Classifying toric and semitoric fans by lifting equations from $\mathrm{SL}_2(\mathbb{Z})$

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To Tudor S. Ratiu on his 65^{th} birthday, with admiration.

Abstract

We present an algebraic method to study four-dimensional toric varieties by lifting matrix equations from the special linear group $\mathrm{SL}_2(\mathbb{Z})$ to its preimage in the universal cover of $\mathrm{SL}_2(\mathbb{R})$. With this method we recover the classification of two-dimensional toric fans, and obtain a description of their semitoric analogue. As an application to symplectic geometry of Hamiltonian systems, we give a concise proof of the connectivity of the moduli space of toric integrable systems in dimension four, recovering a known result, and extend it to the case of semitoric integrable systems.

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1 Introduction

Toric varieties [5, 6, 9, 12, 21, 22] have been extensively studied in algebraic and differential geometry and so have their symplectic analogues, usually called symplectic toric manifolds or toric integrable systems [8].

The relationship between symplectic toric manifolds and toric varieties has been understood since the 1980s, see for instance Delzant [10], Guillemin [14, 15]. The article [11] contains a coordinate description of this relation.

In this paper we present an algebraic viewpoint to study four-dimensional toric varieties, based on the study of matrix relations in the special linear group $\mathrm{SL}_2(\mathbb{Z})$. Indeed, one can associate to a rational convex polygon Δ the collection of primitive integer inwards pointing normal vectors to its faces, called a *toric fan*. This is a d-tuple

$$(v_0 = v_d, v_1, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d,$$

where $d \in \mathbb{Z}$ is the number of faces. A *Delzant polygon* (or *toric polygon*) is one for which $\det(v_i, v_{i+1}) = 1$ for each $0 \leqslant i \leqslant d-1$. This determinant condition forces the vectors to satisfy certain linear equations which are parameterized by integers $a_0, \ldots, a_{d-1} \in \mathbb{Z}$. These integers satisfy

$$\begin{pmatrix} 0 & -1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_{d-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

(this equation appears in [12], page 44) but in fact not all integers satisfying Equation (1) correspond to a Delzant polygon. This is an equation in $\mathrm{SL}_2(\mathbb{Z})$ and in this paper we lift it to the group G presented as

$$G = \langle S, T \mid STS = T^{-1}ST^{-1} \rangle$$

where $\mathrm{SL}_2(\mathbb{Z}) \cong G/(S^4)$. The group G is the pre-image of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$, and thus we can define what we call the *winding number* of an element of $g \in G$ that evaluates to the identity in $\mathrm{SL}_2(\mathbb{Z})$. Roughly speaking, we view g as a word in S and T and by applying this word to a vector one term at a time we produce a path around the origin. We define the winding number of g to be the winding number of this path in the classical sense.

The equation in G analogous to Equation (1) has the property that a collection of integers $a_0, \ldots, a_{d-1}$ satisfy the equation if and only if they correspond to a toric fan, and from the integers it is straightforward to recover the fan. As such, this is a method to translate problems about toric fans into equivalent problems about the algebraic structure of the group G. This allows us to simplify the proofs of some classical results about toric fans (Section 4), and generalize these results to the semitoric case (Section 5). Associated to a semitoric system there is also a collection of vectors $(v_0, \ldots, v_{d-1})$ which satisfy more complicated equations (given explicitly in Definition 2.5) known as a semitoric fan. A semitoric fan can be thought of as a toric fan which includes additional information coming from the focus-focus singularities (called nodal singularities in the context of Lefschetz fibrations and algebraic geometry) of semitoric integrable systems. Indeed, focus--focus singularities appear in algebraic geometry [13], symplectic topology [20, 31, 34], and many simple physical models such as the spherical pendulum [1] and the Jaynes-Cummings system [7, 17, 30]. Roughly speaking, semitoric fans encode the singular affine structure induced by the singular fibration associated to a semitoric integrable system. This affine structure also plays a role in parts of symplectic topology, see for instance Borman-Li-Wu [3], and mirror symmetry, see Kontsevich-Soibelman [18].

We present two theorems in this paper which are applications of the algebraic method we introduce:

- Theorem 2.8 gives a classification of semitoric fans;
- Theorem 2.11 is an application of Theorem 2.8 which describes the path-connected components of the moduli space of semitoric integrable systems.

The paper is divided into two blocks. The first one concerns toric and semitoric fans and requires no prior knowledge of symplectic or algebraic geometry, while the second block, which consists only of Section 6, contains applications to symplectic geometry and will probably be most interesting to those working on differential geometry or Hamiltonian systems. The structure of the paper is as follows. In Section 2 we state our main results, and the applications to symplectic geometry. In Section 3 we define the necessary algebraic structures and prove several general algebraic results. In Section 4 we use this new algebraic approach to recover classical results and about toric fans and in Section 5 we generalize these results to semitoric fans. The last part of this paper contains applications to the symplectic geometry of semitoric integrable systems, where we will prove connectivity of certain moduli spaces analogous to the result in [24] that the moduli space of symplectic toric manifolds is path-connected. Indeed, in Section 6 we use the results of Section 5 to study the connectivity of the moduli space of semitoric systems.

2 Main results

2.1 Toric fans

A toric variety is a variety which contains an algebraic torus as a dense open subset such that the standard action of the torus on itself can be extended to the whole variety. That is, a toric variety is the closure of an algebraic torus orbit [21]. By an algebraic torus we mean the product $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is well known that the geometry of a toric variety is completely classified by the associated fan. In general, a fan is set of rational strongly convex cones in a real vector space such that the face of each cone is also a cone and the intersection of any two cones is a face of each. In this paper we will be concerned with two-dimensional nonsingular complete toric varieties and their associated fans, which we will simply call *toric fans*. These fans are given by a sequence of lattice points

$$(v_0 = v_d, v_1, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$$

labeled in counter-clockwise order such that each pair of adjacent vectors generates all of $\mathbb{Z}^2$ and the angle between any two adjacent vectors is less than π radians. That is, $\det(v_i, v_{i+1}) = 1$ for $i = 0, \ldots, d-1$.

Definition 2.1. A *Delzant polygon*[1] (or *toric polygon*) is a convex polygon Δ in $\mathbb{R}^2$ which is simple, rational, and smooth. Recall $v \in \mathbb{Z}^2$ is primitive if $v = kw$ for some $k \in \mathbb{Z}_{>0}$ and $w \in \mathbb{Z}^2$ implies $k = 1$ and $w = v$.

1. Δ is *simple* if there are exactly two edges meeting at each vertex;

2. Δ is *rational* if for each face f of Δ there exists a vector in $\mathbb{Z}^2$ which is normal to f;

3. Δ is *smooth* if the inwards pointing primitive vectors normal to any pair of adjacent edges form a basis of $\mathbb{Z}^2$.

Delzant polygons were introduced in the work of Delzant [10] in symplectic geometry, who built on the work of Atiyah [2], Kostant [19], and Guillemin-Sternberg [16] to give a classification of symplectic toric 4-manifolds in terms of the Delzant polygon (in fact, their work was in any dimension). Delzant polygons are similar to Newton polygons except that the vertices of Delzant polygons do not have to have integer coordinates. Just as in the case with Newton polygons, a toric fan may be produced from a Delzant polygon considering the collection of inwards pointing normal vectors of the polygon[2].

In [12] Fulton stated the following result for toric fans. We have adapted the statement to relate it to Delzant polygons.

Theorem 2.2 (Fulton [12], page 44)**.** *Up to the action of* $\mathrm{SL}_2(\mathbb{Z})$*, every Delzant polygon can be obtained from a Delzant triangle, rectangle, or Hirzebruch trapezoid by a finite number of corner chops.*

The corner chop operation is defined in Definition 2.7 and the minimal models (the Delzant triangle, rectangle, and Hirzebruch trapezoid) are defined in Definition 4.9 and depicted in Figure 1. The proof of Theorem 2.2 sketched by Fulton in [12], which uses only two-dimensional geometry and basic combinatorial arguments, is relatively long and does not immediately generalize. In Section 4 we provide an alternative proof using $\mathrm{SL}_2(\mathbb{Z})$-relations. This proof may be easily extended to the semitoric case.

As a consequence of Theorem 2.2 in [24] it was recently proved:

Theorem 2.3 ([24])**.** *The moduli space of toric polygons is path-connected.*

That is, any two toric polygons may be deformed onto each other continuously via a path of toric polygons. One shows this by first knowing how to generate all toric polygons as in Theorem 2.2. Then one shows, using elementary analysis,

[1] Higher dimensional toric integrable systems correspond to higher dimensional *Delzant polytopes* which satisfy analogous conditions to those given in this definition.

[2] Notice that since Delzant polygons are required to be convex we automatically have that the angle between adjacent vectors in the fan is less than π.

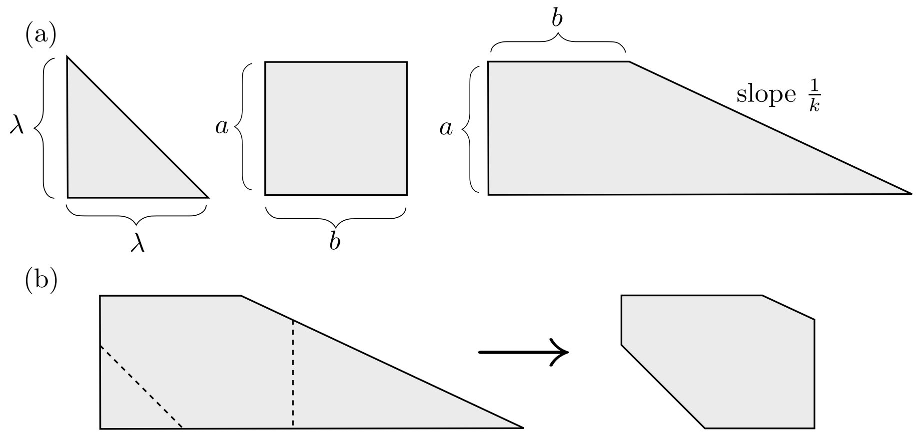

Figure 1: (a) The three minimal models from Theorem 2.2. (b) An illustration of a Delzant polygon produced by corner chopping the Hirzebruch trapezoid.

that the four minimal models can be continuously transformed into one another and that the corner chop operation is continuous. Again, let us emphasize that the results about toric fans are not new, we have just used a new viewpoint to arrive at classical results. We will see that this new viewpoint allows us to generalize the known results.

2.2 Semitoric fans

In analogy with semitoric polygons (originally defined in [27, Definition 2.5]) we define semitoric fans.

Definition 2.4. Let $v, w \in \mathbb{Z}^2$. The ordered pair (v, w) of vectors is:

1. *on the top boundary* if both vectors are in the open lower half-plane;

2. *Delzant* if $\det(v, w) = 1$;

3. *hidden* if $\det(v, Tw) = 1$; and

4. *fake* if $\det(v, Tw) = 0$.

Definition 2.5. Let $d \in \mathbb{Z}$ with $d > 2$ and let $c \in \mathbb{Z}_{\geqslant 0}$. A *semitoric fan of complexity* c is a collection of primitive vectors $(v_0 = v_d, v_1, \dots, v_{d-1}) \in (\mathbb{Z}^2)^d$ labeled in counter-clockwise order such that c of the each adjacent pairs of vectors $(v_i, v_i + 1)$ for $i = 0, \dots, d-1$ are on the top boundary and are hidden or fake corners and all remaining pairs of vectors are Delzant.

Definition 2.5 is inspired by the toric case[3]. Theorem 2.2 states that any toric fan can be produced from a minimal model using only corner chops. Similarly, our

[3] The polygons associated to semitoric fans, known as *semitoric polygons*, are momentum images of semitoric manifolds (also called semitoric integrable systems).

goal is to use a series of transformations to relate any semitoric fan to a standard form up to the action of the appropriate symmetry group.

Definition 2.6. The *symmetry group of semitoric fans* is given by

$$\mathcal{G}' = \left\{ T^k \mid k \in \mathbb{Z} \right\}$$

where $\mathcal{G}'$ acts on a semitoric fan by acting on each vector in the fan.

Definition 2.7.

1. Let $c \in \mathbb{Z}_{\geqslant 0}$. The *standard semitoric fan of complexity* c is the fan $(u_0, \ldots, u_{c+3}) \in (\mathbb{Z}^2)^{c+4}$ given by

$$u_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} c \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

   and

$$u_{4+n} = \begin{pmatrix} -c+n \\ -1 \end{pmatrix}$$

   for $n = 0, \ldots, c-1$.

2. Let $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ be a semitoric fan. The following are called the *four fan transformations*:

   (a) Suppose that (v_i, v_{i+1}) is a Delzant corner for some $i \in \{0, \ldots, d-1\}$. Then

$$(v_0, \ldots, v_i, v_i + v_{i+1}, v_{i+1}, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^{d+1}$$

   obtained by inserting the sum of two adjacent vectors between them. The process of producing this new fan from the original is known as *corner chopping*.

   (b) A *reverse corner chop* is the procedure by which a single vector which is the sum of its adjacent vectors is removed from the fan. This is the inverse of a corner chop.

   (c) Suppose that the pair (v_i, v_{i+1}) is a hidden corner. Then

$$(v_0, \ldots, v_i, Tv_{i+1}, v_{i+1}, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^{d+1}$$

   is a semitoric fan with (v_i, Tv_{i+1}) a Delzant corner and (Tv_{i+1}, v_{i+1}) a fake corner (Lemma 5.1). The process of producing this fan is known as *removing the hidden corner* (v_i, v_{i+1}).

   (d) Suppose that the pair (v_i, v_{i+1}) is a fake corner and the pair (v_{i+1}, v_{i+2}) is a Delzant corner. Then

$$(v_0, \ldots, v_i, Tv_{i+2}, v_{i+2}, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$$

   is a semitoric fan with (v_i, Tv_{i+2}) a Delzant corner and (Tv_{i+2}, v_{i+2}) a fake corner (Lemma 5.2). The process of producing this fan is known as *commuting a fake and a Delzant corner*.

Using the algebraic results from Section 3 we show the following.

Theorem 2.8. *Let $d \geqslant 1$ be an integer. Any semitoric fan $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ of complexity $c \in \mathbb{Z}_{\geqslant 0}$ may be transformed into a semitoric fan $\mathcal{G}'$-equivalent to the standard semitoric fan of complexity c by using the four fan transformations.*

Remark 2.9. The method we are using to study semitoric manifolds is analogous to the method we use to study toric manifolds. Theorem 2.2 explains how to generate the toric polygons and is used to prove that the space of toric polygons is path-connected (Theorem 2.3) which implies that the space of toric manifolds is connected (Theorem 2.10). Similarly, Theorem 2.8 shows how to generate the semitoric polygons, and as an application we prove Lemma 6.23 which describes the connected components in the space of semitoric ingredients (Definition 6.9) and this implies Theorem 2.11, which describes the connected components of the moduli space of semitoric systems. $\oslash$

2.3 Algebraic tools: the winding number

It is shown in Lemma 3.1 that the special linear group $\mathrm{SL}_2(\mathbb{Z})$ may be presented as

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid T^{-1}ST^{-1} = STS,\ S^4 = I \rangle$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus Equation (1) becomes

$$ST^{a_0} \cdots ST^{a_{d-1}} = I$$

where $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ and I denotes the 2×2 identity matrix. Given $v_0, v_1 \in \mathbb{Z}^2$ with $\det(v_1, v_2) = 1$ a set of vectors

$$(v_0, v_1, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$$

may be produced by

$$v_{i+2} = -v_i + a_i v_{i+1}$$

for $i = 0, \ldots, d-1$ where we define $v_d = v_0$ and $v_{d+1} = v_1$. In this way associated to each list of integers satisfying Equation (1) there is an ordered collection of vectors unique up to $\mathrm{SL}_2(\mathbb{Z})$. It can be seen that the determinant between any adjacent pair of these vectors is one and thus if these vectors are labeled in counter-clockwise order, then they are a toric fan. The reason that not all sequences of integers which satisfy Equation (1) correspond to a toric fan is that the vectors $v_0, \ldots, v_{d-1} \in \mathbb{Z}^2$ may circle more than once around the origin, and thus not be labeled in counter-clockwise order (see Figure 2). Thus, we see that viewing $ST^{a_0} \cdots ST^{a_{d-1}}$ as an element of $\mathrm{SL}_2(\mathbb{Z})$ is losing too much information.

Let $K = \ker(\mathbb{Z} * \mathbb{Z} \to \mathrm{SL}_2(\mathbb{Z}))$ where $\mathbb{Z} * \mathbb{Z}$ denotes the free group with generators S and T and the map $\mathbb{Z} * \mathbb{Z} \to \mathrm{SL}_2(\mathbb{Z})$ is the natural projection. For any word in K a sequence of vectors may be produced by letting the word act on a vector $v \in \mathbb{Z}^2$ one term at a time. We know we will end back at v, but the sequence of vectors produced contains more information about the word. This sequence can be used to define a path in $\mathbb{R}^2 \setminus \{(0,0)\}$. Of particular interest, especially when studying toric and semitoric fans, is the *winding number* of such a path. That is, the number of times the path, and hence the collection of vectors, circles the origin. This construction is explained in detail in Section 4, and in particular Definition 4.4 given a precise definition of the number of times an ordered collection of vectors circles the origin. We find that the winding number of a word $\sigma \in K$ is given by

$$W_K(\sigma) = \frac{3s - t}{12}$$

where s is the number of appearances of S in σ and t is the number of appearances of T in σ. We present the group

$$G = \langle S, T \mid STS = T^{-1}ST^{-1} \rangle$$

on which the winding number descends to a well-defined function $W : G \to \mathbb{Z}$. In fact, G is isomorphic to the pre-image of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$ (Proposition 3.7). Thus, if K' is the image of K projected to G, then given some $g \in K'$ there is an associated closed loop in $\mathrm{SL}_2(\mathbb{R})$. The fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is $\mathbb{Z}$ and the classical winding number of this loop in $\mathrm{SL}_2(\mathbb{R})$ coincides with $W(\sigma)$ for any $\sigma \in \mathbb{Z} * \mathbb{Z}$ which projects to g. Finally, in Corollary 4.6 we show that integers $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ correspond to a toric fan if and only if the equality

$$ST^{a_0} \cdots ST^{a_{d-1}} = S^4$$

is satisfied in G. This correspondence is the basis of our method to study toric and semitoric fans.

2.4 Applications to symplectic geometry

While Delzant polygons are in correspondence with toric manifolds, semitoric polygons are associated with the so called semitoric integrable systems. A *semitoric integrable system* (or *semitoric manifold*) is given by a triple $(M, \omega, F : M \to \mathbb{R}^2)$ where M is a connected symplectic 4-manifold and $F : M \to \mathbb{R}^2$ is an integrable system given by two maps $J, H : M \to \mathbb{R}^2$ such that J is a proper map which generates a periodic flow (see Section 6 for the precise definition).

Semitoric integrable systems are of interest in mathematical physics, symplectic geometry and spectral theory, because they exhibit rich features from dynamical, geometric, and topological viewpoints. Examples of semitoric systems permeate the physics literature. For instance the Jaynes-Cummings system [7, 17], which is one of the most thoroughly studied examples of semitoric system [30], models simple physical phenomena. It is obtained by coupling a spin and an oscillator, and its phase space is $S^2 \times \mathbb{R}^2$. Many papers have been written in the past ten years about the symplectic geometry of semitoric integrable systems, see for instance [32, 33, 26, 27, 28, 29, 25]. In [27] the authors prove a result analogous to Delzant's, classifying semitoric systems via a list of ingredients which includes a family of polygons (there is an overview of this result in Section 6.1).

One question which had remained open is whether the set of all semitoric systems has a reasonable underlying geometry, and whether as such one can view it as a topological space or a metric space. In [23] the second author defines a metric space structure for the moduli space of semitoric systems using [27], and a natural question is whether, with respect to such structure, the space is path-connected. That is, can two semitoric systems be continuously deformed onto one another, via a path of semitoric systems?

This is preceded by [24] in which the authors construct a natural metric on the moduli space of symplectic toric manifolds and prove Theorem 2.3, which is used to conclude the following.

Theorem 2.10 ([24])**.** *The moduli space of toric manifolds is path-connected.*

Similarly, Theorem 2.8 implies the following statement. The number of focus-focus singularities and the twisting index of a semitoric system are explained in Section 6.

Theorem 2.11. *If (M, ω, F) and (M', ω', F') are semitoric integrable systems such that:*

(i) *they have the same number of focus-focus singularities;*

(ii) *they have the same sequence of twisting indices,*

then there exists a continuous (with respect to the topology defined in [23]*) path of semitoric systems with the same number of focus-focus points and same twisting indices between them. That is, the space of semitoric systems with fixed number of focus-focus points and twisting index invariant is path-connected.*

Theorem 6.24 is a refined version of Theorem 2.11.

3 Algebraic set-up: matrices and $\mathrm{SL}_2(\mathbb{Z})$ relations

The 2×2 special linear group over the integers, $\mathrm{SL}_2(\mathbb{Z})$, is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will see that to each toric (resp. semitoric) integrable system there is an associated toric (resp. semitoric) fan and we will use the algebraic structure of $\mathrm{SL}_2(\mathbb{Z})$ to study these fans. For our purposes, the following presentation of $\mathrm{SL}_2(\mathbb{Z})$ will be the most natural way to view the group.

Lemma 3.1. *The 2×2 special linear group over the integers,* $\mathrm{SL}_2(\mathbb{Z})$*, may be presented as*

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid T^{-1}ST^{-1} = STS,\ S^4 = I\rangle.$$

Proof. It is well-known that

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid (ST)^3 = S^2 = -I\rangle \tag{2}$$

(see for instance [4]) where $(-I)^2 = I$ and $-I$ is in the center of $\mathrm{SL}_2(\mathbb{Z})$. To obtain the relations in the statement of the lemma from those in Equation (2) notice

$$(ST)^3 = S^2 \Leftrightarrow STS = T^{-1}ST^{-1} \text{ and } S^2 = -I \Rightarrow S^4 = I.$$

To obtain the relations of Equation (2) from those in the statement of the lemma we only have to show that S^2 behaves as $-I$. That is, we must show that S^2 squares to the identity and is in the center of the group. We have that $(S^2)^2 = I$ and that S^2 commutes with S. To show that S^2 commutes with T notice

$$\begin{aligned} TS^2 &= TS(TSTST) \\ &= (TSTST)ST \\ &= S^2T. \end{aligned}$$

This concludes the proof. □

For $v, w \in \mathbb{Z}^2$ let $[v, w]$ denote the 2×2 matrix with v as the first column and w as the second and let $\det(v, w)$ denote the determinant of the matrix $[v, w]$.

Lemma 3.2. *Let $u, v, w \in \mathbb{Z}^2$ and $\det(u, v) = 1$. Then $\det(v, w) = 1$ if and only if there exists some $a \in \mathbb{Z}$ such that $w = -u + av$.*

Proof. In the basis (u, v) we know that $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write $w = \begin{pmatrix} b \\ a \end{pmatrix}$ for some $a, b \in \mathbb{Z}$. Then we can see that $\det(v, w) = -b$ so $\det(v, w) = 1$ if and only if $b = -1$. That is, $w = -u + av$. □

The result of Lemma 3.2 can be easily summarized in a matrix equation, as we will now show. Let

$$(v_0 = v_d, v_1 = v_{d+1}, v_2, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$$

be a toric fan and define $A_i = [v_i, v_{i+1}]$ for $i = 0, \ldots, d$. Note that $A_d = A_0$.

Lemma 3.3. *For each $i \in 0, \ldots, d-1$ there exists an integer $a_i \in \mathbb{Z}$ such that $A_{i+1} = A_iST^{a_i}$.*

Proof. By the definition of a toric fan we know that for each $0 \leqslant i < d-2$ we have that

$$\det(v_i, v_{i+1}) = \det(v_{i+1}, v_{i+2}) = 1$$

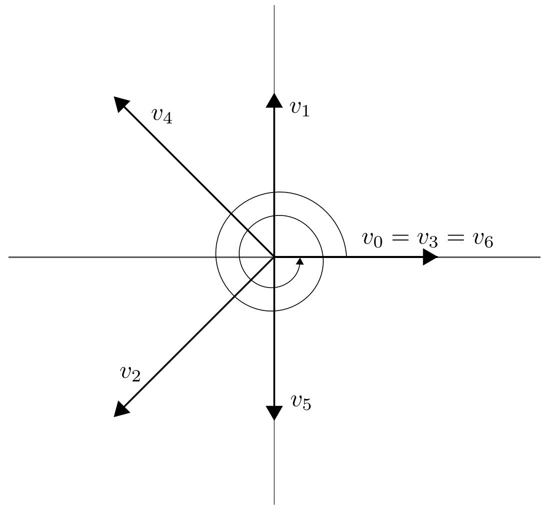

Figure 2: These vectors do not form a fan because they are not labeled in counter-clockwise order.

so by Lemma 3.2 there exists $a_i \in \mathbb{Z}$ such that $v_{i+2} = -v_i + a_i v_{i+1}$. Then

$$\begin{aligned} A_i S T^{a_i} &= [v_{i+1}, -v_i + a_i v_i] \\ &= [v_{i+1}, v_{i+2}] \\ &= A_{i+1}, \end{aligned}$$

and this concludes the proof. □

It follows that

$$A_d = A_{d-1} S T^{a_{d-1}} = A_{d-2} S T^{a_{d-2}} S T^{a_{d-1}} = \cdots = A_0 S T^{a_0} \cdots S T^{a_{d-1}}$$

which means

$$A_0 = A_d = A_0 S T^{a_0} \cdots S T^{a_{d-1}},$$

and so

$$S T^{a_0} \cdots S T^{a_{d-1}} = I. \tag{3}$$

This is a restatement of Equation (1) which is from [12]. So to each toric fan of d vectors there is an associated d-tuple of integers which satisfy Equation (3), but having a tuple of integers which satisfy Equation (3) is not enough to assure that they correspond to an toric fan. The determinant of the vectors will be correct but, roughly speaking, if the vectors wind around the origin more then once then they will not be labeled in the correct order to be a toric fan, as it occurs in the following example.

Example 3.4. Consider the sequence of integers $a_0 = -1, a_1 = -1, a_2 = -2, a_3 = -1, a_4 = -1, a_5 = 0$ and notice that

$$S T^{-1} S T^{-1} S T^{-2} S T^{-1} S T^{-1} S T^{0} = I$$

so Equation (3) is satisfied for $c = 0$ but these integers do not correspond to a toric fan. This is because the vectors they produce:

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$
$$v_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

travel twice around the origin[4], see Figure 2. $\oslash$

So we need extra information that is not captured by viewing this word in S and T as an element of $\mathrm{SL}_2(\mathbb{Z})$. For a more obvious example notice that even though they are equal in $\mathrm{SL}_2(\mathbb{Z})$ we can see that S^4 corresponds to a toric fan while S^8 does not. From [12] we know that integers $(a_0, \ldots, a_d) \in \mathbb{Z}^d$ which satisfy Equation (1) correspond to a toric fan if and only if

$$a_0 + \ldots + a_{d-1} = 3d - 12$$

so we would like to prove that

$$\frac{3d - \sum_{i=0}^{d-1} a_i}{12}$$

is the number of times that the vectors corresponding to $(a_0, \ldots, a_{d-1})$ circle the origin. In order to prove this we will need some more terminology, and in order to keep track of the extra information about circling the origin we will need to consider a group which is larger than $\mathrm{SL}_2(\mathbb{Z})$.

Consider instead the free group with generators S and T. This group is isomorphic to the free product of $\mathbb{Z}$ with itself so we will denote it by $\mathbb{Z}*\mathbb{Z}$. We know $\mathrm{SL}_2(\mathbb{Z})$ is a quotient of $\mathbb{Z}*\mathbb{Z}$ by Lemma 3.1 so there exists a natural projection map $\pi_1 : \mathbb{Z}*\mathbb{Z} \to \mathrm{SL}_2(\mathbb{Z})$. Also define a map $w : \mathbb{Z}*\mathbb{Z} \to \mathbb{Z}$ by

$$w(S^{b_0}T^{a_0} \cdots S^{b_\ell}T^{a_\ell}) = 3\sum_{i=0}^{\ell} b_i - \sum_{i=0}^{\ell} a_i$$

for any $S^{b_0}T^{a_0} \cdots S^{b_\ell}T^{a_\ell} \in \mathbb{Z}*\mathbb{Z}$ where $a_0, \ldots, a_\ell, b_0, \ldots, b_\ell \in \mathbb{Z}$. Given a toric fan with associated integers $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ we will show that $w(ST^{a_0} \cdots ST^{a_{d-1}}) = 12$. Both π_1 and w factor over the same group G which is the fiber product of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}$ over $\mathbb{Z}/(12)$. Now we can see that we wanted the particular presentation of $\mathrm{SL}_2(\mathbb{Z})$ from Lemma 3.1 so that the relationship between G and $\mathrm{SL}_2(\mathbb{Z})$ would be clear. This discussion made precise in the following proposition.

[4]For a formal definition of this, see Definition 4.4

Proposition 3.5. *The following diagram commutes.*

$$\begin{tikzcd}
 & \mathbb{Z} * \mathbb{Z} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"'] \arrow[dd, bend left, "w"] \\
\mathrm{SL}_2(\mathbb{Z}) \arrow[d, "w_{\mathrm{SL}_2(\mathbb{Z})}"'] & G \arrow[l, "\pi_3"'] \arrow[d, "w_G"'] \\
\mathbb{Z}/(12) & \mathbb{Z} \arrow[l, "\pi_4"]
\end{tikzcd} \tag{4}$$

The group G is the fiber product of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}$ over $\mathbb{Z}/(12)$ and is given by

$$G = \langle S, T \mid STS = T^{-1}ST^{-1} \rangle,$$

each of π_1, π_2, π_3, and π_4 is a projection, and $w : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$, $w_G : G \to \mathbb{Z}$, $w_{\mathrm{SL}_2(\mathbb{Z})} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/(12)$ are given by the same formal expression

$$S^{b_0}T^{a_0} \cdots S^{b_\ell}T^{a_\ell} \mapsto 3\sum_{i=0}^{\ell} b_i - \sum_{i=0}^{\ell} a_i.$$

Proof. It can be seen that the map $w_{\mathrm{SL}_2(\mathbb{Z})} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/(12)$ is well-defined by noting that both relations in $\mathrm{SL}_2(\mathbb{Z})$ as presented in Lemma 3.1 preserve the value of the formula up to a multiple of 12. Similarly, since the relation $ST^{-1}S = STS$ preserves the value of the equation we know that w_G is well-defined. Since each of these functions to $\mathbb{Z}$ or $\mathbb{Z}/(12)$ is given by the same formal expression and since each π is a quotient map, the diagram commutes.

To show that G with the associated maps is the fiber product of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}$ over $\mathbb{Z}/(12)$ we must only show that w_G restricted to the fibers is bijective. That is, we must show that

$$w_G \restriction_{\pi_3^{-1}(A)} : \pi_3^{-1}(A) \to \pi_4^{-1}(w_{\mathrm{SL}_2(\mathbb{Z})}(A))$$

is a bijection for each $A \in \mathrm{SL}_2(\mathbb{Z})$. To show it is surjective, notice that for any $g \in G$

$$\pi_3(S^{4k}g) = \pi_3(g) \text{ and } w_G(S^{4k}g) = w_G(g) + 12k$$

for any $k \in \mathbb{Z}$. To show it is injective it is sufficient to consider only $A = I$. Since S^4 is in the center of G we know that $\mathrm{SL}_2(\mathbb{Z}) = G/(S^4)$ so $\pi_3^{-1}(I) = \{S^{4k} \mid k \in \mathbb{Z}\}$. Since $w_G(S^{4k}) = 12k$ we know for each choice of k this maps to a distinct element of $\mathbb{Z}$.

□

Notation 3.6. We have several groups with generators S and T. To denote the different equalities in these groups we will use an equal sign with the group in question as a subscript. That is, if an equality holds in the group H we will write $=_H$. For example, $S^4 =_{\mathrm{SL}_2(\mathbb{Z})} I$ but $S^4 \neq_G I$. ⊘

There is another useful sense in which G is an unwinding of $\mathrm{SL}_2(\mathbb{Z})$. While $\mathrm{SL}_2(\mathbb{Z})$ is discrete, and thus does not have a natural cover, it sits inside the group $\mathrm{SL}_2(\mathbb{R})$, which has a universal cover. We claim that G is the preimage of $\mathrm{SL}_2(\mathbb{Z})$ inside of the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 3.7. *The group G is isomorphic to the preimage of $\mathrm{SL}_2(\mathbb{Z})$ within the universal cover of $\mathrm{SL}_2(\mathbb{R})$.*

Proof. Let G' be the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$. We note that there exists a homomorphism, ϕ from G to G' defined by

$$\phi(S) = \left(\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right)_{0\leqslant\theta\leqslant\pi/2}, \qquad \phi(T) = \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right)_{0\leqslant x\leqslant 1}$$

where the paths given are to represent elements of the universal cover of $\mathrm{SL}_2(\mathbb{R})$. It is easy to show that $\phi(S)\phi(T)\phi(S)$ equals $\phi(T)^{-1}\phi(S)\phi(T)^{-1}$, and thus ϕ does actually define a homomorphism. We have left to show that ϕ defines an isomorphism. To show this, we note that each of G and G' have obvious surjections π and π' to $\mathrm{SL}_2(\mathbb{Z})$. Furthermore it is clear that $\pi = \pi' \circ \phi$. Thus, to show that ϕ is an isomorphism, it suffices to show that $\phi : \ker(\pi) \to \ker(\pi')$ is an isomorphism.

However, it is clear that $\ker(\pi)$ is $\langle S^4 \rangle$. On the other hand, $\ker(\pi') \cong \pi_1(\mathrm{SL}_2(\mathbb{R})) = \mathbb{Z}$, and is generated by $\phi(S)^4$. This completes the proof. $\square$

We will see that there is a one to one correspondence between toric fans up to the action of $\mathrm{SL}_2(\mathbb{Z})$ and lists of integers $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ satisfying

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4 \tag{5}$$

(Corollary 4.6). Equation (5) is a refinement of Equation (3) which implies both that the successive pairs of vectors form a basis of $\mathbb{Z}^2$ and that the vectors are labeled in counter-clockwise order. In Proposition 5.4 we produce an analogous equation for semitoric fans.

Now we would like to simplify these toric fans. We will understand which integers $a_0, \ldots, a_{d-1}$ are possible in an element $ST^{a_0} \cdots ST^{a_{d-1}} \in G$ corresponding to toric fan by studying $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/(-I)$. The following lemma is important for this and will also be useful later on when classifying semitoric fans. If $ST^{a_0} \cdots ST^{a_{d-1}} \in G$ projects to the identity in $\mathrm{SL}_2(\mathbb{Z})$ then by Lemma 3.8 we can see when one of the exponents must be in the set $\{-1, 0, 1\}$. In any of these cases, we will be able to use relations in G to help simplify the expression.

Lemma 3.8. *Suppose that*

$$ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{PSL}_2(\mathbb{Z})} I \tag{6}$$

for some $d \in \mathbb{Z}$, $d > 0$. Then if $d \geqslant 3$ there exist $i, j \in \mathbb{Z}$ satisfying $0 \leqslant i < j \leqslant d-1$ such that $a_i, a_j \in \{-1, 0, 1\}$. Furthermore:

- *If $d > 3$ then i, j can be chosen such that $i \neq j - 1$ and $(i, j) \neq (0, d-1)$.*

- *If $d = 3$ then $a_0 = a_1 = a_2 = 1$ or $a_0 = a_1 = a_2 = -1$.*
- *If $d = 2$ then $a_0 = a_1 = 0$.*
- *If $d = 1$ then Equation (6) cannot hold.*

Proof. It is well known that $\mathrm{PSL}_2(\mathbb{Z})$ acts on the real projective line $\mathbb{R} \cup \{\infty\}$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d} \text{ for } x \in \mathbb{R} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty) = \frac{a}{c}.$$

Let $d > 3$. Suppose that at most two of $a_0, \ldots, a_{d-1}$ are in $\{-1, 0, 1\}$ and if there are two in $\{-1, 0, 1\}$ that they are consecutive or indexed by 0 and $d - 1$. Notice that

$$ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{PSL}_2(\mathbb{Z})} I \text{ implies that } ST^{a_1} \cdots ST^{a_{d-1}} ST^{a_0} =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

by conjugating each side with ST^{a_0}. This conjugation method and renumbering the integers can be used to assure that $a_i \notin \{-1, 0, 1\}$ for $i = 1, \ldots, d-2$. Since this expression is equal to the identity in $\mathrm{PSL}_2(\mathbb{Z})$ it acts trivially on $\mathbb{R} \cup \{\infty\}$. In particular, we have

$$ST^{a_0} \cdots ST^{a_{d-1}}(\infty) = \infty.$$

Notice that $S(x) = {}^{-1}\!/_{x}$ and $T^a(x) = x + a$ for $a \in \mathbb{Z}$. Further notice that for any $a \in \mathbb{Z} \backslash \{-1, 0, 1\}$ and $x \in (-1, 1) \backslash \{0\}$ we have $ST^a(x) \in (-1, 1) \backslash \{0\}$. We see that $ST^{a_{d-1}}(\infty) = 0$ and since $a_{d-2} \notin \{-1, 0, 1\}$ we know $ST^{a_{d-2}}(0) \in (-1, 1) \setminus \{0\}$. Putting these facts together we have

$$\begin{aligned} ST^{a_0} \cdots ST^{a_{d-1}}(\infty) &= ST^{a_0} \cdots ST^{a_{d-2}}(0) \\ &= ST^{a_0} \cdots ST^{a_{d-3}}(x) && \text{for some } x \in (-1, 1) \setminus \{0\} \\ &= ST^{a_0}(y) && \text{for some } y \in (-1, 1) \setminus \{0\} \\ &= \frac{-1}{y + a_0} \neq \infty \end{aligned}$$

This contradiction finishes the $d > 3$ case.

If $d = 3$ essentially the same result holds except that it is not possible to choose two elements that are non-consecutive. Notice

$$ST^{a_0} ST^{a_1} ST^{a_2}(\infty) = ST^{a_0} ST^{a_1}(0) = -\left(\frac{-1}{a_1} + a_0\right)^{-1}.$$

For this function to be the identity we would need

$$\frac{-1}{a_1} + a_0 = 0.$$

This implies that $a_0 a_1 = 1$ so since they are both integers we have $a_0 = a_1 = \epsilon$ where $\epsilon \in \{-1, 1\}$. Conjugating by ST^{a_0}, we find symmetrically, that $a_1 a_2 = 1$, and thus that $a_0 = a_1 = a_2 = \pm 1$.

If $d = 2$ then we have

$$ST^{a_0}ST^{a_1}(\infty) = ST^{a_0}(0) = \frac{-1}{a_0}$$

so we must have $a_0 = 0$ and then

$$ST^0ST^{a_1}(x) = S^2T^{a_1}(x) = x + a_1,$$

so we are also forced to have that $a_1 = 0$, as stated in the lemma. If $d = 1$ then $ST^{a_0}(\infty) = 0$ for any choice of $a_0 \in \mathbb{Z}$, so there are no solutions. $\square$

4 Toric fans

Let $a_0, a_1, \ldots, a_{d-1} \in \mathbb{Z}$ be a collection of integers such that

$$ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} I.$$

This means that

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^{4k} \text{ for some } k \in \mathbb{Z}$$

by Proposition 3.5. We claim that these integers correspond to a toric fan if and only if $k = 1$.

The idea is that $k = 1$ precisely when the vectors in the corresponding fan are labeled in counter-clockwise order, and the only relation in the group G, which is $STS = T^{-1}ST^{-1}$, preserves the number of times the vectors circle the origin. Now we make this idea precise.

Lemma 4.1. *Let $g \in ker(\pi_3)$. Then $\frac{w_G(g)}{12} \in \mathbb{Z}$.*

Proof. Since $\pi_3(g) = I$ we know $w_{\mathrm{SL}_2(\mathbb{Z})} \circ \pi_3(g) = 0$, so by Proposition 3.5 $\pi_4 \circ w_G(g) = 0$. Thus $w_G(g) \in \ker(\pi_4) = \{12k \mid k \in \mathbb{Z}\}$. $\square$

Recall that G is isomorphic to the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$ by Proposition 3.7. Let ϕ be the isomorphism from G to the universal cover of $\mathrm{SL}_2(\mathbb{R})$ with

$$\phi(S) = \left(\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right)_{0 \leqslant \theta \leqslant \pi/2}, \quad \phi(T) = \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right)_{0 \leqslant x \leqslant 1}$$

This means to each element of the kernel of π_3 we can associate a closed loop based at the identity in $\mathrm{SL}_2(\mathbb{R})$ denoted $\phi(g)$. The fundamental group $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ is isomorphic to $\mathbb{Z}$ and is generated as $\langle \phi(S^4) \rangle$, so let $\psi : \pi_1(\mathrm{SL}_2(\mathbb{R})) \to \mathbb{Z}$ be the homomorphism with $\psi(\phi(S^4)) = 1$.

Lemma 4.2. *Let $g \in ker(\pi_3 : G \to \mathrm{SL}_2(\mathbb{Z}))$. Then*

$$\psi \circ \phi(g) = \frac{w_G(g)}{12}.$$

Proof. Since $\ker(\pi_3)$ is generated by S^4, it suffices to check that $\psi(\phi(S^4)) = \frac{w_G(S^4)}{12} = 1$, but this holds by definition. □

Definition 4.3. Define $W : \ker(\pi_3) \to \mathbb{Z}$ by

$$W(g) = \frac{w_G(g)}{12}.$$

We call $W(g)$ the *winding number* of $g \in \ker(\pi_3)$.

Definition 4.4. Let

$$(v_0 = v_d, v_1, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$$

with $\det(v_i, v_{i+1}) > 0$ for $i = 0 \ldots, d-1$. We define *the number of times* $(v_0, \ldots, v_{d-1})$ *circles the origin* to be the winding number of the piecewise linear path in $(\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{(0,0)\}$ produced by concatenating the linear paths between v_i and v_{i+1} for $i = 0, \ldots, d-1$.

Lemma 4.5. *Let $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ such that $ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} I$ and let $v_0, v_1 \in \mathbb{Z}^2$ such that* $\det(v_1, v_2) = 1$. *Define $v_2, \ldots v_{d-1}$ by*

$$v_{i+2} = -v_i + a_i v_{i+1}$$

where $v_d = v_0$ and $v_{d+1} = v_1$. Then the winding number $W(ST^{a_0} \cdots ST^{a_{d-1}}) \in \mathbb{Z}$ is the number of times that $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ circles the origin.

Proof. Let A_i be the matrix $[v_i, v_{i+1}]$ and recall that $A_{i+1} = A_i ST^{a_i}$. Thinking of the elements ST^{a_i} as elements of G, and thus as elements of the universal cover of $\mathrm{SL}_2(\mathbb{R})$, this gives a path from A_i to A_{i+1}, and concatenating these paths gives a path from A_0 to itself in $\mathrm{SL}_2(\mathbb{R})$. Projecting this path into the first column vector of the appropriate matrix gives a path in $(\mathbb{R}^2)^*$. We claim that this path is homotopic to the path formed by taking line segments between v_i and v_{i+1}. This is easily verified because both paths between v_i and v_{i+1} travel counterclockwise less than a full rotation.

We know that $W(ST^{a_0} \cdots ST^{a_{d-1}})$ equals ψ of the path in $\mathrm{SL}_2(\mathbb{R})$, and now we need to show that this equals the winding number in $(\mathbb{R}^2)^*$ of the first column vectors. To show this we note that the element of $\pi_1((\mathbb{R}^2)^*)$ (the group is abelian, so we may ignore basepoint) given by the image of the element of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ under the natural projection map. Thus, we merely need to show that the this projection acts correctly on a generator of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$, but it is easy to see that the image of $\phi(S^4)$ yields a path of winding number 1. □

Corollary 4.6. *There exists a bijection from the set of all sequences $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$, $d > 0$, which satisfy*

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4$$

to the collection of all toric fans modulo the action of $\mathrm{SL}_2(\mathbb{Z})$*. This bijection sends* $(a_0, \ldots, a_d) \in \mathbb{Z}^d$ *to the equivalence class of fans*

$$\{(v_0 = v_d, v_1 = v_{d+1}, v_2, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d \mid v_0, v_1 \in \mathbb{Z}^2, \det(v_0, v_1) = 1\}$$

in which

$$v_{i+2} = -v_i + a_i v_{i+1}$$

for $i = 0, \ldots, d-1$*.*

Proof. Let $(v_0 = v_d, v_1, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ be a toric fan. That is, $\det(v_i, v_{i+1}) = 1$ for each $i = 0, \ldots, d-1$ and the vectors are labeled in counter-clockwise order. It is shown in Section 3 Equation (3) that associated integers $(a_0, \ldots, a_{d-1}) \in (\mathbb{Z})^d$ exist such that

$$ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} I$$

which means

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^{4k}$$

for some $k \in \mathbb{Z}$ with $k \geqslant 0$. By Lemma 4.5 we know

$$W(ST^{a_0} \cdots ST^{a_{d-1}}) = 1$$

so that the vectors will be labeled in the correct order for it to be a fan. Thus, $W(S^{4k}) = 1$ but $W(S^{4k}) = k$ so $k = 1$.

Now suppose that $(a_0, \ldots, a_{d-1})^d \in \mathbb{Z}$ satisfy $ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4$ and define $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ by

$$v_{i+2} = -v_i + a_i v_{i+1}$$

where $v_0, v_1 \in \mathbb{Z}^2$ are any two vectors for which $\det(v_0, v_1) = 1$. Then for each $i = 0, \ldots, d-1$ we have

$$\det(v_{i+1}, v_{i+2}) = \det \begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix} \det(v_i, v_{i+1}) = \det(v_i, v_{i+1}),$$

so by induction all of these determinants are 1. By Lemma 4.5, the path connecting adjacent vectors wraps around the origin only once, and since each v_{i+1} is located counterclockwise of v_i, we have that the v_i's must be sorted in counterclockwise order.

□

Now that we have set up the algebraic framework the following results are straightforward to prove. First we prove that any fan with more than four vectors can be reduced to a fan with fewer vectors.

Lemma 4.7. *If* $(v_0 = v_d, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ *is a toric fan with* $d > 4$ *then there exists some* $i \in \{0, \ldots, d-1\}$ *such that* $v_i = v_{i-1} + v_{i+1}$*.*

Proof. By Corollary 4.6 we know that to the fan $(v_0 = v_d, \dots, v_{d-1})$ there is an associated list of integers $a_0, \dots, a_{d-1} \in \mathbb{Z}$ such that $v_{i+2} = -v_i + a_i v_{i+1}$ and

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4. \tag{7}$$

We must only show that for some $i \in \mathbb{Z}$ we have $a_i = 1$. Since $S^4 =_{\mathrm{PSL}_2(\mathbb{Z})} I$ we can use Lemma 3.8 to conclude that there exist $i, j \in \mathbb{Z}$ satisfying $0 \leqslant i < j-1 \leqslant d-2$ such that $a_i, a_j \in \{-1, 0, 1\}$ and $(i,j) \neq (0, d-1)$. By way of contradiction assume that $a_i, a_j \in \{-1, 0\}$. Conjugate Equation (7) by ST^{a_n} for varying $n \in \mathbb{Z}$ to assure that $i \neq 0$ and $j \neq d-1$. Then at each of these values we may use either $ST^0S =_G S^2$ or $ST^{-1}S =_G S^2TST$ to reduce the number of ST-pairs by one and produce a factor of S^2. These reductions do not interfere with one another because the values in question are not adjacent. So we end up with

$$S^4 ST^{b_0} \cdots ST^{b_{\ell-1}} =_G S^4$$

where $\ell > 2$ because we started with at least five ST-pairs and have only reduced by two. This means

$$ST^{b_0} \cdots ST^{b_{\ell-1}} =_G I$$

with $\ell > 2$.

This implies that $W(ST^{b_0} \cdots ST^{b_{\ell-1}}) = 0$ and thus, by Lemma 4.5, that the corresponding collection of vectors winds no times about the origin. However, this is impossible since for such a sequence of vectors v_{i+1} is always counterclockwise of v_i.

□

The case in which a vector in the fan is the sum of the adjacent vectors is important because this means the fan is the result of corner chopping a fan with fewer vectors in it. Now that we have the proper algebraic tools, we will be clear about the specifics of the corner chopping and reverse corner chopping operations.

Suppose $(v_0 = v_d, v_2, \dots, v_{d-1}) \in (\mathbb{Z}^2)^d$ is a toric fan with associated integers $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$. Then

$$v_{i+2} = -v_i + a_i v_{i+1}$$

so if $a_i = 1$ then we have that $v_{i+1} = v_i + v_{i+2}$. Now we see that in this case

$$\det(v_i, v_{i+2}) = \det(v_i, -v_i) + \det(v_i, v_{i+1}) = 1$$

so

$$(w_0 = v_0, \dots, w_i = v_i, w_{i+1} = v_{i+2}, \dots, w_{i-2} = v_{d-1}) \in (\mathbb{Z}^2)^{d-1}$$

is also a fan. Next notice

$$-w_i + (a_{i+1} - 1)w_{i+1} = -(v_i + v_{i+2}) + a_{i+1}v_{i+2} = -v_{i+1} + a_{i+1}v_{i+2} = w_{i+2}$$

and

$$-w_{i-1} + (a_{i-1} - 1)w_i = (-v_{i-1} + a_{i-1}v_i) - v_i = v_{i+1} - v_i = w_{i+1}$$

so this new fan has associated to it the tuple of integers $(a_0, \ldots, a_{i-1} - 1, a_{i+1} - 1, \ldots, a_{d-1}) \in \mathbb{Z}^{d-1}$. An occurrence of 1 from the original tuple of integers has been removed and the adjacent integers have been reduced by 1. Algebraically, this move corresponds to the relation $STS =_G T^{-1}ST^{-1}$. Geometrically this move corresponds to the inverse of chopping a corner from the associated polygon (as is shown in Figure 1). The corner chopping of a toric polygon is done such that the new face of the polygon produced has inwards pointing normal vector given by the sum of the adjacent inwards pointing primitive integer normal vectors.

Now we can see that Lemma 4.7 tells us that fans with five or more vectors are the result of corner chopping a fan with fewer vectors. We will next classify all possible fans with fewer than five vectors.

Lemma 4.8. *Suppose that integers $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ satisfy*

$$ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4 \tag{8}$$

for some $d \in \mathbb{Z}$, $d \geqslant 0$.

1. *If $d = 4$ then up to a cyclic reordering the set of integer quadruples which satisfy this equation is exactly $a_0 = 0, a_1 = k, a_2 = 0, a_3 = -k$ for each $k \in \mathbb{Z}$.*

2. *If $d = 3$ then $a_0 = a_1 = a_2 = -1$.*

3. *If $d < 3$ then there do not exist integers satisfying Equation* (8)*.*

Proof. Notice $ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4$ implies that

$$ST^{a_0} \cdots ST^{a_{d-1}} =_{\mathrm{PSL}_2(\mathbb{Z})} I.$$

From Lemma 3.8 we know that if $d = 1$ then this equality is impossible and if $d = 2$ the only possibility is $ST^0ST^0 =_{\mathrm{PSL}_2(\mathbb{Z})} I$ but $ST^0ST^0 \neq_G S^4$. Now assume that $d = 3$. Again by Lemma 3.8 we know the only possibilities are $a_0 = a_1 = a_2 = \pm 1$. If $a_0 = a_1 = a_2 = 1$ then notice that

$$STSTST =_G S^2 \neq_G S^4.$$

Next notice that

$$ST^{-1}ST^{-1}ST^{-1} =_G S^4$$

so that is the only possibility for $d = 3$.

Now suppose that $d = 4$. Lemma 3.8 tells us that at least one of the a_i is in the set $\{-1, 0, 1\}$. By conjugation (which cyclically permutes the order of the integers) we may assume that $a_0 \in \{-1, 0, 1\}$. If $a_0 = 1$ then

$$STST^{a_1}ST^{a_2}ST^{a_3} =_G ST^{a_1-1}ST^{a_2}ST^{a_3-1}$$

so for this to equal S^4 in G we must have $a_1 - 1 = a_2 = a_3 - 1 = -1$ by the $d = 3$ argument above. It is straightforward to check that $STSST^{-1}S =_G S^4$ so we have found a solution.

If $a_0 = -1$ then notice

$$ST^{-1}ST^{a_1}ST^{a_2}ST^{a_3} =_G S^4 \text{ implies } ST^{a_1+1}ST^{a_2}ST^{a_3+1} =_G S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

so by Lemma 3.8 we must have $a_1 + 1 = a_2 = a_3 + 1 = \pm 1$. This time, if $a_1 + 1 = a_2 = a_3 + 1 = -1$ then Equation (8) does not hold, since the left side will equal S^6, but if $a_1 + 1 = a_2 = a_3 + 1 = 1$ then the equation holds. So we have found another solution which is $ST^{-1}SSTS =_G S^4$.

Finally suppose that $a_0 = 0$. Notice

$$ST^0ST_{a_1}ST^{a_2}ST^{a_3} =_G S^4 \text{ implies } ST^{a_2}ST^{a_1+a_3} =_G S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

so we can use Lemma 3.8 to conclude that we need $a_2 = a_1 + a_3 = 0$. Let $a_1 = k \in \mathbb{Z}$. Now we have that

$$ST^0ST^kST^0ST^{-k} =_G S^4$$

for any $k \in \mathbb{Z}$. Finally, observe that the other two possibilities we derived in the $d = 4$ case are just reorderings of this one with $k = 1$. $\square$

Definition 4.9. A *Delzant triangle* is the convex hull of the points $(0, 0)$, $(0, \lambda)$, $(\lambda, 0)$ in $\mathbb{R}^2$ for any $\lambda > 0$. A *Hirzebruch trapezoid* with parameter $k \in \mathbb{Z}_{\geqslant 0}$ is the convex hull of $(0, 0)$, $(0, a)$, (b, a), and $(b + ak, 0)$ in $\mathbb{R}^2$ where $a, b > 0$. A Hirzebruch trapezoid with parameter zero is a *rectangle*.

These are shown in Figure 1. So we see that the fan corresponding to any Delzant triangle is

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)$$

with associated integers $(-1, -1, -1)$ and the fan corresponding to a Hirzebruch trapezoid with parameter k is

$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -k \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

with associated integers $(0, k, 0, -k)$. The following Theorem is immediate from Lemma 4.7 and Lemma 4.8.

Theorem 4.10 ([12])**.** *Every Delzant polygon can be obtained from a polygon* $\mathrm{SL}_2(\mathbb{Z})$*-equivalent to a Delzant triangle, a rectangle, or a Hirzebruch trapezoid by a finite number of corner chops.*

Proof. Let Δ be any Delzant polygon with d edges, let $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ be the associated fan of inwards pointing primitive normal vectors, and let $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ be the integers associated to this fan. By Lemma 4.7 if $d > 4$ then $a_i = 1$ for some $i \in \{0, \ldots, d-1\}$ so the fan is the result of a corner chop for some fan with $d - 1$ vectors. That is, Δ is the result of a corner chop of some Delzant polygon with $d - 1$ edges. If $d < 5$ then Lemma 4.8 lists each possibility. If $d = 4$ and $a_0 = a_1 = a_2 = a_3 = 0$ then Δ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a rectangle, if $a_0 = 0, a_1 = k, a_2 = 0$, and $a_3 = -k$ for $k \in \mathbb{Z} \setminus \{0\}$ then Δ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a Hirzebruch trapezoid, and if $d = 3$ with $a_0 = a_1 = a_2 = -1$ then Δ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a Delzant triangle. $\square$

5 Semitoric fans

Now we will apply the method from Section 4 to classify semitoric fans (Definition 2.5).

The first step in the classification is given by a series of lemmas which we will use to manipulate the semitoric fans in a standard form.

Lemma 5.1. *If $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ is a semitoric fan and (v_i, v_{i+1}) is a hidden corner, then*

$$(w_0 = v_0, \ldots, w_i = v_i, w_{i+1} = Tv_{i+1}, w_{i+2} = v_{i+1}, \ldots, w_d = v_{d-1}) \in (\mathbb{Z}^2)^{d+1}$$

is a semitoric fan in which (w_i, w_{i+1}) is a Delzant corner and (w_{i+1}, w_{i+2}) is a fake corner.

Proof. We know that $\det(v_i, Tv_{i+1}) = 1$ because that pair of vectors forms a hidden corner. Notice that

$$\det(w_i, w_{i+1}) = \det(v_i, Tv_{i+1}) = 1$$

and

$$\det(w_{i+1}, Tw_{i+2}) = \det(Tv_{i+1,}Tv_{i+1}) = 0,$$

which concludes the proof. □

Lemma 5.2. *If $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ is a semitoric fan and (v_i, v_{i+1}) is a fake corner and (v_{i+1}, v_{i+2}) is a Delzant corner, then*

$$(w_0 = v_0, \ldots, w_i = v_i, w_{i+1} = Tv_{i+2}, w_{i+2} = v_{i+2}, \ldots, w_{d-1} = v_{d-1}) \in (\mathbb{Z}^2)^d$$

is a semitoric fan in which (w_i, w_{i+1}) is a Delzant corner and (w_{i+1}, w_{i+2}) is a fake corner.

Proof. We know $\det(v_i, Tv_{i+1}) = 0$ so $v_i = Tv_{i+1}$ since they are both on the top boundary and we also know $\det(v_{i+1}, v_{i+2}) = 1$. Now we can check that

$$\det(w_i, w_{i+1}) = \det(v_i, Tv_{i+2}) = \det(Tv_{i+1}, Tv_{i+2}) = \det(v_{i+1}, v_{i+2}) = 1$$

and

$$\det(w_{i+1}, Tw_{i+2}) = \det(Tv_{i+2}, Tv_{i+2}) = 0,$$

which concludes the proof. □

In Lemma 5.1 we have described the process of removing a hidden corner and in Lemma 5.2 we have described the process of commuting a fake and Delzant corner. Both of these processes are defined in Definition 2.7.

Lemma 5.3. *Suppose that $(v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$ is a semitoric fan. Then after a finite number of corner choppings the fan will be $\mathcal{G}'$-equivalent to one in which two adjacent vectors are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*

Proof. Let $v_d = v_0$. If

$$v_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for some $i \in \{0, \dots, d-1\}$ then notice

$$v_{i-1} = \begin{pmatrix} a \\ -1 \end{pmatrix}$$

for some $a \in \mathbb{Z}$. This is because (v_{i-1}, v_i) is not on the upper boundary so it must be a Delzant corner. Then by the action of $T^{-a} \in \mathcal{G}'$, which does not change v_i, we can attain the required pair of vectors.

Otherwise, renumber so that v_0 is in the lower half plane and v_1 is in the upper half plane. Then insert the vector $v_0 + v_1$ between them. This new vector will have a second component with a smaller magnitude than that of v_0 or of v_1. Repeat this process until the new vector lies on the x-axis. Since it is a primitive vector it must be $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$. However since it is the sum of two vectors of opposite sides of the x-axis with the one above being counterclockwise about the origin of the one on bottom, it must be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. ☐

Now we can put Lemmas 5.1, 5.2, and 5.3 together to produce a standard form for semitoric fans (see Figure 3). This standard form will be important to us in Section 6 because it can be obtained from any semitoric fan of complexity c by only using transformations which are continuous in the space of semitoric polygons.

Proposition 5.4. *Let $(v_0, \dots, v_{d-1}) \in (\mathbb{Z}^2)^d$ be a semitoric fan of complexity $c \in \mathbb{Z}$.*

1. *By only corner chopping, removing hidden corners, and commuting fake and Delzant corners we can obtain a new fan $(w_0 = w_{\ell+c}, \dots, w_{\ell+c-1}) \in (\mathbb{Z}^2)^{\ell+c}$ with $\ell + c \geqslant d$ such that*
   - $w_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ *and* $w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
   - *each corner (w_i, w_{i+1}) for $i = 0, \dots, \ell - 1$ is Delzant;*
   - *each corner (w_i, w_{i+1}) for $i = \ell, \dots, \ell + c - 1$ is fake; and*
   - $w_\ell = T^c w_0$ *so* $\det(w_{\ell-1}, T^c w_0) = 1$.

2. *The fan $(w_0, \dots, w_{c+\ell-1})$ has associated integers $b_0, \dots, b_{\ell-1}$ such that*

$$\begin{aligned} w_2 &= -w_\ell + b_0 w_1, \\ w_{i+2} &= -w_i + b_i w_{i+1} \text{ for } i = 1, \dots, \ell - 2, \text{ and} \\ w_1 &= -w_{\ell-1} + b_{\ell-1} w_\ell. \end{aligned}$$

   These integers satisfy

$$ST^{b_0} ST^{b_1} \cdots ST^{b_{\ell-1}} =_G S^4.$$

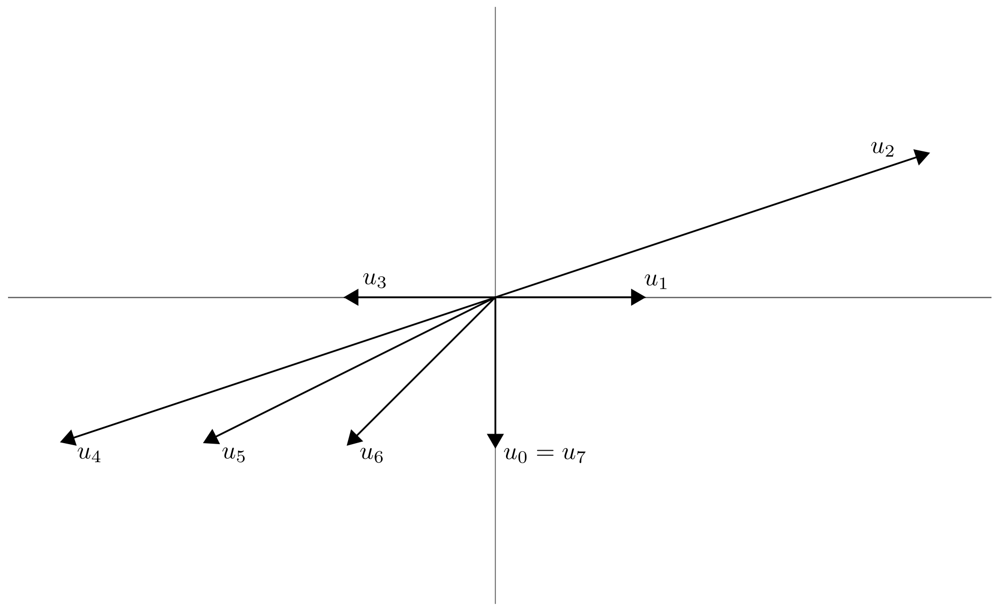

Figure 3: Any semitoric fan with complexity $c \in \mathbb{Z}_{\geqslant 0}$ can be transformed into the standard fan of complexity c. This image has $c = 3$.

3. *The fan* $(w_0, \ldots, w_{c+\ell-1})$ *can be obtained via a finite number of corner chops and reverse corner chops from a fan* $(u_0, \ldots, u_{c+3}) \in (\mathbb{Z}^2)^{c+4}$ *where*

$$u_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} c \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and

$$u_{4+n} = \begin{pmatrix} -c+n \\ -1 \end{pmatrix}$$

for $n = 0, \ldots, c-1$.

Proof. The first part is immediate from Lemmas 5.1, 5.2, and 5.3. By Lemma 5.3 we know after a finite number of cuts and renumbering it can be arranged that the first two vectors in the fan are

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then we invoke Lemma 5.1 to remove all of the hidden corners, and finally use Lemma 5.2 commute all of the fake corners to be adjacent and arrive at the fan $(w_0, \ldots, w_{\ell+c-1}) \in (\mathbb{Z}^2)^{\ell+c}$ in the statement of the proposition. Notice that (w_i, w_{i+1}) being fake for $i = \ell, \ldots, \ell+c-1$ implies that

$$\det(w_\ell, Tw_{\ell+1}) = \cdots = \det(w_{\ell+c-1}, Tw_0) = 0.$$

Now, we know both vectors in a fake corner must have negative second component by definition, so this implies that

$$w_\ell = Tw_{\ell+1} = T^2 w_{\ell+2} = \cdots = T^c w_0.$$

Now $(T^c w_0 = w_\ell, w_1, \ldots, w_{\ell-1}) \in (\mathbb{Z}^2)^\ell$ is a toric fan (we know the vectors are in counter-clockwise order because we started with a semitoric fan) so there must exist $b_0, \ldots, b_{\ell-1} \in \mathbb{Z}$ as in Part 2 of the statement of this theorem. Since

$$ST^{b_0}ST^{b_1}\cdots ST^{b_{\ell-1}} =_G S^4$$

if $\ell > 4$ we can use Lemma 3.8 to conclude two nonconsecutive exponents are in $\{-1,0,1\}$ and one of these exponents is not b_0 or $b_{\ell-1}$. Let $b_i \in \{-1,0,1\}$ for some $0 < i < \ell - 1$. If $b_i = 1$ then we can remove the vector w_{i+1} from the semitoric fan to produce a new semitoric fan via the reverse corner chop operation. In this way we remove a vector from the fan and reduce the number of ST-pairs in the corresponding element of G via $STS =_G T^{-1}ST^{-1}$. If $b_i = -1$ then we use the relation $ST^{-1}S =_G SSTST$ and since

$$SSTST =_G S(STS)T =_G S(T^{-1}ST^{-1})T =_G ST^{-1}S$$

this relation can actually be realized by a corner chop, which we know corresponds to a legal move at the level of fans. If $b_i = 0$ then we actually have a factor of S^2 in the word. Notice that $TSTSTST$ reduces to either TSS or SST via a corner chop depending on where it is cut. In particular, if our word contains the subword $T^{a+1}SST^b$, we can perform a corner chop to obtain $T^{a+1}STSTST^{b+1}$, and then a reverse corner chop to reduce to T^aSST^{b+1}. Note that this can be done even if the T^a was in the ST^{b_0} term. By repeating this operation as necessary, we can move any factors of SS to the front of our word. So we see that in any case we can do an algebraic reduction which will reduce the number of ST-pairs by one and also corresponds to a fan transformation.

Repeat this process until there are only four ST-pairs. Then by Lemma 4.8 we know we must have reduced this equation to

$$ST^kS^2T^{-k}S^2 =_G S^4$$

and thus we use the relation $TSS = SST$ to end up with S^4. This means the corresponding integers are $b_0' = 0, b_1' = 0, b_2' = 0$, and $b_3' = 0$ which produces the desired fan $(u_0, \ldots, u_{c+3})$.

If $\ell < 4$ by Lemma 4.8 we must have $\ell = 3$ and $b_0 = -1, b_1 = -1, b_2 = -1$. In this case we can do a reverse corner chop to again achieve S^4. $\square$

Theorem 2.8 is immediate from Proposition 5.4.

Remark 5.5. Notice that Theorem 4.10 is different from Theorem 2.8 because in Theorem 4.10 the minimal models of the Delzant polygons may be achieved through only corner chops. In Theorem 2.8 we use instead a variety of transformations (all of which are continuous, as we show in Section 6). $\oslash$

6 Application to symplectic geometry

The results of Section 5 have an interpretation in symplectic geometry of toric manifolds and semitoric integrable systems. In Section 6.1 we will review the Classification Theorem of Pelayo-Vũ Ngọc [27], in Section 6.2 we define the metric on the moduli space of semitoric systems given by the second author in [23], and in Section 6.3 we prove the connectivity result for semitoric systems using Theorem 2.8.

6.1 Invariants of semitoric systems

A single polytope is enough to classify toric integrable systems [10]. Semitoric systems are classified in terms of a list of five invariants [27]. Roughly speaking, the complete invariant is a semitoric polygon together with a set of interior points each labeled with extra information (encoding singularities of so called *semitoric type*, which semitoric systems posses, but toric systems do not) modulo an equivalence relation. Even without the extra information semitoric polygons are more complicated than toric polygons because they have fake corners and hidden corners (as defined in Definition 6.4). In this section we will define each of invariants of semitoric systems and state the Pelayo-Vũ Ngọc Classification Theorem. In this section we give the necessary definitions to state the classification theorem and define the metric in [23]. Readers interested in further details may consult [32, 33, 26, 27, 28, 29].

6.1.1 The number of focus-focus points invariant

While a toric integrable system can only have transversally-elliptic and elliptic-elliptic singularities a semitoric integrable system can also have focus-focus singularities (for a definition of these types of singularities see for instance [27]). In [33, Theorem 1] Vũ Ngọc proves that any semitoric system has at most finitely many focus-focus singular points. The first invariant is a nonnegative integer m_f known as the *number of singular points invariant*.

6.1.2 The Taylor series invariant

In [32] Vũ Ngọc proves that the semi-global[5] structure of a focus-focus singular point is completely determined by a Taylor series.

Definition 6.1. Let $\mathbb{R}[[X,Y]]$ denote the algebra of real formal power series in two variables and let $\mathbb{R}[[X,Y]]_0 \subset \mathbb{R}[[X,Y]]$ be the subspace of series $\sum_{i,j\geqslant 0} \sigma_{i,j} X^i Y^j$ which have $\sigma_{0,0} = 0$ and $\sigma_{0,1} \in [0, 2\pi)$.

The *Taylor series invariant* is one element of $\mathbb{R}[[X,Y]]_0$ for each of the m_f focus-focus points.

[5] i.e. in the neighborhood of the fiber over the critical point.

6.1.3 The affine invariant and the twisting index invariant

The affine invariant is the polygon which is directly analogous to the Delzant polygon in the toric case and the twisting index represents how the different focus-focus singular points relate to one another. The twisting index will be represented by an integer label on each focus-focus point, but up to the appropriate group action this sequence of integers will only be defined up to the addition of a common integer. These two invariants are described together because the choice of common integer added to the twisting index effects the polygon.

In general, the momentum map image of a semitoric system does not have to be a polygon and does not even have to be convex, but in [33] the author was able to recover a family of convex polygons which take the place of the Delzant polygon in the semitoric case.

Definition 6.2. A *convex polygonal set*, which we will refer to as a polygon, is the intersection in $\mathbb{R}^2$ of finitely or infinitely many closed half planes such that there are at most finitely many corner points in each compact subset of the intersection. A polygon is *rational* if each edge is directed along a vector with integer coefficients. Let $\mathrm{Polyg}(\mathbb{R}^2)$ denote the set of all rational convex polygons.

Notice that a convex polygonal set may be noncompact. For any $\lambda \in \mathbb{R}$ we will use the notation

$$\ell_\lambda = \{(x, y) \in \mathbb{R}^2 \mid x = \lambda\} \text{ and } \mathrm{Vert}(\mathbb{R}^2) = \{\ell_\lambda \mid \lambda \in \mathbb{R}\}.$$

Definition 6.3. A *labeled weighted polygon of complexity* $m_f \in \mathbb{Z}_{\geqslant 0}$ is defined to be

$$(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) \in \mathrm{Polyg}(\mathbb{R}^2) \times (\mathrm{Vert}(\mathbb{R}^2) \times \{-1, +1\} \times \mathbb{Z})$$

with

$$\min_{s\in\Delta} \pi_1(s) < \lambda_1 < \cdots < \lambda_{m_f} < \max_{s\in\Delta} \pi_1(s)$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is projection onto the first coordinate. We denote the space of labeled weighted polygons of complexity m_f by $\mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$.

This is not yet the affine invariant. The affine invariant will be a subset of a quotient of $\mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$. Recall

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})^6 \text{ so } (T^t)^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

for $k \in \mathbb{Z}$.

Definition 6.4.

1. For $\Delta \in \mathrm{Polyg}(\mathbb{R}^2)$ a point $(x_0, y_0) \in \mathbb{R}^2$ is said to be in the *top boundary of* Δ if $y_0 = \max\{y \in \mathbb{R} \mid (x_0, y) \in \Delta\}$. We denote this by $(x_0, y_0) \in \partial^{\mathrm{top}}\Delta$.

[6] The authors of [26, 27] denote by T the transpose of this matrix. Since in this paper we have discovered the connection to $\mathrm{SL}_2(\mathbb{Z})$ we have choosen to instead use the notation standard to $\mathrm{SL}_2(\mathbb{Z})$-presentations.

2. Let $\Delta \in \text{Polyg}(\mathbb{R}^2)$. A *vertex* of Δ is a point $p \in \partial\Delta$ such that the edges meeting at p are not co-linear. For a vertex $p \in \Delta$ let $u, v \in \mathbb{R}^2$ be primitive inwards pointing normal vectors to the edges of Δ adjacent to p in the order of positive orientation. Then p is

   (a) a *Delzant corner* if $\det(u, v) = 1$;

   (b) a *hidden corner* if $p \in \partial^{\text{top}}\Delta$ and $\det(u, Tv) = 1$; and

   (c) a *fake corner* if $p \in \partial^{\text{top}}\Delta$ and $\det(u, Tv) = 1$.

3. We say $\Delta \in \text{Polyg}(\mathbb{R}^2)$ has *everywhere finite height* if the intersection of Δ with any vertical line is either compact or empty.

Now we are ready to define the polygons which correspond to semitoric fans.

Definition 6.5. An element $(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) \in \mathcal{LW}\text{Polyg}(\mathbb{R}^2)$ is a *primitive semitoric polygon* if

1. Δ has everywhere finite height;
2. $\epsilon_j = +1$ for $j = 1, \ldots m_f$;
3. each ℓ_{λ_j} intersects the top boundary of Δ;
4. any point in $\partial^{\text{top}}\Delta \cap \ell_{\lambda_j}$ for some $j \in \{1, \ldots, m_f\}$ is either a hidden or fake corner; and
5. all other corners are Delzant corners.

In the case that $(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})$ is a primitive semitoric polygon and Δ is compact then there exists an associated semitoric fan.

6.1.4 The action of $G_{m_f} \times \mathcal{G}$

Now we will define a group and the way that its elements act on a labeled weighted polygon. For any $\ell \in \text{Vert}(\mathbb{R}^2)$ and $k \in \mathbb{Z}$ fix an origin in ℓ and let $t_\ell^k : \mathbb{R}^2 \to \mathbb{R}^2$ act as the identity on the half-space to the left of ℓ and as $(T^t)^k$, with respect to the origin in ℓ, on the half-space to the right of ℓ. For $\vec{u} = (u_1, \ldots, u_{m_f}) \in \{-1, 0, 1\}^{m_f}$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{m_f}) \in \mathbb{R}^{m_f}$ let

$$t_{\vec{\lambda}}^{\vec{u}} = t_{\ell_{\lambda_1}}^{u_1} \circ \cdots \circ t_{\ell_{\lambda_{m_f}}}^{u_{m_f}}.$$

Definition 6.6. For any nonnegative $m_f \in \mathbb{Z}$ let $G_{m_f} = \{-1, 1\}^{m_f}$ and let $\mathcal{G} = \{(T^t)^k \mid k \in \mathbb{Z}\}$. We define the action of $((\epsilon'_j)_{j=1}^{m_f}, (T^t)^k) \in G_{m_f} \times \mathcal{G}$ on an element of $\mathcal{LW}\text{Polyg}(\mathbb{R}^2)$ by

$$((\epsilon'_j)_{j=1}^{m_f}, (T^t)^k) \cdot (\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f}) = (t_{\vec{\lambda}}^{\vec{u}}((T^t)^k\Delta), (\ell_{\lambda_j}, \epsilon'_j\epsilon_j, k + k_j)_{j=1}^{m_f})$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{m_f})$ and $\vec{u} = (\frac{\epsilon_j - \epsilon_j \epsilon_j'}{2})_{j=1}^{m_f}$.

Definition 6.7. A *semitoric polygon* is the orbit under $G_{m_f} \times \mathcal{G}$ of a primitive semitoric polygon. That is, given a primitive semitoric polygon $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}) \in \mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$ the associated semitoric polygon is the subset of $\mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$ given by

$$[\Delta_w] = \{(t_{\vec{\lambda}}^{\vec{u}}((T^t)^k(\Delta)), (\ell_{\lambda_j}, 1 - 2u_j, k_j + k)_{j=1}^{m_f}) \mid \vec{u} \in \{0,1\}^{m_f}, k \in \mathbb{Z}\}.$$

The collection of semitoric polygons is denoted $\mathcal{D}\mathrm{Polyg}(\mathbb{R}^2)$.

In general the action of $G_{m_f} \times \mathcal{G}$ may not preserve the convexity of the polygons but it is shown in [27, Lemma 4.2] that $[\Delta_w] \subset \mathcal{LW}\mathrm{Polyg}(\mathbb{R}^2)$ for any primitive semitoric polygon Δ_w.

Remark 6.8. The affine invariant is a family of polygons. These polygons are determined by choosing a single primitive semitoric polygon. $\oslash$

6.1.5 The volume invariant

The next invariant is also not independent of the polygon. Suppose that Δ is a primitive semitoric polygon. For each $j = 1, \ldots, m_f$ we define a real number $h_j \in (0, \mathrm{length}(\pi_2(\Delta \cap \ell_{\lambda_j})))$. This is called the volume invariant because it is the Liouville volume of a submanifold that has to be removed from M during the procedure which produces these polygons (the details are in [26, 27]).

The complete invariant can be represented by a primitive semitoric polygon Δ with m_f vertical lines $\ell_{\lambda_1}, \ldots, \ell_{\lambda_{m_f}}$ and for each $j = 1, \ldots, m_f$ there is a distinguished point on ℓ_{λ_j} a distance h_j from the bottom of Δ. Each of these points is also labeled with an integer k_j and a Taylor series $(S_j)^\infty$. This is shown in Figure 4. It is important to note that in this rough description we have omitted the action of group $G_{m_f} \times \mathcal{G}$ which is key to understanding the classification.

6.1.6 The classification theorem

Definition 6.9. We define a *semitoric list of ingredients* to be

1. a nonnegative integer m_f;
2. a labeled Delzant semitoric polygon $[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f})]$ of complexity m_f;
3. a collection of m_f real numbers $h_1, \ldots, h_{m_f} \in \mathbb{R}$ such that $0 < h_j < \mathrm{length}(\pi_2(\Delta \cap \ell_{\lambda_j}))$ for each $j = 1, \ldots, m_f$; and
4. a collection of m_f Taylor series $(S_1)^\infty, \ldots, (S_{m_f})^\infty \in \mathbb{R}[[X, Y]]_0$.

Let $\mathcal{M}$ denote the collection of all semitoric lists of ingredients and let $\mathcal{M}_{m_f}$ be lists of ingredients with Ingredient (1) equal to the nonnegative integer m_f.

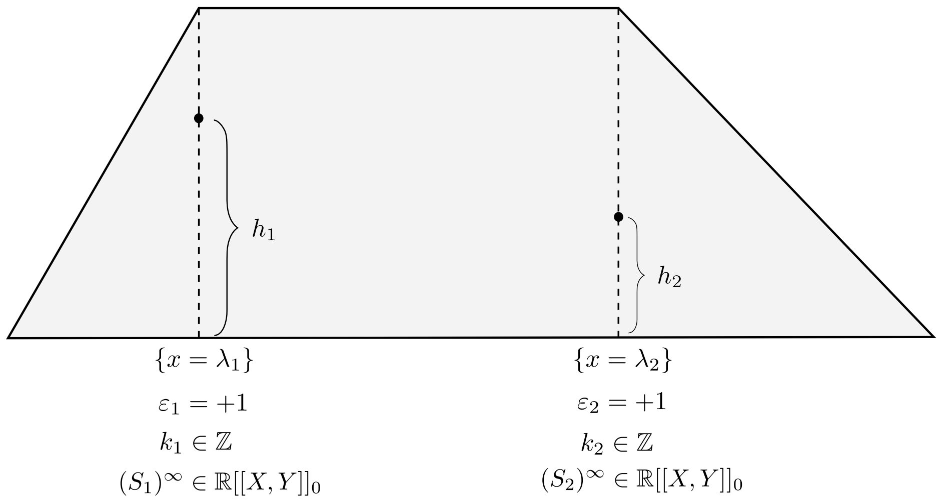

Figure 4: The complete invariant of semitoric systems is the orbit under $G_{m_f} \times \mathcal{G}$ of this object.

A *semitoric integrable system* is a 4-dimensional connected symplectic manifold (M, ω) with an integrable Hamiltonian system $F = (J, H) : M \to \mathbb{R}^2$ such that J is a proper momentum map for a Hamiltonian S^1-action on M and F has only non-degenerate singularities which have no real-hyperbolic blocks. Such a system is said to be a *simple semitoric integrable system* is one for which there is at most one focus-focus critical point in $J^{-1}(x)$ for any $x \in \mathbb{R}$.

An *isomorphism of semitoric systems* is a symplectomorphism $\phi : M_1 \to M_2$, where $(M_1, \omega_1, F_1 = (J_1, H_1))$ and $(M_2, \omega_2, F_2 = (J_2, H_2))$ are semitoric systems, such that $\phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function such that $\frac{\partial f}{\partial H_1}$ is everywhere nonzero. We denote the moduli space of simple semitoric systems modulo semitoric isomorphism by $\mathcal{T}$.

Semitoric systems and the associated isomorphisms are defined in [26]. We use the following result to study the moduli space of semitoric systems by instead studying the set of semitoric lists of ingredients.

Theorem 6.10 ([27])**.** *There exists a bijection between the set of simple semitoric integrable systems modulo semitoric isomorphism and $\mathcal{M}$, the set of semitoric lists of ingredients. In particular,*

$$\mathcal{T} \cong \mathcal{M}$$
$$[(M, \omega, (H, J))] \leftrightarrow \big([\Delta_w], (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f}\big),$$

where the invariants m_f, $[\Delta_w]$, h_j, and $(S_j)^\infty$ are as defined above.

6.2 Metric and topology on the moduli space

In [23] the second author defines a metric space structure for $\mathcal{T}$ using Theorem 6.10. This metric is constructed by combining metrics on each ingredient. We reproduce this construction briefly in this section.

Remark 6.11. It is important to notice that the metric we are using on $\mathcal{T}$ in this article is not the same as the metric defined in [23]. Since these two metrics induce the same topology on $\mathcal{T}$ [23, Section 2.6] it is suggested in [23, Remark 1.31(3)] that the metric in the present article be used when studying the topological properties of $\mathcal{T}$, such as connectedness.

The metric defined in [23] gives $\mathcal{T}$ the appropriate metric space structure, which can be seen when the completion is computed in that article. That metric is defined as the minimum of a collection of functions, one of which is the metric used in this article, so it is immediate that the distance between two systems using Definition 6.18 will never be smaller than the distance between those two systems using the metric studied in [23]. $\oslash$

6.2.1 The metric on Taylor series

Recall that the Taylor series in $\mathbb{R}[[X,Y]]_0$ have the coefficient to X^0Y^1 in the interval $[0,2\pi)$. From the construction in [32] we can see that when defining the topology the endpoints of this interval should be associated. We define a topology on $\mathbb{R}[[X,Y]]_0$ such that a sequence of Taylor series converges if and only if each term converges in the appropriate space. This is the topology induced by the following metric.

Definition 6.12. A sequence $\{b_n\}_{n=0}^{\infty}$ with $b_n \in (0,\infty)$ for each $n \in \mathbb{Z}_{\geqslant 0}$ is called *linear summable* if $\sum_{n=0}^{\infty} n b_n < \infty$. Let $\{b_n\}_{n=0}^{\infty}$ be such a sequence and define $d_{\mathbb{R}[[X,Y]]_0} : (\mathbb{R}[[X,Y]]_0)^2 \to \mathbb{R}$ by

$$\begin{aligned} d_{\mathbb{R}[[X,Y]]_0}((S)^\infty,(S')^\infty) = & \sum_{i,j\geqslant 0,(i,j)\neq(0,1)}^{\infty} \min\{\left|\sigma_{i,j}-\sigma'_{i,j}\right|, b_{i+j}\} \\ & + \min\{\left|\sigma_{0,1}-\sigma'_{0,1}\right|, 2\pi - \left|\sigma_{0,1}-\sigma'_{0,1}\right|, b_1\} \end{aligned}$$

where $(S)^\infty, (S')^\infty \in \mathbb{R}[[X,Y]]_0$ with

$$(S)^\infty = \sum_{i,j\geqslant 0} \sigma_{i,j} X^i Y^j \text{ and } (S')^\infty = \sum_{i,j\geqslant 0} \sigma'_{i,j} X^i Y^j.$$

In order for a series to converge with respect to this metric each term except for the coefficient to X^0Y^1 must converge in $\mathbb{R}$, and the coefficient to X^0Y^1 must converge in $\mathbb{R}/2\pi\mathbb{Z}$.

6.2.2 The metric on the affine invariant

In [24] the authors use the Lebesgue measure of the symmetric difference of the moment polytopes to define a metric on the space of toric systems. For the portion of the semitoric metric related to the affine invariant something similar is done in [23].

Definition 6.13. Let ν be a measure on $\mathbb{R}^2$. We say that it is *admissible* if:

1. it is in the same measure class as the Lebesgue measure on $\mathbb{R}^2$ (that is, $\mu \ll \nu$ and $\nu \ll \mu$ where μ is the Lebesgue measure);
2. the Radon-Nikodym derivative of ν with respect to Lebesgue measure depends only on the x-coordinate, that is, there exists a $g : \mathbb{R} \to \mathbb{R}$ such that ${}^{\mathrm{d}\nu}\!/\!{}_{\mathrm{d}\mu}(x, y) = g(x)$ for all $(x, y) \in \mathbb{R}^2$;
3. the function g from Part (2) satisfies $xg \in \mathrm{L}^1(\mu, \mathbb{R})$ and g is bounded and bounded away from zero on any compact interval.

Since the polygons may be noncompact an admissible measure on $\mathbb{R}^2$ is used in place of the Lebesgue measure. Notice that if ν is an admissible measure and Δ is a convex noncompact polygon with everywhere finite height then $\nu(\Delta) < \infty$. This is essentially because the convex polygon can grow at most linearly and the admissible measure is designed to shrink faster than this by Part (3). Since there is a family of polygons they must all be compared. Let $*$ denote the symmetric difference of two sets. That is, for $A, B \subset \mathbb{R}^2$ we have

$$A * B = (A \setminus B) \cup (B \setminus A).$$

Definition 6.14. Let $\mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2) \subset \mathcal{D}\mathrm{Polyg}(\mathbb{R}^2)$ denote orbits under $G_{m_f} \times \mathcal{G}$ of primitive semitoric polygons with twisting index given by $\vec{k} \in \mathbb{Z}^{m_f}$. Then

$$\mathcal{D}\mathrm{Polyg}(\mathbb{R}^2) = \bigcup_{\substack{m_f \in \mathbb{Z}_{\geqslant 0} \\ \vec{k} \in \mathbb{Z}^{m_f}}} \mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2).$$

Remark 6.15. Let m_f be a nonnegative integer and $\vec{k} \in \mathbb{Z}^{m_f}$. Suppose that a primitive semitoric polygon $[\Delta_w]$ has twisting index $\vec{k'} \in \mathbb{Z}^{m_f}$ where there exists some $c \in \mathbb{Z}$ such that $k_j = k'_j + c$ for $j = 1, \ldots, m_f$. Then $[\Delta_w] \in \mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$ because the set $[\Delta_w]$ is also the orbit of a primitive semitoric polygon with twisting index $\vec{k}$. ⊘

Definition 6.16. Let $m_f \in \mathbb{Z}_{\geqslant 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$. Let $\Delta_w = (\Delta, (\ell_{\lambda_j}, +1, k_j)_{j=1}^{m_f}), \Delta'_w = (\Delta', (\ell_{\lambda'_j}, +1, k'_j)_{j=1}^{m_f}) \in \mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$ be primitive semitoric polygons so $[\Delta_w]$ and $[\Delta'_w]$ are semitoric polygons. Then, if $m_f > 0$ we define the distance between them to be

$$d^{\nu}_{\mathcal{P}}([\Delta_w], [\Delta'_w]) = \sum_{\vec{u} \in \{0,1\}^{m_f}} \nu(t^{\vec{u}}_{\vec{\lambda}}(\Delta) * t^{\vec{u}}_{\vec{\lambda'}}(\Delta')).$$

If $m_f = 0$ then this sum will be empty so we instead use

$$d^{\nu}_{\mathcal{P}}([\Delta_w], [\Delta'_w]) = \nu(\Delta * \Delta').$$

To find the distance between two families of polygons we take the sum of the symmetric differences of all of them and in the case that $m_f = 0$ there is a unique polygon.

6.2.3 The metric on the moduli space of semitoric systems

The total metric will be formed by combining the metrics from Section 6.2.1 and Section 6.2.2.

Definition 6.17. Let $\mathcal{M}_{m_f,\vec{k}} \subset \mathcal{M}_{m_f}$ denote those lists of ingredients with the polygon invariant in the set $\mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$.

The following is from [23].

Definition 6.18. Let ν be an admissible measure on $\mathbb{R}^2$ and let $\{b_n\}_{n=0}^{\infty}$ be a linear summable sequence. Let m_f, m_f' be nonnegative integers and $\vec{k} \in \mathbb{Z}^{m_f}$.

1. We define *the metric on* $\mathcal{M}_{m_f,\vec{k}}$ to be given by

$$d_{m_f,\vec{k}}^{\nu,\{b_n\}_{n=0}^{\infty}}(m,m') = d_{\mathcal{P}}^{\nu}([\Delta_w],[\Delta_w'])+\sum_{j=1}^{m_f}\left(d_{\mathbb{R}[[X,Y]]_0}((S_j)^{\infty},(S_j')^{\infty})+\left|h_j-h_j'\right|\right)$$

   where $m, m' \in \mathcal{M}_{m_f,\vec{k}}$ are given by

$$m = ([\Delta_w],(h_j)_{j=1}^{m_f},((S_j)^{\infty})_{j=1}^{m_f}), m' = ([\Delta_w'],(h_j')_{j=1}^{m_f},((S_j')^{\infty})_{j=1}^{m_f}).$$

2. We define *the metric on* $\mathcal{M}$ to be given by

$$d^{\nu,\{b_n\}_{n=0}^{\infty}}(m,m') = \begin{cases} d_{m_f,\vec{k}}^{\nu,\{b_n\}_{n=0}^{\infty}}(m,m') & \text{, if } m,m' \in \mathcal{M}_{m_f,\vec{k}} \text{ for} \\ & \text{some } m_f \in \mathbb{Z}_{\geqslant 0}, \vec{k} \in \mathbb{Z}^{m_f} \\ 1 & \text{, otherwise.} \end{cases}$$

   where $m, m' \in \mathcal{M}$.

3. Let $\Phi : \mathcal{T} \to \mathcal{M}$ be the correspondence from Theorem 6.10. Then we define the *metric on* $\mathcal{T}$ by $\mathcal{D}^{\nu,\{b_n\}_{n=0}^{\infty}} = \Phi^* d^{\nu,\{b_n\}_{n=0}^{\infty}}$.

The metric on $\mathcal{T}$ depends on the choice of admissible measure and linear summable sequence, but the topology it induces does not.

Proposition 6.19 ([23, Theorem A])**.** *Let ν be an admissible measure on $\mathbb{R}^2$ and $\{b_n\}_{n=0}^{\infty}$ a linear summable sequence. Then the space $(\mathcal{T}, \mathcal{D}^{\nu,\{b_n\}_{n=0}^{\infty}})$ is a metric space and the topology induced by $\mathcal{D}^{\nu,\{b_n\}_{n=0}^{\infty}}$ on $\mathcal{T}$ does not depend on the choice of ν or the choice of $\{b_n\}_{n=0}^{\infty}$.*

Let $\mathcal{T}_{m_f,\vec{k}} = \Phi^{-1}(\mathcal{M}_{m_f,\vec{k}})$. From Definition 6.18 it can be seen that each $\mathcal{T}_{m_f,\vec{k}}$ is a separate component of $\mathcal{T}$, so it is natural to wonder if $\mathcal{T}_{m_f,\vec{k}}$ is connected for each $m_f \in \mathbb{Z}_{\geqslant 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$. From Theorem 2.11 we know that these are in fact path-connected.

6.3 The connectivity of the moduli space of semitoric integrable systems

Lemma 6.20. *Each relation in Theorem 2.8 corresponds to some continuous transformation of the polygons. More specifically, suppose that two fans $\mathcal{F}_0, \mathcal{F}_1 \in (\mathbb{Z}^2)^d$ are related by*

1. *performing corner chops;*
2. *performing reverse corner chops;*
3. *removing hidden corners; or*
4. *commuting fake and Delzant corners;*

(see Definition 2.7). Then there exists a continuous family of (compact) primitive semitoric polygons Δ_t, $t \in [0,1]$, such that the fan related to Δ_0 is $\mathcal{F}_0$ and the fan related to Δ_1 is $\mathcal{F}_1$.

Proof. Suppose that Δ is a polygon with associated fan $\mathcal{F} = (v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d$, let $i \in \{0, 1, \ldots, d-1\}$, and let $(x_0, y_0) \in \mathbb{R}^2$ be the vertex between the edge normal to v_i and the edge normal to v_{i+1}. Let $\varepsilon > 0$ be smaller than the length of either of these edges. Now let $H_t(v)$ denote the half-space given by

$$H_t(v) = \{(x,y) \in \mathbb{R} \mid ((x,y) - (x_0,y_0) - \varepsilon t v) \cdot v \geqslant 0\}$$

where $\cdot$ denotes the usual vector dot product

First we consider the corner chop operation. Suppose (v_i, v_{i+1}) is a Delzant corner. For $t \in [0,1]$ let

$$\Delta_t = \Delta \cap H_t(v_i + v_{i+1}).$$

Since $H_t(v_i + v_{i+1})$ is deforming continuously we see that Δ_t is a continuous family and since the edges of Δ_t are parallel to the edges of Δ except for the new edge with inwards pointing normal vector given by $v_i + v_{i+1}$ we see that the fan of Δ_t is the corner chop of the fan for Δ if $t \in (0,1]$. Since a reverse corner chop is the inverse of this operation, we can use the same path backwards.

Now suppose that (v_i, v_{i+1}) is a hidden corner. For $t \in [0,1]$ let

$$\Delta_t = \Delta \cap H_t(T v_{i+1}).$$

The continuity of $H_t(T v_{i+1})$ implies that Δ_t is a continuous family and by construction it has the desired fan. Thus Δ_t is the required family for the operation of removing hidden corners.

Finally, suppose that (v_i, v_{i+1}) is a fake corner and (v_{i+1}, v_{i+2}) is a Delzant corner. Here we can see that this fan is the result of removing the hidden corner (v_i, v_{i+2}) from the fan

$$\mathcal{F}' = (v_0, \ldots, v_i, v_{i+2}, \ldots, v_{d-1}) \in (\mathbb{Z})^{d-1}.$$

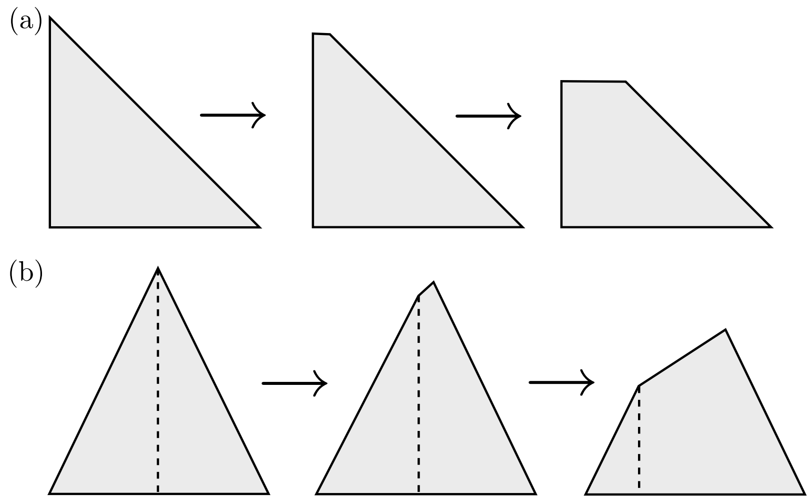

Figure 5: For each fan transformation there is a continuous path of semitoric polygons which transitions between the fans. In (a) we show the corner chop and in (b) we show the removal of a hidden corner (which is replaced by a fake and a Delzant corner).

We know (v_i, v_{i+2}) is a hidden corner because it is given that (v_i, v_{i+1}) is fake, which means $v_i = Tv_{i+1}$. Then we compute

$$\det(v_i, Tv_{i+2}) = \det(Tv_{i+1}, Tv_{i+2}) = 1$$

because (v_{i+1}, v_{i+2}) is Delzant. Thus there is a continuous path from any polygon with fan $\mathcal{F}$ to any polygon with fan $\mathcal{F}'$. Let Δ' have fan $\mathcal{F}'$. But now, making ε smaller if needed, we can consider

$$\Delta_t = \Delta' \cap \{(x,y) \in \mathbb{R} \mid ((x,y) - (x_1,y_1) - \varepsilon t T v_{i+2}) \cdot (Tv_{i+2}) \geqslant 0\}$$

where (x_1, y_1) is the vertex between the edges corresponding to v_i and v_{i+2} and $t \in [0,1]$. This is a continuous path to a polygon with the required fan. This completes the proof.

$\square$

Recall that toric polygons are precisely the compact primitive semitoric polygons with complexity zero. Thus, by Proposition 5.4 and Lemma 6.20 we have recovered Theorem 2.3.

In light of Lemma 5.4 and Proposition 6.20 the only difficulty remaining to prove the following lemma is incorporating the case of semitoric systems which have noncompact polygons as invariants.

Lemma 6.21. *Let $m_f \in \mathbb{Z}_{\geqslant 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$. Then $\mathcal{D}Polyg_{m_f,\vec{k}}(\mathbb{R}^2)$ is path-connected.*

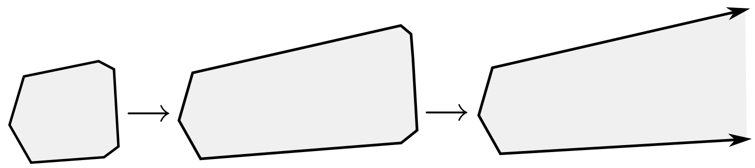

Figure 6: The continuous path from a compact primitive semitoric polygon to a noncompact primitive semitoric polygon with finitely many vertices.

Proof. We must only consider the primitive semitoric polygons, because if the primitive polygons converge so do all of the polygons in the family. Any two compact primitive semitoric polygons with the same fan can be connected by a continuous path. This path is made by continuously changing the lengths of the edges because the angles of the two polygons must all be the same since they have the same fan. So, by Proposition 6.20 given two elements of $\mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$ which are compact we know that the corresponding fans are related by the moves listed in that proposition and then by Lemma 5.4 we know these moves correspond to continuous paths of polygons. So we have established that any two compact elements of $\mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$ are connected by a continuous path.

Next assume that

$$[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})] \in \mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$$

is such that Δ is noncompact but has only finitely many vertices. Choose $N \in \mathbb{R}$ such that all of the vertices of Δ are in the region $\{(x,y) \in \mathbb{R} \mid -N < x < N\}$. The set $\Delta \cap [-N-1, N+1]$ is a polygonal subset of $\mathbb{R}^2$ but the corners which intersect $\ell_{N+1} \cup \ell_{-N-1}$ may not be Delzant. By [24, Remark 23] we may change the set on arbitrarily small neighborhoods of these corners to produce a new set, Δ', which is equal to $\Delta \cap [-N-1, N+1]$ outside of those small neighborhoods and has only Delzant corners inside of those neighborhoods. Thus Δ' is a primitive semitoric polygon and by choosing the neighborhoods small enough we can assure that $\Delta \cap [-N, N] = \Delta' \cap [-N, N]$. For $t \in [0, 1)$ let $\Delta(t)$ be the polygon with the same fan as Δ', the property that

$$\Delta(t) \cap [-N, N] = \Delta' \cap [-N, N],$$

and which has all of the same edge lengths as Δ' with the exception of the two or four edges which intersect $\ell_N \cup \ell_{-N}$. These edges are extended horizontally by a length of $\frac{1}{t} - 1$. By this we mean that if an edge of Δ' which intersected $\{(x,y) \in \mathbb{R}^2 \mid x = N\}$ had as one of its endpoints (x_0, y_0) with $x_0 > N$ then the corresponding edge of $\Delta(t)$ would have as its endpoint $(x_0 + \frac{1}{t} - 1, y_0)$. Then define $\Delta(1) = \Delta$ and we can see that $\Delta(t)$ for $t \in [0,1]$ is a path from Δ', which is compact, to Δ so $[(\Delta(t), (\lambda_j, +1, k_j)_{j=1}^{m_f})]$ is a continuous path which connects a compact semitoric polygon to $[\Delta_w]$. This process is shown in Figure 6.

Now we have connected all of the elements except for those with an infinite amount of vertices. Suppose that

$$[\Delta_w] = [(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})] \in \mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$$

is such that Δ is noncompact and has infinitely many vertices. We will connect $[\Delta_w]$ to a polygon which has only finitely many vertices to finish the proof. Since Δ has everywhere finite height and is the intersection of infinitely many half-planes we can choose two of these planes which are not horizontal and are not parallel to one another. Denote the intersection of these two half-planes by A and notice $\Delta \subset A$. Since the boundaries of these two half-planes must intersect we can see that A can only be unbounded in either the positive or negative x-direction, but not both. Without loss of generality assume that A is unbounded in the positive x-direction.

Let ν be any admissible measure. For any $n \in \mathbb{Z}_{\geqslant 0}$ since $\nu(A) < \infty$ we know there exists some $x_n \in \mathbb{R}$ such that

$$\nu(A \cap [x_n, \infty)) < 1/n,$$

Δ does not have a vertex on the line ℓ_{x_n}, and $x_n > |\lambda_j|$ for all $j = 1, \ldots, m_f$. Let Δ_n denote the polygon which satisfies

$$\Delta_n \cap [-\infty, x_n] = \Delta \cap [-\infty, x_n]$$

and has no vertices with x-coordinate greater than x_n.

For each $n \in \mathbb{Z}_{\geqslant 0}$ and $t \in (0, 1]$ define $\Delta_n(t)$ to have the same fan as Δ_{n+1} and to have all the same edge lengths as Δ_{n+1} except for the two edges which intersect ℓ_{x_n}. Extend those two edges horizontally by $1/t - 1$. Define $\Delta_n(0) = \Delta_n$. Now $\Delta_n(t)$ for $t \in [0, 1]$ is a C^0 path which takes Δ_n to Δ_{n+1}. Moreover,

$$\Delta * \Delta_n(t) \subset A \cap [x_n, \infty] \text{ so } \nu(\Delta * \Delta_n(t)) < 1/n$$

for each $t \in [0, 1]$. Each of these paths for $n \in \mathbb{Z}_{\geqslant 0}$ can be concatenated to form a continuous path Δ_t, $t \in [0, 1]$, from Δ_0 to Δ and we know that Δ_0 has only finitely many vertices. It is important not only that each Δ_n be getting closer to Δ but also that the path from Δ_n to Δ_{n+1} stays close to Δ. Then we define $[(\Delta(t), (\lambda_j, +1, k_j)_{j=1}^{m_f})]$ which is a continuous path from a semitoric polygon with finitely many vertices to $[\Delta_w]$. This is shown in Figure 7.

□

Remark 6.22. According to [33, Theorem 3] all of the polygons in $\mathcal{D}\mathrm{Polyg}_{m_f,\vec{k}}(\mathbb{R}^2)$ are compact if $m_f > 1$. ⊘

Now we can classify the connected components of $\mathcal{M}$ and $\mathcal{T}$. Recall that $\mathcal{M}_{m_f,\vec{k}} = \mathcal{M}_{m_f,\vec{k}'}$ if $k_j = k'_j + c$ for some $c \in \mathbb{Z}$ so when stating the following lemma we require that the first component of the twisting index be 0. This is done only to make sure that there are no repeats in the list of components. Recall $\mathcal{M}_0$ is the collection of semitoric lists of ingredients with $m_f = 0$ and $\mathcal{T}_0 = \Phi^{-1}(\mathcal{M}_0)$ is the collection of semitoric systems with no focus-focus singularities.

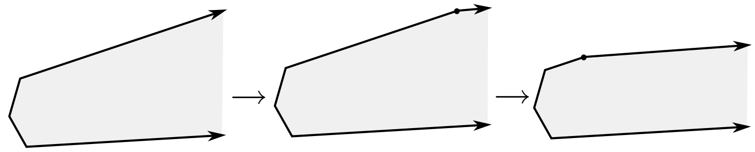

Figure 7: The continuous path which adds new vertices to a noncompact primitive semitoric polygon. This process is repeated to form a path to systems with infinitely many vertices.

Lemma 6.23. *The connected components of $\mathcal{M}$ are*

$$\{\mathcal{M}_{m_f,\vec{k}} \mid m_f \in \mathbb{Z}_{>0}, \vec{k} \in \mathbb{Z}^{m_f} \text{ with } k_1 = 0\} \cup \mathcal{M}_0$$

and they are each path-connected.

Proof. It is sufficient to prove that $\mathcal{M}_{m_f,\vec{k}}$ is path-connected for each choice of $m_f \in \mathbb{Z}_{\geqslant 0}$ and $\vec{k} \in \mathbb{Z}^{m_f}$. Let $m, m' \in \mathcal{M}_{m_f,\vec{k}}$ with

$$m = ([\Delta_w], (h_j)_{j=1}^{m_f}, ((S_j)^\infty)_{j=1}^{m_f}) \text{ and } m' = ([\Delta'_w], (h'_j)_{j=1}^{m_f}, ((S'_j)^\infty)_{j=1}^{m_f}).$$

By Lemma 6.21 we know there exists a continuous path

$$[\Delta_w(t)] = [(\Delta(t), (\ell_{\lambda_j(t)}, +1, k_j)_{j=1}^{m_f})],$$

$t \in [0,1]$, from $[\Delta_w]$ to $[\Delta'_w]$ and by [23, Proposition 1.20] we know $\mathbb{R}[[X,Y]]_0$ is path-connected so there exists a continuous path $(S_j(t))^\infty$ from $(S_j)^\infty$ to $(S'_j)^\infty$ for each $j = 1, \dots, m_f$. For $j = 1, \dots, m_f$ let

$$\begin{aligned}
\mathrm{len}_j &= \mathrm{length}(\pi_2(\Delta \cap \ell_{\lambda_j})) \\
\mathrm{len}'_j &= \mathrm{length}(\pi_2(\Delta' \cap \ell_{\lambda'_j})) \\
\mathrm{len}_j(t) &= \mathrm{length}(\pi_2(\Delta(t) \cap \ell_{\lambda_j(t)}))
\end{aligned}$$

for $t \in [0,1]$ and define

$$h_j(t) = \left(\frac{(1-t)h_j}{\mathrm{len}_j} + \frac{th_j}{\mathrm{len}'_j} \right) \mathrm{len}_j(t).$$

Now we have that $0 < h_j(t) < \mathrm{len}_j(t)$ and $t \mapsto h_j(t)$ is a continuous function from $[0,1]$ to $\mathbb{R}$ because it is impossible for a semitoric polygon to have a vertical boundary at ℓ_{λ_j} for any $j \in \{1, \dots, m_f\}$. Now define

$$m(t) = ([\Delta_w(t)], (h_j(t))_{j=1}^{m_f}, ((S_j(t))^\infty)_{j=1}^{m_f})$$

for $t \in [0,1]$ which is a continuous path from m to m'. ☐

Thus we have established the following result.

Theorem 6.24. *The set of connected components of $\mathcal{T}$ is*

$$\{\mathcal{T}_{m_f,\vec{k}} \mid m_f \in \mathbb{Z}_{>0}, \vec{k} \in \mathbb{Z}^{m_f} \text{ with } k_1 = 0\} \cup \mathcal{T}_0$$

and each $\mathcal{T}_{m_f,\vec{k}}$ is path-connected.

Theorem 2.11 follows immediately.

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