Sets

Let $A = \{a, b, c, \ldots, z\}$, $|A| = 26$

Let $B = \{0, 1\}$, $|B| = 2$

Let $E = \{\text{all even integers}\}$, $|E| = \infty$

Let $S = \{x \in E : x \text{ is a multiple of 3}\}$

Let $I = [0, 1] = \{x : 0 \leq x \leq 1\}$

In a set, the order of elements doesn’t matter:

$\{0, 1, 2\} = \{2, 0, 1\}$

and there are no duplicates.

Tuples

Let $C = \{H, T\}$.

All pairs of elements from $C$:

$\{(H,H),(H,T),(T,H),(T,T)\} = C \times C = C^2$

All triples of elements of $C$:

$\{(H,H,H),(H,H,T),(H,T,H),\ldots\} = C \times C \times C = C^3$

All sequences of $k$ elements from $C$: denoted $C^k = C \times C \times \cdots \times C$.

How many sequences of length $k$ are there? $|C^k| = |C|^k = 2^k$.

In a sequence, the order of elements matters:

$(H, T) \neq (T, H)$.

Let $A = \{a, b, c, \ldots, z\}$.

How many sequences of length 2? $26^2$

How many sequences of length 10? $26^{10}$

How many sequences of length $n$? $26^n$

An alien language has an alphabet of size 10. Every sequence of $\leq 5$ of these characters is a valid word. How many words are there in this language?

$10^1 + 10^2 + 10^3 + 10^4 + 10^5 = 10 + 100 + 1000 + 10000 + 100000 = 111110$. 

Review of basic probability

CSE 101

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$S = \{x \in E : x \text{ is a multiple of 3}\}$

$I = [0, 1] = \{x : 0 \leq x \leq 1\}$

In a set, the order of elements doesn’t matter:

$\{0, 1, 2\} = \{2, 0, 1\}$

and there are no duplicates.
### Union and intersection

- **$A \cup B$** = \{any element in $A$ or in $B$ or in both\}
- **$A \cap B$** = \{any element in $A$ and in $B$\}

$M = \{2, 3, 5, 7, 11\}$ and $N = \{1, 3, 5, 7, 9\}$

$M \cup N = \{1, 2, 3, 5, 7, 9, 11\}$

$M \cap N = \{3, 5, 7\}$

$S = \{\text{all even integers}\}$ and $T = \{\text{all odd integers}\}$

$S \cup T = \{\text{all integers}\}$

$S \cap T = \emptyset$

### Permutations

How many ways to order the three letters $A, B, C$?

$ABC, ACB, BAC, CAB, CBA$

3 choices for the first, 2 choices for the second, 1 choice for the third

$3 \times 2 \times 1 = 6$. Call this $3!$

How many ways to order $A, B, C, D, E$?

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

How many ways to place 6 men in a line-up?

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

How many possible outcomes of shuffling a deck of cards?

52!

General rule: The number of ways to order $n$ distinct items is:

$$n! = n(n-1)(n-2) \cdots 1$$

### Combinations

An ice-cream parlor has flavors \{chocolate, vanilla, strawberry, pecan\}. You are allowed to pick two of them. How many options do you have?

$CV, CS, CP, VS, VP, SP$

In general, the number of ways to pick $k$ items out of $n$ is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

For instance, $\binom{4}{2} = \frac{4 \cdot 3}{2!} = 6$.

How many ways to pick three ice-cream flavors?

$$\binom{4}{3} = 4$$

Pick any 4 of your favorite 100 songs. How many ways to do this?

$$\binom{100}{4} = \frac{100 \cdot 99 \cdot 98 \cdot 97}{4 \cdot 3 \cdot 2 \cdot 1}$$

### Probability spaces

How to interpret a statement like:

*The chance of getting a flush in a five-card poker hand is about 0.20\%*. (Flush = five of the same suit.)

The underlying probability space has two components:

1. The **sample space** (the space of outcomes).
   In the example, $\Omega = \{\text{all possible five-card hands}\}$.

2. The **probabilities of outcomes**.
   In the example, all hands are equally likely: probability $1/|\Omega|$.

Note: $\sum_{\omega \in \Omega} \Pr(\omega) = 1$.

**Event** of interest: the set of outcomes $A = \{\omega : \omega \text{ is a flush}\} \subset \Omega$.

$$\Pr(A) = \sum_{\omega \in A} \Pr(\omega) = \frac{|A|}{|\Omega|}$$
Roll a die. What is the chance of getting a number > 3?

Sample space \( \Omega = \{1, 2, 3, 4, 5, 6\} \).

Probabilities of outcomes: \( \Pr(\omega) = \frac{1}{6} \).

Event of interest: \( A = \{4, 5, 6\} \)

\[
\Pr(A) = \Pr(4) + \Pr(5) + \Pr(6) = \frac{1}{2}.
\]

Socks in a drawer. A drawer has three blue socks and three red socks. You put your hand in and pull out two socks at random. What is the probability that they match?

Think of grabbing one sock first, then another.

\( \Omega = \{(B, B), (B, R), (R, B), (R, R)\} = \{B, R\}^2 \).

Probabilities:

\[
\begin{align*}
\Pr((B, B)) &= \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5} \\
\Pr((B, R)) &= \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} \\
\Pr((R, B)) &= \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} \\
\Pr((R, R)) &= \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}
\end{align*}
\]

Event of interest: \( A = \{(B, B), (R, R)\} \). \( \Pr(A) = \frac{2}{5} \).

Roll three dice. What is the chance that their sum is 3?

Sample space

\[
\Omega = \{(1, 1, 1), (1, 1, 2), (1, 1, 3), \ldots , (6, 6, 6)\}
\]

\[
= \Omega_o \times \Omega_o \times \Omega_o
\]

where \( \Omega_o = \{1, 2, 3, 4, 5, 6\} \).

Probabilities of outcomes:

\[
\Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{216}
\]

Event of interest: \( A = \{(1, 1, 1)\} \). \( \Pr(A) = \frac{1}{216} \).

Roll \( n \) dice.

Then \( \Omega = \Omega_o \times \cdots \times \Omega_o = \Omega_o^n \), where \( \Omega_o = \{1, 2, 3, 4, 5, 6\} \).

What is \( |\Omega|? \) \( 6^n \).

Probability of an outcome: \( \Pr(\omega) = \frac{1}{6^n} \).

Socks in a drawer, cont’d. This time the drawer has three blue socks and two red socks. You put your hand in and pull out two socks at random. What is the probability that they match?

Sample sample space, \( \Omega = \{(B, B), (B, R), (R, B), (R, R)\} = \{B, R\}^2 \).

Different probabilities:

\[
\begin{align*}
\Pr((B, B)) &= \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10} \\
\Pr((B, R)) &= \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10} \\
\Pr((R, B)) &= \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10} \\
\Pr((R, R)) &= \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}
\end{align*}
\]

Event of interest: \( A = \{(B, B), (R, R)\} \). \( \Pr(A) = \frac{2}{5} \).
Shuffle a pack of cards.

Sample space $\Omega = \{\text{all possible orderings of 52 cards}\}$.

What is $|\Omega|$?

$$52! = 52 \cdot 51 \cdot 50 \cdot 49 \cdots 3 \cdot 2 \cdot 1$$

Toss a fair coin 10 times. What is the chance none are heads?

Again, sample space $\Omega = \{H, T\}^{10}$, with $|\Omega| = 2^{10} = 1024$.

For any sequence of coin tosses $\omega \in \Omega$, we have $\Pr(\omega) = 1/1024$.

Event of interest: $A = \{(T, T, T, T, T, T, T, T, T, T)\}$. $\Pr(A) = 1/1024$.

What is the probability of exactly one head?

Event of interest: $A = \{\omega \in \Omega : \omega \text{ has exactly one } H\}$.

What is $|A|$? 10.

Each sequence in $A$ can be specified by the location of the one $H$, and there are 10 choices for this.

What is $\Pr(A)$? $10/1024$.

Toss a fair coin 10 times. What is the chance exactly two heads?

Again, sample space $\Omega = \{H, T\}^{10}$, with $|\Omega| = 2^{10} = 1024$.

For any sequence of coin tosses $\omega \in \Omega$, we have $\Pr(\omega) = 1/1024$.

Event of interest: $A = \{(T, T, T, T, T, T, T, T, T, T)\}$. $\Pr(A) = 1/1024$.

What is $|A|$?

$$\binom{10}{2} = \frac{10 \cdot 9}{2} = 45.$$ Each sequence in $A$ can be specified by the locations of the two $H$'s and there are $\binom{10}{2}$ choices for these locations.

What is $\Pr(A)$? $45/1024$.

What is the probability of exactly $k$ heads?

Event of interest: $A = \{\omega \in \Omega : \omega \text{ has exactly } k \text{ H's}\}$.

What is $|A|$? $\binom{10}{k}$.

What is $\Pr(A)$? $\binom{10}{k}/1024$.

Birthday paradox. A room contains $k$ people. What is the chance that they all have different birthdays?

Number the people 1, 2, ..., $k$.
Number the days of the year 1, 2, ..., 365.

Let $\omega = (\omega_1, \ldots, \omega_k)$, where $\omega_i \in \{1, 2, \ldots, 365\}$ is the birthday of person $i$. Thus $\Omega = \{1, 2, \ldots, 365\}^k$.

What is $|\Omega|$? $365^k$.

Event of interest: $A = \{(\omega_1, \ldots, \omega_k) : \text{all } \omega_i \text{ different}\}$.

What is $|A|$? $365 \cdot 364 \cdot 363 \cdots (365 - k + 1)$.

Therefore,

$$\Pr(A) = \frac{365 \cdot 364 \cdots (365 - k + 1)}{365^k} = \frac{365 \cdot 364 \cdot 363 \cdots (365 - k + 1)}{365 \cdot 365 \cdot 365 \cdots 365}.$$

For $k = 23$, this is less than $1/2$. In other words, in a group of 23 random people, chances are some pair of them have a common birthday!
People’s probability judgements

Experiment by Kahneman-Tversky. Subjects were told:

Linda is 31, single, outspoken, and very bright. She majored in philosophy in college. As a student, she was deeply concerned with racial discrimination and other social issues, and participated in anti-nuclear demonstrations.

They were then asked to rank three possibilities:
(a) Linda is active in the feminist movement.
(b) Linda is a bank teller.
(c) Linda is a bank teller and is active in the feminist movement.

Over 85% respondents chose (a) > (c) > (b).

But:
Pr(bank teller, feminist) ≤ Pr(bank teller).

In a city, 60% of people have a car, 20% of people have a bike, and 10% of people have a motorcycle. Anyone without at least one of these walks to work. What is the minimum fraction of people who walk to work?

Let \( \Omega = \{ \text{people in the town} \} \). Let \( C = \{ \text{has car} \}, B = \{ \text{has bike} \}, M = \{ \text{has motorcycle} \}, W = \{ \text{walks} \} \).

General picture:

\[
\Pr(W) \geq 1 - \Pr(C \cup B \cup M)
\]

\[
\Pr(C \cup B \cup M) \leq \Pr(C) + \Pr(B) + \Pr(M) = 0.6 + 0.2 + 0.1 = 0.9
\]

and thus \( \Pr(W) \geq 0.1 \).

Complements and unions

The complement of an event.

Let \( \Omega \) be a sample space and \( E \subset \Omega \) an event.
Write \( E^c \) for the event that \( E \) does not occur, that is, \( E^c = \Omega \setminus E \).

\[
\Pr(E^c) = 1 - \Pr(E).
\]

The union bound.

For any events \( E_1, \ldots, E_k \):

\[
\Pr(E_1 \cup \cdots \cup E_k) \leq \Pr(E_1) + \cdots + \Pr(E_k).
\]

This inequality is exact when the events are disjoint.

Coupon-collector problem

Each cereal box has one of \( k \) action figures. How many boxes do you need to buy so that you are likely to get all \( k \) figures?

Say we buy \( n \) boxes.
Let \( A_i \) be the event that the \( i \)th action figure is not obtained.

\[
\Pr(A_i) = \Pr(\text{not in 1st box}) \cdot \Pr(\text{not in 2nd box}) \cdots \Pr(\text{not in } n\text{th box})
= \left(1 - \frac{1}{k}\right)^n \leq e^{-n/k}
\]

By union bound, the probability of missing some figure is

\[
\Pr(A_1 \cup \cdots \cup A_k) \leq \Pr(A_1) + \cdots + \Pr(A_k) \leq ke^{-n/k}.
\]

Setting \( n \geq k \ln 2k \) makes this \( \leq 1/2 \).

Therefore: enough to buy \( O(k \log k) \) cereal boxes.
Independence

Two events $A, B$ are independent if the probability of $B$ occurring is the same whether or not $A$ occurs.

Example: toss two coins.
$A = \{\text{first coin is heads}\}$
$B = \{\text{second coin is heads}\}$

Formally, we say $A, B$ are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

Independent or not?

1. You have two children.
   $A = \{\text{first child is a boy}\}$, $B = \{\text{second child is a girl}\}$.
   Independent.

2. You throw two dice.
   $A = \{\text{first is a six}\}$, $B = \{\text{sum} > 10\}$.
   Not independent.

3. You get dealt two cards at random from a deck of 52.
   $A = \{\text{first is a heart}\}$, $B = \{\text{second is a club}\}$.
   Not independent.

Random variables

Roll a die.
Define $X = \begin{cases} 1 & \text{if die is} \geq 3 \\ 0 & \text{otherwise} \end{cases}$

Here the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

- $\omega = 1, 2 \Rightarrow X = 0$
- $\omega = 3, 4, 5, 6 \Rightarrow X = 1$

Roll $n$ dice.

$X = \# \text{ of 6's}$
$Y = \# \text{ of 1's before the first 6}$

Both $X$ and $Y$ are defined on the same sample space,
$\Omega = \{1, 2, 3, 4, 5, 6\}^n$. For instance,

- $\omega = (1,1,1,\ldots,1,6) \Rightarrow X = 1, Y = n - 1$.

In general, a random variable (r.v.) is a defined on a probability space.
It is a mapping from $\Omega$ to $\mathbb{R}$. We’ll use capital letters for r.v.’s.

The distribution of a random variable

Roll a die. Define $X = 1$ if die is $\geq 3$, otherwise $X = 0$.

$X$ takes values in $\{0, 1\}$ and has distribution:

$\Pr(X = 0) = \frac{1}{3}$ and $\Pr(X = 1) = \frac{2}{3}$.

Roll $n$ dice. Define $X = \# \text{ of 6's}$.

$X$ takes values in $\{0, 1, 2, \ldots, n\}$. The distribution of $X$ is:

$\Pr(X = k) = \#(\text{sequences with k 6's}) \cdot \Pr(\text{one such sequence})$

$= \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k}$
Expected value, or mean

The expected value of a random variable $X$ is

$$E(X) = \sum_x x \cdot \Pr(X = x).$$

Roll a die. Let $X$ be the number observed.

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5 \text{ (average)}$$

Biased coin. A coin has heads probability $p$. Let $X$ be 1 if heads, 0 if tails.

$$E(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$  

Toss a coin with bias $p$ repeatedly, until it comes up heads. Let $X$ be the number of tosses.

$$E(X) = \frac{1}{p}.$$  

Pascal’s wager

Pascal: I think there is some chance ($p > 0$) that God exists. Therefore I should act as if he exists.

Let $X = my level of suffering.$

- Suppose I behave as if God exists (that is, I behave myself). Then $X$ is some significant but finite amount, like 100 or 1000.

- Suppose I behave as if God doesn’t exist (I do whatever I want to). If indeed God doesn’t exist: $X = 0$. But if God exists: $X = \infty$ (hell). Therefore, $E(X) = 0 \cdot (1 - p) + \infty \cdot p = \infty$. The first option is much better!

Linearity of expectation

- If you double a set of numbers, how is the average affected? It is also doubled.

- If you increase a set of numbers by 1, how much does the average change? It also increases by 1.

- Rule: $E(aX + b) = aE(X) + b$ for any random variable $X$ and any constants $a, b$.

- But here’s a more surprising (and very powerful) property: $E(X + Y) = E(X) + E(Y)$ for any two random variables $X, Y$.

- Likewise: $E(X + Y + Z) = E(X) + E(Y) + E(Z)$, etc.

Linearity: examples

Roll 2 dice and let $Z$ denote the sum. What is $E(Z)$?

Method 1

Distribution of $Z$:

<table>
<thead>
<tr>
<th>$z$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr(Z = z)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

Now use formula for expected value:

$$E(Z) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots = 7.$$  

Method 2

Let $X_1$ be the first die and $X_2$ the second die. Each of them is a single die and thus (as we saw earlier) has expected value 3.5. Since $Z = X_1 + X_2$, 

$$E(Z) = E(X_1) + E(X_2) = 3.5 + 3.5 = 7.$$  

The first option is much better!
Toss $n$ coins of bias $p$, and let $X$ be the number of heads. What is $\mathbb{E}(X)$?

Let the individual coins be $X_1, \ldots, X_n$.
Each has value 0 or 1 and has expected value $p$.

Since $X = X_1 + X_2 + \cdots + X_n$,
$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = np.$$ Roll a die $n$ times, and let $X$ be the number of 6’s. What is $\mathbb{E}(X)$?
Let $X_1$ be 1 if the first roll is a 6, and 0 otherwise.
$$\mathbb{E}(X_1) = \frac{1}{6}.$$ Likewise, define $X_2, X_3, \ldots, X_n$.
Since $X = X_1 + \cdots + X_n$, we have
$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = \frac{n}{6}.$$ Coupon collector, again
Each cereal box has one of $k$ action figures. What is the expected number of boxes you need to buy in order to collect all the figures?

Suppose you’ve already collected $i - 1$ of the figures. Let $X_i$ be the time to collect the next one.
Each box you buy will contain a new figure with probability $(k - (i - 1))/k$. Therefore,
$$\mathbb{E}(X_i) = \frac{k}{k - i + 1}.$$ Total number of boxes bought is $X = X_1 + X_2 + \cdots + X_k$, so
$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_k)$$
$$= \frac{k}{k} + \frac{k}{k - 1} + \frac{k}{k - 2} + \cdots + \frac{k}{1}$$
$$= k \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \approx k \ln k.$$ Independent random variables

Random variables $X, Y$ are independent if
$$\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y).$$

Independent or not?

► Pick a card out of a standard deck. $X =$ suit and $Y =$ number.

Independent.

► Flip a fair coin $n$ times. $X =$ # heads and $Y =$ last toss.

Not independent.

► $X, Y$ take values $\{-1, 0, 1\}$, with the following probabilities:

<table>
<thead>
<tr>
<th>$Y$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.4</td>
<td>0.16</td>
<td>0.24</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.02</td>
<td>0.03</td>
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</table>

<table>
<thead>
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<th>$X$</th>
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<th>1</th>
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</thead>
<tbody>
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<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.8</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.5</td>
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Independent.