Weak Mixing of a Transformation Similar to Pascal

Daniel M. Kane
June 19, 2006

Abstract

We construct a class of transformations similar to the Pascal transformation, except for the use of spacers, and show that these transformations are weakly mixing.

1 Introduction

The Pascal transformations arise as natural examples of adic transformations. Adic transformations were studied by Vershik as models for measure-preserving transformations [2], [3], [4]. Vershik conjectured that the Pascal transformations are weakly mixing, and while they are known to be totally ergodic this conjecture remains open [1], [5]. In this paper we define a class of transformations that are towers over the Pascal transformations and show that they are weakly mixing (Theorem 1).

For each $0 < \alpha < 1$, define the Pascal transformation $S = S_\alpha^{-1}$ in the following way as a cutting and stacking transformation (see for example [5]):

We proceed by inductively defining a sequence of columns. Start by letting column $C_{0,0} = (B^{(0)}_0)$ consist of one interval, called a level, of total mass 1 (we normalize the measure at the end). In the $n$th generation of columns we will have $n+1$ columns, where column $m$ is denoted by $C_{n,m} = (B^{(0)}_m, \ldots, B^{(h_{n,m} - 1)}_m)$ for $0 \leq m \leq n$. The columns in generation $(n+1)$ are obtained from those in generation $n$ in the following way. First cut level $B^{(i)}_{n,m}$ into sublevels $B^{(i)}_{n,m,0}, B^{(i)}_{n,m,1}$ with mass ratio $\alpha : 1 - \alpha$. Then define

$$C_{n+1,m} = (B^{(0)}_{n,m,0}, \ldots, B^{(h_{n,m} - 1)}_{n,m,0}, B^{(0)}_{n,m-1,1}, \ldots, B^{(h_{n,m} - 1)}_{n,m-1,1}),$$

where levels $B^{(i)}_{n,m,i}$ with indices $m < 0$ or $m > n$ are ignored. We define the action of $S$ on $B^{(i)}_{n,m}$ by sending it (using the standard translation of an interval to another of the same length) to $B^{(i+1)}_{n,m}$ when $i \neq h_{n,m} - 1$. In the limit this defines a finite measure-preserving transformation $S$, known as the

\footnote{while $S$ depends on $\alpha$ we do not write this explicitly}
Pascal transformation. While $S$ is totally ergodic [1], it is an open problem as to whether $S$ is weakly mixing.

Now we define a new transformation $T_k$, the Pascal with spacers transformation, in a similar way, only with additional spacers (an additional piece of our measure space of the correct total mass that is not part of any previous generation’s column) placed on top of column $C_{n,m}$ if $n$ is a multiple of $k$. Notice that all of the levels in column $C_{n,m}$ have mass $\alpha^{n-m}(1-\alpha)^m$ so $T_k$ is measure preserving. Notice also that the amount of mass added by spacers on generation $n$ columns is 0 if $k$ does not divide $n$ and $\sum_{i=0}^{n} \alpha^i(1-\alpha)^{n-i} \leq (n+1)\max(\alpha,1-\alpha)^n$ otherwise, hence $T$ is defined on a space with finite mass. Therefore, we can instead consider the transformation $T_k$ on the renormalized measure space so that the total mass of the space is 1. We shall prove the following theorem:

**Theorem 1.** The transformation $T_k$ is weakly mixing if $k > 1$.

We also note, though that not all patterns of spacers are weakly mixing. In fact we can show that $T_1$ is not weak mixing.

**Proposition 2.** $T_1$ is not weakly mixing.

**Proof.** In $T_1$, all of the column heights are congruent to 2 modulo 3 and hence the function that assigns $e^{2\pi i (m+h)/3}$ to any point in the level $B_{n,m}^{(h)}$ is well defined and clearly has eigenvalue $e^{2\pi i/3}$. \qed

Notice that the above proof also shows that $T_1$ is not even totally ergodic.

The transformation $T_k$ can be expressed symbolically as follows: the space, $X$ is the subset of $\{0,1\}^\omega \times \mathbb{Z}$ consisting of elements of the form $((0^a1^b0S,n),n)$ where $S$ is some string of 1’s and 0’s, $b > 0$, and $0 \leq n \leq \frac{a+b}{k} + 1$. The measure on $X$ is generated by the cylinder sets $[0^a1^b0S,n] = \{(0^a1^b0SS',n) : S' \in \{0,1\}^\omega\}$, where $b > 0$, $S$ is any finite string and $0 \leq n \leq \frac{a+b}{k} + 1$, by $\mu[0^a1^b0S,n] = \alpha^x(1-\alpha)^y$, where $x$ and $y$ are the number of 0’s and 1’s respectively in the string $0^a1^b0S$. The transformation $T_k$ acts on $X$ by

$$T_k((0^a1^b0S,n)) = \begin{cases} 
((0^a1^b0S),n+1) & \text{if } n \leq \frac{a+b}{k} \\
((1^b-10^{a+1}1S),0) & \text{otherwise}
\end{cases}.$$

**Acknowledgements.** This paper is based on research in the Ergodic Theory group of the 2004 SMALL Undergraduate Research Program at Williams College, with C. Silva as faculty advisor. Support for this project was provided by a National Science Foundation REU Grant and the Bronfman Science Center of Williams College.

### 2 Ergodicity

Here we will prove the following Lemma:

**Lemma 3.** $T_k$ is ergodic.
Proof. Notice that the induced map of $T_k$ on the complement of the spacers in $X$ is $S$. Since for any point $x \in X$, $T_k^n(x)$ is in this set for some $n$, the result follows from the well-known ergodicity of $S$. \hfill \Box

3 Some Machinery involving Column Heights and Copy Heights

We will use the convention that $h_{n,m} = 0$ if $m < 0$ or $m > n$.

Lemma 4. For $n \geq 0$,

$$h_{n+1,m} = \begin{cases} h_{n,m} + h_{n,m-1} & \text{if } n + 1 \not\equiv 0 \pmod{k} \\ h_{n,m} + h_{n,m-1} + 1 & \text{if } n + 1 \equiv 0 \pmod{k} \end{cases}.$$ 

Proof. This follows immediately from the construction of the columns. \hfill \Box

Definition 1. If $I$ is a level in some column $C_{r,s}$, and $0 \leq m \leq n$ with $n \geq r$, let $P_{n,m}(I)$ denote the set of copies of $I$ in column $C_{n,m}$. In other words, $P_{n,m}(I)$ is the set of levels, $J$, in $C_{n,m}$ so that $J \subset I$.

Definition 2. If $I$ is a level in column $C_{n,m}$ where $I = B^{(H)}_{n,m}$, let $h(I) = H$.

Definition 3. If $I$ is a level in $C_{r,s}$, $n, m \in \mathbb{Z}$, with $n \geq m \geq 0$ and $\lambda \in \mathbb{C}$, let

$$S_{n,m}(I, \lambda) = \sum_{I' \in P_{n,m}(I)} \mu(I') \lambda^{h(I')}.$$ 

If $m \not\in [0,n]$, let $S_{n,m}(I, \lambda) = 0$.

The idea of the proof will be to assume for sake of contradiction that $T_k$ has an eigenfunction, $f$, with eigenvalue $\lambda \neq 1$. We will then look at some interval $I$ on which $f$ is nearly constant. We will then consider the values of $f$ on the generation-$N$ copies of $I$. Since if two copies of $I$ are both in column $C_{N,M}$, their $f$-values are proportional to $\lambda$ raised to the power of their heights, we have that the integral of $f$ over $P_{N,M}(I)$ is bounded above by $|S_{N,M}(I, \lambda)|$. We will produce a contradiction by proving that $\lim_{N \to \infty} \sum_M |S_{N,M}(I, \lambda)| = 0$. To do this we will use the following lemmas:

Lemma 5. If $I$ is a level of a column of generation at most $n$, then

$$S_{n+1,m}(I, \lambda) = \alpha S_{n,m}(I, \lambda) + (1 - \alpha) \lambda^{h_{n,m}} S_{n,m-1}(I, \lambda).$$ 

Proof. Let the heights of the copies of $I$ in $C_{n,m}$ be $H_i$ for $1 \leq i \leq k_1$. Let the heights of the copies of $I$ in $C_{n,m-1}$ be $G_i$ for $1 \leq i \leq k_2$. Then the heights of
Lemma 5 Implies that
Proof. the copies of Lemma 6.

Let \((p, x)\) be the sum of pairs \((n, m)\) so that \(p\) takes a step from \((n + 1, m)\) to \((n', m')\). For such a path, \(p\), let \(e(p)\) be the second coordinate of the end of the path. Let \(h(p)\) be the sum of \(h_n, m\) over pairs \((n, m)\) so that \(p\) takes a step from \((n + 1, m)\) to \((n', m')\). By repeated use of Lemma 5, we have that

\[
S_{n+1,m}(I, \lambda) = \sum_{p \in P} \lambda^{h(p)} \alpha^{e(p)+4-m} (1 - \alpha)^{m-e(p)} S_{n,e(p)}(I, \lambda).
\]

For \(n \equiv -2 \pmod{k}\), we wish to show that

\[
|S_{n+4,m}(I, \lambda)| \leq \alpha^4 |S_{n,m}(I, \lambda)| + 4\alpha^3(1 - \alpha)|S_{n,m-1}(I, \lambda)| + \alpha^2(1 - \alpha)^2 (6 - 2 \cos(\theta/6)) |S_{n,m-2}(I, \lambda)| + 4\alpha(1 - \alpha)^3|S_{n,m-3}(I, \lambda)| + (1 - \alpha)^4|S_{n,m-4}(I, \lambda)|,
\]

and our result will follow from summing over \(m\). We will do this by showing that there exist two paths \(p_1, p_2 \in P\) with \(\lambda^{h(p_1)}\) far from \(\lambda^{h(p_2)}\), and using Equation 1. In particular, if \(\lambda^{h(p_1)-h(p_2)}\) is at least as far from 1 as \(e^{i\theta/3}\), our result will follow.
Consider two paths \( p_1, p_2 \in P \) from \((n+4, m)\) to \((n, m-2)\) that each pass through \((a, b)\) and \((a-1, b-1)\), identical except that the \( p_1 \) passes through \((a-1, b)\) and \( p_2 \) passes through \((a-1, b-1)\). Then \( h(p_1) - h(p_2) = h_{a-1, b} - h_{a-2, b} \).

If we let \( p_{i,1}, p_{i,2} \) for \( i = 1, 2, 3 \) be such pairs of paths with \((a, b)\) equal to \((n+3, m-1), (n+3, m)\) and \((n+4, m-1)\) respectively we have that

\[
E_1 = \lambda^{h(p_{1,1}) - h(p_{1,2})},
\]

we have that \( E_1^{-1} E_2^{-1} E_3 = \lambda \), and hence one of the \( E_i \) is at least as far from \( 1 \) as \( e^{i\theta/3} \).

\[\Box\]

### 4 Weak Mixing

Here we prove Theorem 1.

**Proof.** Suppose for sake of contradiction that \( T_k \) has an eigenvalue of \( \lambda = e^{i\theta} \) where \( \theta \in (-\pi, \pi] \), and \( \theta \neq 0 \). Let \( f : X \to \mathbb{C} \) be the associated eigenfunction. Since \( T_k \) is ergodic by Lemma 3, and since \( |f| \) is \( T_k \)-invariant, we may assume that \( |f| = 1 \) a.e. We may also assume that \( \log(f)/i \in (-\pi/3, \pi/3) \) on a set of positive measure, \( G \). Let \( I \) be a level of stage \( n \) with \( n \) odd, which is at least \((\frac{3}{4})\) full of \( G \) (i.e. \( \mu(I \cap G) \geq \left(\frac{3}{4}\right) \mu(I) \)). We have that

\[
\int_I \Re f(x) dx \geq \frac{1}{2} \cdot \frac{3}{4} \mu(I) - 1 \cdot \frac{1}{4} \mu(I) = \frac{1}{8} \mu(I).
\]

Therefore,

\[
\left| \int_I f(x) dx \right| \geq \frac{1}{8} \mu(I).
\]

We have that

\[
\sum_m |S_{n,m}(I, \lambda)| = \mu(I)
\]

since if \( I \) is a level of stage \( n \), \( S_{n,m}(I, \lambda) \) is either \( \mu(I) \lambda^h(I) \), if \( I \) is a level in \( C_{n,m} \), and 0 otherwise. Therefore, by Lemma 6,

\[
\sum_m |S_{n+a \max(k,4),m}(I, \lambda)| \leq \mu(I)(1 - (2 - 2 \cos(\theta/6)) \alpha^2(1 - \alpha)^2)^a.
\]
Notice that for any integers \( N \) and \( M \), and for a level \( J \) in column \( C_{N,M} \), which has bottom level \( J' \), that

\[
\int_J f(x)dx = \lambda^{h(J)} \int_{J'} f(x)dx
\]

since \( J = T^h(J)(J') \).

Therefore, we have that

\[
\left| \int_{\bigcup_{I' \in P_{N,M}(I)}} f(x)dx \right| = \left| \left( \int_{J'} f(x)dx \right) \left( \sum_{I' \in P_{N,M}(I)} \lambda^{h(I')} \right) \right| \leq |S_{N,M}(I,\lambda)|.
\]

Hence we have that

\[
\frac{1}{8} \mu(I) \leq \left| \int_I f(x)dx \right|
\]

\[
= \left| \sum_m \left| \int_{\bigcup_{I' \in P_{n+a \max(4,k),m}(I)}} f(x)dx \right| \right|
\]

\[
\leq \sum_m \left| \int_{\bigcup_{I' \in P_{n+a \max(4,k),m}(I)}} f(x)dx \right|
\]

\[
\leq \sum_m |S_{n+a \max(4,k),m}(I,\lambda)|
\]

\[
\leq \mu(I)(1 - (2 - 2 \cos(\theta/6))\alpha^2(1 - \alpha)^2)^a,
\]

which does not hold for sufficiently large values of \( a \). Hence we have a contradiction. Therefore, \( T_k \) has no eigenvalues other than 1, and is ergodic. Therefore \( T_k \) is weakly mixing.

\[\square\]

References


