

AN EXAMPLE CONCERNING SET ADDITION IN \mathbb{F}_2^n

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ABSTRACT. We construct sets A, B in a vector space over \mathbb{F}_2 with the property that A is “statistically” almost closed under addition by B in the sense that $a + b$ almost always lies in A when $a \in A, b \in B$, but which is extremely far from being “combinatorially” almost closed under addition by B : if $A' \subset A, B' \subset B$ and $A' + B'$ is comparable in size to A' then $|B'| \lesssim |B|^{1/2}$.

1. INTRODUCTION

The aim of this note is to prove the following result, which is a precise version of the claim made in the abstract.

Theorem 1.1. *Let $0 < \delta < 1 < K$. Then there are arbitrarily large finite sets A, B in some vector space over \mathbb{F}_2 with the property that $\mathbb{P}(a + b \in A | a \in A, b \in B) \geq 1 - \delta$, but such that if $A' \subset A$ and $B' \subset B$ satisfy $|A' + B'| \leq K|A'|$ then $|B'| = O_{K,\delta}(|B|^{1/2})$.*

The sets are constructed as follows. For positive integer m we take A to consist of a direct product of m “bands” (union of a few consecutive Hamming layers) in copies of \mathbb{F}_2^m . Then we take B to consist of the standard basis vectors in $(\mathbb{F}_2^m)^m$. The details may be found in Section 3.

Whilst the construction is fairly simple, neither the construction nor its analysis seemed quite obvious to us and so we thought it worth recording in this short note.

2. A CRUDE ISOPERIMETRIC INEQUALITY FOR BANDS

By a *band* of width k in \mathbb{F}_2^D we mean the union $L_\ell \cup \dots \cup L_{\ell+k-1}$ of k consecutive layers, where we define the layer $L_d := \{x \in \mathbb{F}_2^D : |x| = d\}$, where $|x|$ is the number of nonzero coordinates of x .

Lemma 2.1. *Suppose that $A \subset \mathbb{F}_2^D$ is contained in some band Σ of width k , where $D > 10k^2$. Let $S = \{e_1, \dots, e_D\}$ be the set of standard basis vectors in \mathbb{F}_2^D . Then*

$$|(A + kS) \setminus \Sigma| \geq \frac{1}{2}|A|.$$

Proof. We begin by noting that if $x \in L_i$ then there are $\binom{D-i}{k}$ elements $y \in L_{i+k}$ differing from x by an element of kS . Each such element arises from $\binom{i+k}{k}$ choices of x . It follows by double counting that if $A_i \subset L_i$ is some set then

$$|(A_i + kS) \cap L_{i+k}| \geq \frac{\binom{D-i}{k}}{\binom{i+k}{k}} |A_i|.$$

If $i \leq D/2$ then we have

$$\frac{\binom{D-i}{k}}{\binom{i+k}{k}} \geq \left(\frac{\frac{D}{2} - k}{\frac{D}{2} + k}\right)^k \geq \left(1 - \frac{4k}{D}\right)^k > \frac{1}{2}.$$

Thus if $i \leq D/2$ then

$$|(A_i + kS) \cap L_{i+k}| \geq \frac{1}{2} |A_i|.$$

Suppose that $\Sigma = L_\ell \cup \dots \cup L_{\ell+k-1}$, where $\ell \leq D/2$. Then, applying the preceding inequality with $i = \ell, \dots, \ell + k - 1$ in turn and with $A_i := A \cap L_i$, the claimed result follows by noting that all of the layers L_{i+k} lie outside Σ .

If $\ell > D/2$ then a very similar argument works, but we instead use the inequality

$$|(A_i + kS) \cap L_{i-k}| \geq \frac{\binom{i}{k}}{\binom{D+k-i}{k}} |A_i| \geq \frac{1}{2} |A_i|$$

for $i \geq D/2$. □

3. THE EXAMPLE

Let $0 < \delta < 1 < K$, as in the statement of Theorem 1.1. Let $k = 2k' + 1$ be odd and of size $\sim \frac{4}{\delta}$, and let $m = 2m'$ be even with $m > 10k^2$. Consider the vector space $V = (\mathbb{F}_2^m)^m$, and let $B \subset V$ be the standard basis consisting of elements $(e_{i,j})_{i,j \in [m]}$. Let $A \subset V$ consist of the direct product of m copies of the band $\Sigma := L_{m'-k'} \cup \dots \cup L_{m'+k'}$ in \mathbb{F}_2^m .

It is straightforward to show that A is almost closed under addition by B in the “statistical” sense.

Lemma 3.1. *We have $\mathbb{P}(a + b \in A | a \in A, b \in B) \geq 1 - \frac{2}{k} > 1 - \delta$.*

Proof. By symmetry it suffices to show that $\mathbb{P}(a + e_{1,1} \in A | a \in A) \geq 1 - \frac{2}{k}$. If the projection of a to the first copy of \mathbb{F}_2^m lies in $L_{m'-k'+1} \cup$

$\cdots \cup L_{m'+k'-1}$, then it is evidently the case that $a + e_{1,1} \in A$. The probability of this event is precisely

$$\frac{|L_{m'-k'+1} \cup \cdots \cup L_{m'+k'-1}|}{|L_{m'-k'} \cup \cdots \cup L_{m'+k'}|} \geq \frac{2k' - 1}{2k' + 1} = 1 - \frac{2}{k'},$$

since the sizes of the layers decrease away from the middle layer. \square

Shortly we will turn to the somewhat less immediate task of showing that $|A' + B'|$ is much bigger than $|A'|$ unless $|B'|$ is of size $\ll m = |B|^{1/2}$. We note that there are two quite distinct examples of sets B' of size m for which there is some $A' \subset A$ almost closed under addition by B' . Indeed

- If $B' = \{e_{i,1}\}_{i \in [m]}$ then $A + B'$ consists of the product of the band of width $(k + 2)$, $L_{m'-k'-1} \cup \cdots \cup L_{m'+k'+1}$, with $(m - 1)$ copies of the usual band of width k . By a computation very similar to that in the proof of Lemma 3.1, we have $|A + B'| \leq (1 + O(\frac{1}{k}))|A|$.
- If $B' = \{e_{1,j}\}_{j \in [m]}$ then we may take $A' \subset A$ to be the product of some $e_{1,1}$ -invariant set in the first factor with an $e_{1,2}$ -invariant set in the second factor and so on to obtain $A' + B' = A'$. Note that any band in \mathbb{F}_2^m of width ≥ 2 does contain a set invariant under a given basis vector $e_{i,j}$.

These examples, in addition to showing that we cannot improve upon our bound of $|B'| \lesssim |B|^{1/2}$, should also convince the reader that there is nothing to be gained by taking our set A to live inside $(\mathbb{F}_2^m)^r$ with differing values of m, r : if $m > r$ then examples of the first type are worse, whilst if $m < r$ then examples of the second type are bad.

Suppose that

$$|A' + B'| \leq K|A'| \tag{3.1}$$

for some $A' \subset A$ and $B' \subset B$. We begin by applying a variant of the Plünnecke-Ruzsa inequality due to Ruzsa [1, Chapter 1, Proposition 7.3]. A special case of this theorem tells us that if $|A' + B'| \leq K|A'|$ then for any positive integer k there is $A'' \subset A'$,

$$|A''| \geq \frac{1}{2}|A'|, \tag{3.2}$$

such that

$$|A'' + kB'| \leq 2^{k+1}K^k|A''|. \tag{3.3}$$

Now since $B' \subset B$ we have

$$B' = \bigcup_{j=1}^m B'_j,$$

where

$$B'_j := \bigcup_{i \in I_j} \{e_{i,j}\}$$

for some set $I_j \subset [m]$. Say that j is *good* if $|I_j| > 10k^2$, and write $J \subset [m]$ for the set of good J . We claim that if j is good then

$$|(A'' + kB'_j) \setminus A| \geq \frac{1}{2}|A''|. \quad (3.4)$$

The sets $(A'' + kB'_j) \setminus A$ are disjoint for different j , as all the elements of such a set, considered as a subset of $(\mathbb{F}_2^m)^m$, have their j th coordinate, and no other, lying outside the band Σ . It follows from this and (3.3) that

$$J \leq 2^{k+2}K^k,$$

and therefore we have

$$|B'| = \sum_{j=1}^m |I_j| \leq 10k^2m + Jm \ll_{K,\delta} m.$$

It remains to establish the claim (3.4). To do this, identify B'_j with the standard basis vectors in $\mathbb{F}_2^{I_j}$, and write V as a direct product $\mathbb{F}_2^{I_j} \times V'$. Then A, A'' may be fibred as unions

$$A = \bigcup_{v \in V'} (A(v) \times \{v\})$$

$$A'' = \bigcup_{v \in V'} (A''(v) \times \{v\})$$

where $A''(v) \subset A(v) \subset \mathbb{F}_2^{I_j}$ and, crucially, each $A(v)$ is a band of width k in $\mathbb{F}_2^{I_j}$ (but not necessarily the middle one). By Lemma 2.1 (with $D = |I_j| > 10k^2$) it follows that

$$|(A''(v) + kB'_j) \setminus A(v)| \geq \frac{1}{2}|A''(v)|,$$

from which the claim follows by summing over v .

4. ACKNOWLEDGEMENTS

The first author thanks Shachar Lovett for asking a question which led to this note. Lovett (personal communication) subsequently informed us that Hosseini, Impagliazzo and he observed that easier examples are available in other ambient spaces. For example, $A = \{0, 1\}^n$, $B = \{e_1, \dots, e_n\}$ (viewed as subsets of \mathbf{Z}^n , or of \mathbb{F}_p^n for any $p \geq 3$) have similar properties. In fact for this example we have $\mathbb{P}(a + b \in A | a \in$

$A, b \in B) = \frac{1}{2}$ and $|A' + B'| \leq K|A'|$ implies that $|B'| \ll_K 1$. A somewhat similar example is mentioned in [2, Exercise 2.6.2].

The authors thank MSRI and the Simons Institute for providing excellent working conditions while this work was carried out. The first author is a Simons Investigator. The second author is supported by NSF CAREER award number 1553288.

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