Weak Mixing of a Transformation Similar to Pascal

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Abstract

We construct a class of transformations similar to the Pascal transformation, except for the use of spacers, and show that these transformations are weakly mixing.

1 Introduction

The Pascal transformations arise as natural examples of adic transformations. Adic transformations were studied by Vershik as models for measure-preserving transformations [2], [3], [4]. Vershik conjectured that the Pascal transformations are weakly mixing, and while they are known to be totally ergodic this conjecture remains open [1], [5]. In this paper we define a class of transformations that are towers over the Pascal transformations and show that they are weakly mixing (Theorem 1).

For each $0 < \alpha < 1$, define the Pascal transformation $S = S_\alpha$[1] in the following way as a cutting and stacking transformation (see for example [5]):

We proceed by inductively defining a sequence of columns. Start by letting column $C_{0,0} = (B^{(0)}_{0,0})$ consist of one interval, called a *level*, of total mass 1 (we normalize the measure at the end). In the n^{th} generation of columns we will have $n+1$ columns, where column m is denoted by $C_{n,m} = (B^{(0)}_{n,m}, \dots, B^{(h_{n,m}-1)}_{n,m})$ for $0 \le m \le n$. The columns in generation $(n+1)$ are obtained from those in generation n in the following way. First cut level $B^{(i)}_{n,m}$ into sublevels $B^{(i)}_{n,m,0}, B^{(i)}_{n,m,1}$ with mass ratio $\alpha : 1-\alpha$. Then define

$$C_{n+1,m} = (B^{(0)}_{n,m,0}, \dots, B^{(h_{n,m}-1)}_{n,m,0}, B^{(0)}_{n,m-1,1}, \dots, B^{(h_{n,m-1}-1)}_{n,m-1,1}),$$

where levels $B^{(j)}_{n,m,i}$ with indices $m < 0$ or $m > n$ are ignored. We define the action of S on $B^{(i)}_{n,m}$ by sending it (using the standard translation of an interval to another of the same length) to $B^{(i+1)}_{n,m}$ when $i \neq h_{n,m} - 1$. In the limit this defines a finite measure-preserving transformation S, known as the

[1] while S depends on α we do not write this explicitly

Pascal transformation. While S is totally ergodic [1], it is an open problem as to whether S is weakly mixing.

Now we define a new transformation T_k, the *Pascal with spacers transformation*, in a similar way, only with additional *spacers* (an additional piece of our measure space of the correct total mass that is not part of any previous generation's column) placed on top of column $C_{n,m}$ if n is a multiple of k. Notice that all of the levels in column $C_{n,m}$ have mass $\alpha^{n-m}(1-\alpha)^m$ so T_k is measure preserving. Notice also that the amount of mass added by spacers on generation n columns is 0 if k does not divide n and $\sum_{i=0}^{n} \alpha^i(1-\alpha)^{n-i} < (n+1)\max(\alpha, 1-\alpha)^n$ otherwise, hence T is defined on a space with finite mass. Therefore, we can instead consider the transformation T_k on the renormalized measure space so that the total mass of the space is 1. We shall prove the following theorem:

Theorem 1. *The transformation T_k is weakly mixing if $k > 1$.*

We also note, though that not all patterns of spacers are weakly mixing. In fact we can show that T_1 is not weak mixing.

Proposition 2. *T_1 is not weakly mixing.*

Proof. In T_1, all of the column heights are congruent to 2 modulo 3 and hence the function that assigns $e^{2\pi i(m+h)/3}$ to any point in the level $B_{n,m}^{(h)}$ is well defined and clearly has eigenvalue $e^{2\pi i/3}$. □

Notice that the above proof also shows that T_1 is not even totally ergodic.

The transformation T_k can be expressed symbolically as follows: the space, X is the subset of $\{0,1\}^\omega \times \mathbb{Z}$ consisting of elements of the form $((0^a1^b0S), n)$ where S is some string of 1's and 0's, $b > 0$, and $0 \leq n \leq \frac{a+b}{k} + 1$. The measure on X is generated by the cylinder sets $[0^a1^b0S, n] = \{((0^a1^b0SS'), n) : S' \in \{0,1\}^\omega\}$, where $b > 0$, S is any finite string and $0 \leq n \leq \frac{a+b}{k} + 1$, by $\mu[0^a1^b0S, n] = \alpha^x(1-\alpha)^y$, where x and y are the number of 0's and 1's respectively in the string 0^a1^b0S. The transformation T_k acts on X by

$$T_k(((0^a1^b0S), n)) = \begin{cases} ((0^a1^b0S), n+1) & \text{if } n \leq \frac{a+b}{k} \\ ((1^{b-1}0^{a+1}1S), 0) & \text{otherwise} \end{cases}.$$

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2 Ergodicity

Here we will prove the following Lemma:

Lemma 3. *T_k is ergodic.*

Proof. Notice that the induced map of T_k on the complement of the spacers in X is S. Since for any point $x \in X$, $T_k^{-n}(x)$ is in this set for some n, the result follows from the well-known ergodicity of S. $\square$

3 Some Machinery involving Column Heights and Copy Heights

We will use the convention that $h_{n,m} = 0$ if $m < 0$ or $m > n$.

Lemma 4. *For $n \geq 0$,*

$$h_{n+1,m} = \begin{cases} h_{n,m} + h_{n,m-1} & \text{if } n+1 \not\equiv 0 \pmod{k} \\ h_{n,m} + h_{n,m-1} + 1 & \text{if } n+1 \equiv 0 \pmod{k} \end{cases}.$$

Proof. This follows immediately from the construction of the columns. $\square$

Definition 1. *If I is a level in some column $C_{r,s}$, and $0 \leq m \leq n$ with $n \geq r$, let $P_{n,m}(I)$ denote the set of copies of I in column $C_{n,m}$. In other words, $P_{n,m}(I)$ is the set of levels, J, in $C_{n,m}$ so that $J \subset I$.*

Definition 2. *If I is a level in column $C_{n,m}$ where $I = B_{n,m}^{(H)}$, let $h(I) = H$.*

Definition 3. *If I is a level in $C_{r,s}$, $n, m \in \mathbb{Z}$, with $n \geq m \geq 0$ and $\lambda \in \mathbb{C}$, let*

$$S_{n,m}(I, \lambda) = \sum_{I' \in P_{n,m}(I)} \mu(I') \lambda^{h(I')}.$$

If $m \notin [0, n]$, let $S_{n,m}(I, \lambda) = 0$.

The idea of the proof will be to assume for sake of contradiction that T_k has an eigenfunction, f, with eigenvalue $\lambda \neq 1$. We will then look at some interval I on which f is nearly constant. We will then consider the values of f on the generation-N copies of I. Since if two copies of I are both in column $C_{N,M}$, their f-values are proportional to λ raised to the power of their heights, we have that the integral of f over $P_{N,M}(I)$ is bounded above by $|S_{N,M}(I, \lambda)|$. We will produce a contradiction by proving that $\lim_{N\to\infty} \sum_M |S_{N,M}(I, \lambda)| = 0$. To do this we will use the following lemmas:

Lemma 5. *If I is a level of a column of generation at most n, then*

$$S_{n+1,m}(I, \lambda) = \alpha S_{n,m}(I, \lambda) + (1 - \alpha)\lambda^{h_{n,m}} S_{n,m-1}(I, \lambda).$$

Proof. Let the heights of the copies of I in $C_{n,m}$ be H_i for $1 \leq i \leq k_1$. Let the heights of the copies of I in $C_{n,m-1}$ be G_i for $1 \leq i \leq k_2$. Then the heights of the copies of I in $C_{n+1,m}$ are H_i and $G_j + h_{n,m}$. Therefore, we have that

$$\begin{aligned}
S_{n+1,m}(I,\lambda) &= \alpha^{n+1-m}(1-\alpha)^m\left(\sum_{i=1}^{k_1}\lambda^{H_i}+\sum_{i=1}^{k_2}\lambda^{G_i+h_{n,m}}\right)\\
&= \alpha\sum_{i=1}^{k_1}\alpha^{n-m}(1-\alpha)^m\lambda^{H_i}+(1-\alpha)\lambda^{h_{n,m}}\sum_{i=1}^{k_2}\alpha^{n+1-m}(1-\alpha)^{m-1}\lambda^{G_i}\\
&= \alpha S_{n,m}(I,\lambda)+(1-\alpha)\lambda^{h_{n,m}}S_{n,m-1}(I,\lambda).
\end{aligned}$$

□

Lemma 6. *If $k > 1$, $\lambda = e^{i\theta}$ where $\theta \in (-\pi, \pi]$, $n \equiv -2 \pmod k$, and I is a level of generation at most n, then for the transformation T_k,*

$$\sum_{m=0}^{n+4}|S_{n+4,m}(I,\lambda)| \leq \sum_{m=0}^{n}|S_{n,m}(I,\lambda)|(1-(2-2\cos(\theta/6))\alpha^2(1-\alpha)^2);$$

and, regardless of the value of n,

$$\sum_{m=0}^{n+1}|S_{n+1,m}(I,\lambda)| \leq \sum_{m=0}^{n}|S_{n,m}(I,\lambda)|.$$

Proof. Lemma 5 Implies that $S_{n+1,m}(I,\lambda) \leq \alpha|S_{n,m}(I,\lambda)|+(1-\alpha)|S_{n,m-1}(I,\lambda)|$. Our latter result follows from summing this over m.

The idea of the proof will be to use Lemma 5 repeatedly to relate the value of $S_{n+4,m}(I,\lambda)$ to the values of $S_{n,m-i}(I,\lambda)$. In particular, let P be the set of paths of length 4 starting from $(n+4, m)$ and taking steps of the form (x, y) to $(x-1, y)$ or $(x-1, y-1)$. For such a path, p, let $e(p)$ be the second coordinate of the end of the path. Let $h(p)$ be the sum of $h_{n',m'}$ over pairs (n', m') so that p takes a step from $(n'+1, m')$ to $(n', m'-1)$. By repeated use of Lemma 5, we have that

$$S_{n+4,m}(I,\lambda) = \sum_{p\in P}\lambda^{h(p)}\alpha^{e(p)+4-m}(1-\alpha)^{m-e(p)}S_{n,e(p)}(I,\lambda). \qquad (1)$$

For $n \equiv -2 \pmod k$, we wish to show that

$$\begin{aligned}
|S_{n+4,m}(I,\lambda)| \leq &\alpha^4|S_{n,m}(I,\lambda)|+4\alpha^3(1-\alpha)|S_{n,m-1}(I\lambda)|\\
&+\alpha^2(1-\alpha)^2(6-(2-2\cos(\theta/6)))|S_{n,m-2}(I,\lambda)|\\
&+4\alpha(1-\alpha)^3|S_{n,m-3}(I,\lambda)|+(1-\alpha)^4|S_{n,m-4}(I,\lambda)|,
\end{aligned}$$

and our result will follow from summing over m. We will do this by showing that there exist two paths $p_1, p_2 \in P$ with $\lambda^{h(p_1)}$ far from $\lambda^{h(p_2)}$, and using Equation 1. In particular, if $\lambda^{h(p_1)-h(p_2)}$ is at least as far from 1 as $e^{i\theta/3}$, our result will follow.

Consider two paths $p_1, p_2 \in P$ from $(n+4, m)$ to $(n, m-2)$ that each pass through (a, b) and $(a-2, b-1)$, identical except that the p_1 passes through $(a-1, b-1)$ and p_2 passes through $(a-1, b)$. Then $h(p_1) - h(p_2) = h_{a-1,b} - h_{a-2,b}$. If we let $p_{i,1}, p_{i,2}$ for $i = 1, 2, 3$ be such pairs of paths with (a, b) equal to $(n+3, m-1)$, $(n+3, m)$ and $(n+4, m-1)$ respectively we have that

$$h(p_{1,1}) - h(p_{1,2}) = h_{n+2,m-1} - h_{n+1,m-1} = h_{n+1,m-2}$$

,

$$h(p_{2,1}) - h(p_{2,2}) = h_{n+2,m-2} - h_{n+1,m-2} = h_{n+1,m-3}$$

and

$$h(p_{3,1}) - h(p_{3,2}) = h_{n+3,m-1} - h_{n+2,m-1} = h_{n+2,m-2} + 1.$$

Since the last of these is one more than the sum of the first two, letting

$$E_i = \lambda^{h(p_{i,1}) - h(p_{i,2})},$$

we have that $E_1^{-1} E_2^{-1} E_3 = \lambda$, and hence one of the E_i is at least as far from 1 as $e^{i\theta/3}$.

□

4 Weak Mixing

Here we prove Theorem 1.

Proof. Suppose for sake of contradiction that T_k has an eigenvalue of $\lambda = e^{i\theta}$ where $\theta \in (-\pi, \pi]$, and $\theta \neq 0$. Let $f : X \to \mathbb{C}$ be the associated eigenfunction. Since T_k is ergodic by Lemma 3, and since $|f|$ is T_k-invariant, we may assume that $|f| = 1$ a.e. We may also assume that $\log(f)/i \in (-\pi/3, \pi/3)$ on a set of positive measure, G. Let I be a level of stage n with n odd, which is at least $\left(\frac{3}{4}\right)$-full of G (i.e. $\mu(I \cap G) \geq \left(\frac{3}{4}\right) \mu(I)$). We have that

$$\int_I \Re f(x) dx \geq \frac{1}{2} \cdot \frac{3}{4} \mu(I) - 1 \cdot \frac{1}{4} \mu(I) = \frac{1}{8} \mu(I).$$

Therefore,

$$\left| \int_I f(x) dx \right| \geq \frac{1}{8} \mu(I).$$

We have that

$$\sum_m |S_{n,m}(I, \lambda)| = \mu(I)$$

since if I is a level of stage n, $S_{n,m}(I, \lambda)$ is either $\mu(I)\lambda^{h(I)}$, if I is a level in $C_{n,m}$, and 0 otherwise. Therefore, by Lemma 6,

$$\sum_m |S_{n+a\max(k,4),m}(I, \lambda)| \leq \mu(I)(1 - (2 - 2\cos(\theta/6))\alpha^2(1-\alpha)^2)^a.$$

Notice that for any integers N and M, and for a level J in column $C_{N,M}$, which has bottom level J', that

$$\int_J f(x)dx = \lambda^{h(J)} \int_{J'} f(x)dx$$

since $J = T_k^{h(J)}(J')$.

Therefore, we have that

$$\left| \int_{\bigcup_{I' \in P_{N,M}(I)}} f(x)dx \right| = \left| \left(\int_{J'} f(x)dx \right) \left(\sum_{I' \in P_{N,M}(I)} \lambda^{h(I')} \right) \right| \leq |S_{N,M}(I, \lambda)|.$$

Hence we have that

$$\begin{aligned} \frac{1}{8}\mu(I) &\leq \left| \int_I f(x)dx \right| \\ &= \left| \sum_m \int_{\bigcup_{I' \in P_{n+a\max(4,k),m}(I)}} f(x)dx \right| \\ &\leq \sum_m \left| \int_{\bigcup_{I' \in P_{n+a\max(4,k),m}(I)}} f(x)dx \right| \\ &\leq \sum_m |S_{n+a\max(4,k),m}(I, \lambda)| \\ &\leq \mu(I)(1 - (2 - 2\cos(\theta/6))\alpha^2(1-\alpha)^2)^a, \end{aligned}$$

which does not hold for sufficiently large values of a. Hence we have a contradiction. Therefore, T_k has no eigenvalues other than 1, and is ergodic. Therefore T_k is weakly mixing. □

References

[1] Adams and K. E. Petersen, *Binomial coefficient multiples of irrationals*, Monatsh. Math. **125** (1998), no. 4, 269-278.

[2] A. M. Vershik, *Description of invarient measures for actions of some infinite groups*, Dokl. Akad. Nauk SSSR **218** (1974), 749-752 = Soviet Math. Dokl. **15** (1974), 1396-1400.

[3] A. M. Vershik, *Uniform algebraic approximation of shift and multiplication operators*, Dokl. Akad. Nauk SSSR **259** (1981), 526-529 = Soviet Math. Dokl. **24** (1981), 97-100.

[4] A. M. Vershik and A. N. Livshitz, *Adic models of ergodic transformations, spectral theory, substitutions, and related topics*, Adv. Soviet Math. **9** (1992), 185-204.

[5] X. Méla and K. Petersen, *Dynamical properties of the pascal adic transformation*, , Ergodic Theory Dynam. Systems, **25** (2005), 227-256.