The Number of Partitions with no $k$-Sequences

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- Each cell is either On or Off at each time step.

If the plane is initialized with each cell On independently with probability $p$, what happens?
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Plane fills with On cells with probability 1.

- Find a $k \times k$ filled square.
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How hard is it for this to happen?
Probability Model

- Let $C_m$ be the event that there is an On cell on the $m^{th}$ layer on a particular side.
- $\Pr(C_m) = 1 - (1 - p)^m$. 

Require event: $A_k = \bigcap_{i=1}^\infty (C_i \cup \ldots \cup C_{i+k-1})$.

Namely, $A_k$ is the event that we do not miss $k$ of the $C_i$ in a row.

Try to determine $\Pr(A_k)$.

This model is useful for answering questions like:
- If we have a finite grid, what's the probability that it fills with On cells?
- How long does a typical cell take to turn On?
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Partitions

This model also has an interesting interpretation in terms of integer partitions.

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- The number of parts of size $m$ is independent for different $m$.
- The probability that there is no part of size $m$ is $1 - q^m$.
- Thus $\Pr(A_k)$ is the probability that a random partition has no $k$ parts whose sizes are consecutive integers. We call such a partition a partition without $k$-sequences.
Generating functions

Let \( p(n) \) be the number of partitions of size \( n \), and \( p_k(n) \) the number of partitions of size \( n \) with no \( k \)-sequence. Consider the generating functions

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G(q) = \sum_{n=0}^{\infty} p(n)q^n, \quad G_k(q) = \sum_{n=0}^{\infty} p_k(n)q^n.
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We have that

\[
\Pr(A_k) = \frac{G_k(q)}{G(q)}.
\]

Since the asymptotics of \( G(q) \) are well understood, this reduces our problem to that of understanding \( G_k(q) \).
Asymptotics

The asymptotic of $G_k(q)$ was conjectured by George Andrews.

**Conjecture (George Andrews)**

For $k \geq 2$,

$$G_k(e^{-s}) \sim C_k \exp \left( \frac{\pi^2}{6s} \left( 1 - \frac{2}{k(k+1)} \right) \right)$$

as $s \to 0^+$. 
Past Work

Anderws proved the $k = 2$ case of his conjecture using

$$G_2(q) = \prod_{n=1}^{\infty} \frac{1 + q^{3n}}{1 - q^{2n}} \chi(q),$$

where $\chi(q)$ is the mock theta function

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There are several other cases of partitions without 2-sequences with mild extra restrictions (distinct parts, smallest part bigger than 1) that, by work of MacMahon, Rogers, and Ramanujan have generating functions given by modular forms. These generating functions thus also have well-understood asymptotics.
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Past Work

For $k > 2$, Holroyd, Liggett, and Romik show that

**Theorem**

*For fixed $k$, as $s \to 0^+$,*

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\log(G_k(e^{-s})) \sim \frac{\pi^2}{6s} \left(1 - \frac{2}{k(k + 1)}\right).
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This was later strengthened by Bringmann and Mahlburg, who showed that

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*For fixed $k$, as $s \to 0^+$,*

$$\log(G_k(e^{-s})) = \frac{\pi^2}{6s} \left(1 - \frac{2}{k(k+1)}\right) + O_k(\log(s)).$$
We prove:

**Theorem**

*For* $k \geq 2$, $s \geq 0$

$$G_k(e^{-s}) = \frac{1}{k} \exp\left(\frac{\pi^2}{6s} \left(1 - \frac{2}{k(k + 1)}\right) + O_k\left(s^{\frac{1}{2k+3}}\right)\right).$$
Recurrence Relation

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- Obtain $G_k(q)$ via recurrence relation.
- Let $p_{k,r,L}(n)$ be the number of partitions of $n$ with:
  - No $k$-sequences
  - No parts of size bigger than $L$
  - Parts of sizes $L, L - 1, \ldots, L - r + 1$, but no part of size $L - r$
- Let
  $$v_{r,L}^k(q) := \sum_{n=0}^{\infty} p_{k,r,L}(n) q^n.$$
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- We have
  \[ v_{0,L}^k(q) = \sum_{r=0}^{k-1} v_{r,L-1}^k(q) \]
  \[ v_{r,L}^k(q) = \frac{q^L}{1 - q^L} v_{r-1,L-1}^k(q). \]
Recurrence Relation

Thus, letting $z(x) := \frac{x}{1-x}$,

$$v_L(q) := \begin{pmatrix} v_{0,L}^k(q) \\ v_{1,L}^k(q) \\ \vdots \\ v_{k-1,L}^k(q) \\ v_k(q) \end{pmatrix}, \quad M(x) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z(x) & 0 & \cdots & 0 \\ 0 & z(x) & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & z(x) \end{pmatrix},$$

then

$$v_L(q) = M(q^L)v_{L-1}(q).$$
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Since $v_0(q) = (1, 0, \cdots, 0)^T$, and $G_k(q) = \lim_{L \to \infty} (v_L(q))_1$ we have a recurrence relation that yields $G_k$. 
Eigenvalues

- Have linear, homogeneous, recurrence relation with non-constant coefficients.
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- If coefficients were constant, could solve easily using eigenvalues of $M$. 

\[ M(x) = A(x) D(x) A(x)^{-1}, \quad D(x) = \text{Diag}(\lambda_1(x), \ldots, \lambda_k(x)) \]  

for $|\lambda_1(x)| \geq |\lambda_2(x)| \geq \ldots \geq |\lambda_k(x)|$. 

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Eigenvalues

We need to consider

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\[ \cdots T_{L+1}(q)D(q^L)T_L(q)D(q^{L-1}) \cdots T_2(q)D(q)A(q)^{-1}v_0. \]
Starting with $A(q)^{-1}v_0$, we repeatedly multiply by $T_{L+1}(q)D(q^L)$. Multiplying by $D$ decreases the sizes of the other coordinates relative to the first coordinate. Multiplying by $T$ does not affect the vector much. After some point the vector is approximated by its first coordinate. This lets us ignore off-diagonal entries of $T$. Thus our final answer is roughly

$$\prod L \lambda_1(q^L) \prod L [T_L(q)]_{1,1} \approx \exp(s - 1 \int \log(\lambda_1(e^{-x}))) dx + \sum L ([T_L(q)]_{1,1} - 1).$$
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$$\approx \exp \left( s^{-1} \int \log(\lambda_1(e^{-x}))dx + \sum_L ([T_L(q)]_{1,1} - 1) \right).$$
The above argument only works once $v_L$ is dominated by the primary eigenvector of $M$.

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The Run-Up

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- Need a different method for dealing with the early steps.
- Consider partitions $\lambda$ with no $k$-sequences consisting of parts of size $\leq L$, weighted by $q^{|\lambda|}$, with $q^L$ near 1.
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If we have parts of size $s_1 < s_2 < \cdots < s_h$, contribution is $z(q^{s_1})z(q^{s_2})\cdots z(q^{s_h})$. 
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- Since \( z(q^{s_i}) \) is big, want \( h \) large.
- For the most part, skip every \( k^{th} \) size, leaving blocks of size \( k - 1 \).
- Only a few places \( t_1 < t_2 < \cdots \) where you have blocks of smaller size.
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Only a few places $t_1 < t_2 < \cdots$ where you have blocks of smaller size.

Compute contribution for given $t$'s and sum over choices.
Putting it Together

- Use run-up to approximate $v_L$ for $L \approx s^{\frac{-3}{2k+2}}$.
- Use eigenvalue method for the rest.
- A somewhat detailed computation yields the result.
Concluding Remarks

From our asymptotic for $G_k(q)$, we obtain an asymptotic for $p_k(n)$:

**Corollary**

For fixed $k$, as $n \to \infty$ we have

$$p_k(n) \sim \frac{1}{2k} \left( \frac{1}{6} \left( 1 - \frac{2}{k(k+1)} \right) \right)^{\frac{1}{4}} \frac{1}{n^{\frac{3}{4}}} \exp \left( \pi \sqrt{\frac{2}{3} \left( 1 - \frac{2}{k(k+1)} \right) n} \right).$$
Concluding Remarks

We also suspect that these techniques can be used to obtain an asymptotic expansion for $G_k(e^{-s})$ with relative error $O(s^N)$ for any $N$. In fact we conjecture the following first correction term:

**Conjecture**

For $s \geq 0$

$$G_k(e^{-s}) = \frac{1}{k} \exp \left( \frac{\pi^2}{6s} \left( 1 - \frac{2}{k(k+1)} \right) + \sqrt{\frac{2}{9\pi}} s^{\frac{1}{k}} + O_k \left( s^{\frac{2}{k}} \right) \right)$$

This conjecture agrees well with numerical evidence. It is of particular interest because it would imply that $G_k(q)$ is not modular for any $k > 2$. 
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Acknowledgements

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