3.1 Random variables

A random variable (r.v.) is defined on a probability space $\Omega, Pr$ and is a mapping from $\Omega$ to $\mathbb{R}$.

The value of the random variable is fully determined by the outcome $\omega \in \Omega$. Thus the underlying probability space (probabilities $Pr(\omega)$) induces a probability distribution over the random variable. Let’s look at some examples.

Suppose you roll a fair die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$, all outcomes being equally likely. On this space we can then define a random variable $X = \begin{cases} 1 & \text{if die is } \geq 3 \\ 0 & \text{otherwise} \end{cases}$

In other words, the outcomes $\omega = 1, 2$ map to $X = 0$, while the outcomes $\omega = 3, 4, 5, 6$ map to $X = 1$. The r.v. $X$ takes on values $\{0, 1\}$, with probabilities $Pr(X = 0) = 1/3$ and $Pr(X = 1) = 2/3$.

Or say you roll this same die $n$ times, so that the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}^n$. Examples of random variables on this larger space are $X = \text{the number of 6’s rolled}$, $Y = \text{the number of 1’s seen before the first 6}$.

The sample point $\omega = (1, 1, 1, 1, \ldots, 1, 6)$, for instance, would map to $X = 1, Y = n - 1$. The variable $X$ takes values in $\{0, 1, 2, \ldots, n\}$, with $Pr(X = k) = \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k}$

(see you why?).

As a third example, suppose you throw a dart at a dartboard of radius 1, and that it lands at a random location on the board. Define random variable $X$ to be the distance of the dart from the center of the board. Now $X$ takes values in $[0, 1]$, and for any $x$ in this range, $Pr(X \leq x) = x^2$.

Henceforth, we’ll follow the convention of using capital letters for r.v.’s.

3.2 The mean, or expected value

For a random variable $X$ that takes on a finite set of possible values, the mean, or expected value, is $E(X) = \sum_x x \Pr(X = x)$

(where the summation is over all the possible values $x$ that $X$ can have). This is a direct generalization of the notion of average (which is typically defined in situations where the outcomes are equally likely). If $X$
is a continuous random variable, then this summation needs to be replaced by an equivalent integral; but we’ll get to that later in the course.

Here are some examples.

1. **Coin with bias (heads probability) p.**

   Define X to be 1 if the outcome is heads, or 0 if it is tails. Then
   \[
   \mathbb{E}(X) = 0 \cdot \Pr(X = 0) + 1 \cdot \Pr(X = 1) = 0 \cdot (1 - p) + 1 \cdot p = p.
   \]

   Another random variable on this space is \(X^2\), which also takes on values in \(\{0, 1\}\). Notice that \(X^2 = X\), and in fact \(X^k = X\) for all \(k = 1, 2, 3, \ldots\)! Thus, \(\mathbb{E}(X^2) = p\) as well. This simple case shows that in general, \(\mathbb{E}(X^2) \neq \mathbb{E}(X)^2\).

2. **Fair die.**

   Define X to be the outcome of the roll, so \(X \in \{1, 2, 3, 4, 5, 6\}\). Then
   \[
   \mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5.
   \]

3. **Two dice.**

   Let X be their sum, so that \(X \in \{2, 3, 4, \ldots, 12\}\). We can calculate the probabilities of each possible value of X and tabulate them as follows:

<table>
<thead>
<tr>
<th>x</th>
<th>Pr(X = x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/36</td>
</tr>
<tr>
<td>3</td>
<td>1/18</td>
</tr>
<tr>
<td>4</td>
<td>1/9</td>
</tr>
<tr>
<td>5</td>
<td>5/36</td>
</tr>
<tr>
<td>6</td>
<td>5/18</td>
</tr>
<tr>
<td>7</td>
<td>1/6</td>
</tr>
<tr>
<td>8</td>
<td>5/36</td>
</tr>
<tr>
<td>9</td>
<td>5/18</td>
</tr>
<tr>
<td>10</td>
<td>1/9</td>
</tr>
<tr>
<td>11</td>
<td>1/18</td>
</tr>
<tr>
<td>12</td>
<td>1/36</td>
</tr>
</tbody>
</table>

   This gives \(\mathbb{E}(X) = 7\).

4. **Roll n die; how many sixes appear?**

   Let X be the number of 6’s. We’ve already analyzed the distribution of X, so
   \[
   \mathbb{E}(X) = \sum_{k=0}^{n} k \Pr(X = k) = \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} = \frac{n}{6}.
   \]

   The last step is somewhat mysterious; just take our word for it, and we’ll get back to it later!

5. **Toss a fair coin forever; how many tosses to the first heads?**

   Let \(X \in \{1, 2, \ldots\}\) be the number of tosses until you first see heads. Then
   \[
   \Pr(X = k) = \Pr((T, T, T, \ldots, T, H)) = \frac{1}{2^k}.
   \]

   It follows that
   \[
   \mathbb{E}(X) = \sum_{k=1}^{\infty} \frac{k}{2^k} = 2.
   \]

   We saw in class how to do this summation. The technique was based on the formula for the sum of a geometric series: if \(|r| < 1\), then
   \[
   a + ar + ar^2 + \cdots = \frac{a}{1 - r}.
   \]
6. Toss a coin with bias \( p \) forever; how many tosses to the first heads? 

Once again, \( X \in \{1, 2, \ldots\} \), but this time the distribution is different:

\[
\Pr(X = k) = \Pr((T,T,T,\ldots,T,H)) = (1 - p)^{k-1} p.
\]

Using the same technique as before, we get \( E(X) = 1/p \).

There’s another way to derive this expectation. We always need at least one coin toss. If we’re lucky (with probability \( p \)), we’re done; otherwise (with probability \( 1 - p \)), we start again from scratch. Therefore \( E(X) = 1 + (1 - p)E(X) \), so that \( E(X) = 1/p \).

7. Pascal’s wager: does God exist?

Here was Pascal’s take on the issue of God’s existence: if you believe there is some chance \( p > 0 \) (no matter how small) that God exists, then you should behave as if God exists.

Why? Well, let the random variable \( X \) denote your amount of suffering.

Suppose you behave as if God exists (that is, you are good). This behavior incurs a significant but finite amount of suffering (you are not able to do some of the things you would like to). Say \( X = 10 \).

On the other hand, suppose you behave as if God doesn’t exist – that is, you do all the things you want to do. If God really doesn’t exist, you’re fine, and your suffering is \( X = 0 \). But if God exists, then you go straight to hell and your suffering is \( X = \infty \). Thus your expected suffering if you behave badly is \( E(X) = 0 \cdot (1 - p) + \infty \cdot p = \infty \).

So: to minimize your expected suffering, behave as if God exists!