $D$, minimize rate $R$

• Solve simultaneously using an unconstrained Lagrangian cost function:
  \[ J = D + \lambda R \]

• The Information Bottleneck method measures distortion using $I(Y, \tilde{X})$ and compression using $I(X, \tilde{X})$. 
Rate and Distortion Using Mutual Information

- Maximizing \( I(Y, \tilde{X}) \) minimizes distortion.
  - Recall that \( X \) is an observation of the variable of interest, \( Y \).
  - \( I(Y, \tilde{X}) \) measures the quality of the compression.

- Minimizing \( I(X, \tilde{X}) \) maximizes compression.
  - Recall, \( I(X, \tilde{X}) = H(X) - H(X \mid \tilde{X}) \)
  - \( H(X) \) is constant, so minimizing \( I(X, \tilde{X}) \) maximizes \( H(X \mid \tilde{X}) \)
  - \( H(X \mid \tilde{X}) \) defines the minimum additional number of bits needed on average to represent \( x \in X \) after observing \( \tilde{x} \in \tilde{X} \)

\[
\begin{align*}
H(X) & \quad \text{total # of bits needed to encode } X \\
I(X, \tilde{X}) & \quad \text{remaining # of bits needed to encode } X \\
H(X \mid \tilde{X}) & \quad \text{# of bits transmitted}
\end{align*}
\]
The Information Bottleneck Cost Function

Minimize the Lagrangian cost function:

\[ L = I(X, \tilde{X}) - \beta I(Y, \tilde{X}) \]

- **\( \beta \)** is the tradeoff between maximizing compression and minimizing distortion.
  - If \( \beta = 0 \), every \( x \in X \) maps to every \( \tilde{x} \in \tilde{X} \) with the same probability
    - No information is being transmitted
  - In the limit \( \beta \to \infty \), \( I(Y, \tilde{X}) \) is maximized, given the capacity of \( \tilde{X} \).
    - This results in hard clustering

- Note, to find a solution we must fix the cardinality of \( \tilde{X} \).
Solving the Cost Function

Minimize the Lagrangian cost function:

\[ L = I(X, \tilde{X}) - \beta I(Y, \tilde{X}) \]

- Remember, we want to find a mapping between \( x \in X \) and \( \tilde{x} \in \tilde{X} \), characterized by a conditional probability distribution \( p(\tilde{x} | x) \).

Solve \[ \frac{\partial L}{\partial p(\tilde{x} | x)} = 0 \] for \( p(\tilde{x} | x) \).
The Solution

\[
p(\tilde{x} | x) = \frac{p(\tilde{x}) \exp \left[ -\beta \sum_y p(y | x) \log \frac{p(y | x)}{p(y | \tilde{x})} \right]}{\sum_{\tilde{x}} p(\tilde{x}) \exp \left[ -\beta \sum_y p(y | x) \log \frac{p(y | x)}{p(y | \tilde{x})} \right]}
\]

\[
p(\tilde{x} | x) = \frac{p(\tilde{x})}{Z(x, \beta)} \exp \left[ -\beta \mathcal{D}_{KL} \left( p(y | x), p(y | \tilde{x}) \right) \right]
\]

\[
\text{normalization over all } \tilde{x}
\]

\[
\text{tradeoff between } p(y|x) \text{ and } p(y|\tilde{x})
\]
Equations for $p(\tilde{x})$ and $p(y \mid \tilde{x})$

$$p(\tilde{x} \mid x) = \frac{p(\tilde{x})}{Z(x,\beta)} \exp\left[ -\beta D_{KL}\left( p(y \mid x), p(y \mid \tilde{x}) \right) \right]$$

We also need equations for $p(\tilde{x})$ and $p(y \mid \tilde{x})$

$$p(\tilde{x}) = \sum_x p(\tilde{x} \mid x) \ p(x)$$

$$p(y \mid \tilde{x}) = \frac{p(\tilde{x} \mid y) \ p(y)}{p(\tilde{x})}$$

$$= \frac{p(y)}{p(\tilde{x})} \sum_x p(\tilde{x} \mid x) \ p(x \mid y)$$

$$= \frac{1}{p(\tilde{x})} \sum_x p(\tilde{x} \mid x) \ p(y \mid x) \ p(x)$$
Information Bottleneck Iterative Algorithm

• Alternating iterations are run to solve the three equations:

\[ p(\tilde{x} \mid x) = \frac{p(\tilde{x})}{Z(\tilde{x}, \beta)} \exp \left[ -\beta D_{KL} \left( p(y \mid x), p(y \mid \tilde{x}) \right) \right] \]

\[ p(\tilde{x}) = \sum_x p(\tilde{x} \mid x) p(x) \]

\[ p(y \mid \tilde{x}) = \frac{1}{p(\tilde{x})} \sum_x p(\tilde{x} \mid x) p(y \mid x) p(x) \]

• Convergence can be proven.

• Separate solutions exist for different values of \( \beta \) and cardinalities of \( \tilde{X} \), which are fixed for the algorithm.
  – Use deterministic annealing to search for best \( \beta \) and cardinality of \( \tilde{X} \).
Hard Clustering Limit

- Taking the limit $\beta \to \infty$ results in maximizing $I(Y, \tilde{X})$ with no constraint on compression.
- In the limit $\beta \to \infty$, $p(\tilde{x}, x) = 0$ or $1$. Hence, each $x \in X$ will map to exactly one $\tilde{x} \in \tilde{X}$

Recall, $I(Y, \tilde{X}) = H(Y) - H(Y | \tilde{X})$

$H(Y)$ is constant, so maximizing $I(Y, \tilde{X})$ minimizes $H(Y | \tilde{X})$

where

\[
H(Y | \tilde{X}) = -\sum_{\tilde{x}} \sum_{y} p(\tilde{x}, y) \log p(\tilde{x} | y)
\]

\[
H(Y | \tilde{X}) = -\sum_{\tilde{x}} \sum_{y} p(\tilde{x} | y) p(y) \log p(\tilde{x} | y)
\]

Each element of $H(Y | \tilde{X})$ is minimized when $p(\tilde{x} | y) = 0$ or $1$

\[
p(\tilde{x} | y) = \sum_{x} p(\tilde{x} | x) p(x | y) = p(x_1 | y) p(\tilde{x} | x_1) + p(x_2 | y)p(\tilde{x} | x_2) + \ldots + p(x_n | y)p(\tilde{x} | x_n)
\]

where $\sum_{x} p(\tilde{x} | x) = 1$

This is closest to 1 when $p(\tilde{x}, x) = 1$ for the largest $p(x_i | y)$
This is closest to 0 when $p(\tilde{x}, x) = 1$ for the smallest $p(x_i | y)$
New Equations for the Hard Clustering Limit

- Think of each $\tilde{x} \in \tilde{X}$ as a set of elements $x \in X$

  $$p(\tilde{x} \mid x) = \begin{cases} 1 & \text{if } x \in \tilde{x} \\ 0 & \text{otherwise} \end{cases}$$

  $$p(\tilde{x}) = \sum_{x \in \tilde{x}} p(x)$$

  $$p(y \mid \tilde{x}) = \frac{1}{p(\tilde{x})} \sum_{x \in \tilde{x}} p(y \mid x) p(x)$$

- These equations define the properties of the optimal distribution but not the optimal distribution itself.
Agglomerative Algorithm

- Start with a partition of $|X|$ singleton clusters. $|\tilde{X}| = |X|$
- At each step, decrease $|\tilde{X}|$, merging two clusters that locally minimize the loss of mutual information $I(Y, \tilde{X})$.
- Merging $\tilde{x}_i$ and $\tilde{x}_j$ into $x_*$: $(\tilde{x}_i, \tilde{x}_j) \Rightarrow \tilde{x}_*$

$$p(\tilde{x}_* | x) = \begin{cases} 1 & \text{if } x \in \tilde{x}_i \text{ or } x \in \tilde{x}_j \\ 0 & \text{otherwise} \end{cases}$$

$$p(y | \tilde{x}_*) = \frac{p(\tilde{x}_i)}{p(\tilde{x}_* )} p(y | \tilde{x}_i) + \frac{p(\tilde{x}_j)}{p(\tilde{x}_* )} p(y | \tilde{x}_j)$$

$$p(\tilde{x}_*) = p(\tilde{x}_i) + p(\tilde{x}_j)$$

- Merge the pair $\tilde{x}_i$ and $\tilde{x}_j$ that minimizes the decrease in $I(Y, \tilde{X})$, defined as

$$\partial I(\tilde{x}_i, \tilde{x}_j) = I(\tilde{X}_{\text{before}}, Y) - I(\tilde{X}_{\text{after}}, Y)$$
With a Little Algebra...

\[ \partial l(\tilde{x}_i, \tilde{x}_j) = l(\tilde{X}_{\text{before}}, Y) - l(\tilde{X}_{\text{after}}, Y) \]

\[ = \sum_{\tilde{x} \in \tilde{X}_{\text{before}}} \sum_y p(\tilde{x}, y) \log \frac{p(\tilde{x}, y)}{p(\tilde{x})p(y)} - \sum_{\tilde{x} \in \tilde{X}_{\text{after}}} \sum_y p(\tilde{x}, y) \log \frac{p(\tilde{x}, y)}{p(\tilde{x})p(y)} \]

\[ = \left[ \sum_y p(\tilde{x}_i, y) \log \frac{p(\tilde{x}_i, y)}{p(\tilde{x}_i)p(y)} + \sum_y p(\tilde{x}_j, y) \log \frac{p(\tilde{x}_j, y)}{p(\tilde{x}_j)p(y)} \right] - \sum_y p(\tilde{x}_*, y) \log \frac{p(\tilde{x}_*, y)}{p(\tilde{x}_*)p(y)} \]

\[ = p(\tilde{x}_i) \sum_y p(y \mid \tilde{x}_i) \log \frac{p(y \mid \tilde{x}_i)}{p(y \mid \tilde{x}_*)} + p(\tilde{x}_j) \sum_y p(y \mid \tilde{x}_j) \log \frac{p(y \mid \tilde{x}_j)}{p(y \mid \tilde{x}_*)} \]

which is in the form a Jensen-Shannon (JS) divergence

\[ D_{JS}(p_i, p_j) = \pi_i D_{KL}(p_i, p_*) + \pi_j D_{KL}(p_j, p_*) \]
Complexity of the Agglomerative Algorithm

- $O(|X|)$ iterations are required to check all sizes of $\tilde{X}$.
- Each iteration requires calculating $\delta l(\tilde{x}_i, \tilde{x}_j)$ for each possible pair $\tilde{x}_i, \tilde{x}_j$.
  - $O(|x|^2)$ possible pairs
  - Each calculation requires $O(|Y|)$ time
- Total running time is $O(|X|^3 |Y|)$
  - Can be reduced to $O(|X|^2 |Y|)$ by reusing $\delta l(\tilde{x}_i, \tilde{x}_j)$ calculations.

- The agglomerative algorithm does not guarantee a globally optimal solution.
  - $l(Y, \tilde{X})$ is minimized locally for each merge.
Double Clustering Procedure

• First obtain word-clusters to represent documents in a reduced dimensional space.
• Then cluster the documents using the word-cluster representation.
Double Clustering Example

Given a joint probability distribution $p(x, y)$:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.11</td>
<td>0.12</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.09</td>
<td>0.10</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>$y_3$</td>
<td>0.02</td>
<td>0.01</td>
<td>0.13</td>
<td>0.16</td>
</tr>
<tr>
<td>$y_4$</td>
<td>0.03</td>
<td>0.02</td>
<td>0.06</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Find $\tilde{X}$:
- $x_1, x_2, \in \tilde{x}_1$
- $x_3, x_4 \in \tilde{x}_2$

Represent documents using $p(\tilde{x}, y)$:

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{x}_1$</th>
<th>$\tilde{x}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.23</td>
<td>0.03</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.19</td>
<td>0.07</td>
</tr>
<tr>
<td>$y_3$</td>
<td>0.03</td>
<td>0.29</td>
</tr>
<tr>
<td>$y_4$</td>
<td>0.05</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Find $\tilde{Y}$:
- $y_1, y_2, \in \tilde{y}_1$
- $y_3, y_4 \in \tilde{y}_2$

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{x}_1$</th>
<th>$\tilde{x}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{y}_1$</td>
<td>0.42</td>
<td>0.10</td>
</tr>
<tr>
<td>$\tilde{y}_2$</td>
<td>0.08</td>
<td>0.40</td>
</tr>
</tbody>
</table>
Alternative Clustering Methods

• L1 norm estimation of the JS-divergence

\[ L_1(p, q) = \sum_y |p(y) - q(y)| \]

\[ d_{i,j} = (p(\tilde{x}_i) + p(\tilde{x}_j)) L_1(p(y | \tilde{x}_i), p(y | \tilde{x}_j)) \]

• Ward’s method as an alternative to the JS-divergence

\[ d_{i,j} = \frac{p(\tilde{x}_i) p(\tilde{x}_j)}{p(\tilde{x}_i) + p(\tilde{x}_j)} \sum_y (p(y | \tilde{x}_i) - p(y | \tilde{x}_j))^2 \]

• Complete-linkage algorithm
  – Merge the most similar documents
  – Similarity is measured as the cosine of the angle between tf-idf vector representations of documents
Experimental Data

- Measure accuracy by comparing resulting clusters to real document categories.
- 20Newsgroups corpus (Lang 1995)
  - 20,000 articles distributed among 20 UseNet discussion groups
- 10 subsets were used, each containing randomly selected documents:

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Newsgroups included</th>
<th>Documents per group</th>
<th>Total documents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Science</td>
<td>sci.crypt, sci.electronics, sci.med, sci.space</td>
<td>500</td>
<td>2000</td>
</tr>
<tr>
<td>Binary1,2,3</td>
<td>talk.politics.mideast, talk.politics.misc</td>
<td>250</td>
<td>500</td>
</tr>
<tr>
<td>Multi51,2,3</td>
<td>comp.graphics, rec.motorcycles, rec.sport.baseball, sci.space, talk.politics.mideast</td>
<td>100</td>
<td>500</td>
</tr>
<tr>
<td>Multi101,2,3</td>
<td>alt.atheism, comp.sys.mac.hardware, misc.foresale, rec.autos, rec.sport.hockey, sci.crypt, sci.electronics, sci.med, sci.space, talk.politics.gun</td>
<td>50</td>
<td>500</td>
</tr>
</tbody>
</table>
Experimental Setup

• Each data set was represented using 2000 words.
  – The 2000 words with the highest mutual information between words and documents were selected.

• Tested the following algorithms:
  – IB-double - Information bottleneck double clustering procedure
  – IB-single - Information bottleneck, without clustering words
  – L1-double
  – L1-single
  – Ward-double
  – Ward-single
  – Complete tf-idf

• Tested performance using 10, 20, 30, 40, and 50 word-clusters.

• The number of document clusters was identical to the number of real clusters.
Measuring Accuracy

Contingency table for Muti5\textsubscript{2} data set using the IB-double with 10 word-clusters:

<table>
<thead>
<tr>
<th></th>
<th>graphics</th>
<th>motorcycles</th>
<th>baseball</th>
<th>space</th>
<th>mideast</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^?)</td>
<td>78/100</td>
<td>3</td>
<td>11</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>(x^?)</td>
<td>3</td>
<td>68/100</td>
<td>7</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>(x^?)</td>
<td>4</td>
<td>5</td>
<td>59/100</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>(x^?)</td>
<td>6</td>
<td>14</td>
<td>13</td>
<td>68/100</td>
<td>13</td>
</tr>
<tr>
<td>(x^?)</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>13</td>
<td>63/100</td>
</tr>
</tbody>
</table>

Total Accuracy = \[\frac{\frac{78}{100} + \frac{68}{100} + \frac{59}{100} + \frac{68}{100} + \frac{63}{100}}{5}\] = 0.67
Summary of Results

- IB-double gives the best overall performance
  - The IB measure had better performance than all other double clustering methods.
- In 46 out of 50 trials, double-clustering improved performance for all distance measures.
- All algorithms performed best on the binary data sets, and weakest on the Multi10 data sets.
  - Increasing the number of clusters results in decreasing performance
Discussion

• The agglomerative algorithm does not use the general solution found for Lagrangian optimization.
  – It does not guarantee a globally optimal solution
  – It has a running time of $O(|X|^3 |Y|)$.

• Further work may look at simultaneously clustering words and documents.

• The Information Bottleneck method has also been applied to clustering words for text classification.