The Number of Partitions with no $k$-Sequences

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The Rogers-Ramanujan identities tell us that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}.$$
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Or, in other words

\[ \# \{\text{Partitions of } n \text{ into distinct, non-consecutive parts}\} \]
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\[ \# \{\text{Partition of } n \text{ into parts congruent to 1 or 4 modulo 5}\}. \]
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\]

Using the latter formulation and the theory of modular forms, we can get precise asymptotics for the number of such partitions.
Repeated Parts

The distinct parts requirement can be removed by work of MacMahon, who showed that

\[
\# \{\text{Partitions of } n \text{ into non-consecutive parts bigger than } 1\} \\
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\# \{\text{Partition of } n \text{ into parts not congruent to } 1 \text{ or } 5 \text{ modulo } 6\}. 
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$k$-Sequences

- We have a good understanding of the number of partitions of $n$ so that no pair of parts are consecutive integers.
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Generalization: Consider partitions of $n$ so that the sizes of no $k$ parts are consecutive integers (i.e. for no $m$ are there parts of sizes $m, m+1, \ldots, m+k-2$ and $m+k-1$). We call these partitions of $n$ with no $k$-sequences.
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- Let

\[
p_k(n) := \# \{ \text{Partitions of } n \text{ with no } k\text{-sequences} \}
\]

\[
G_k(q) := \sum_{n=0}^{\infty} p_k(n)q^n.
\]
Finding the asymptotics in the $k > 2$ case seems more difficult.

**Conjecture (George Andrews)**

For $k \geq 2$,

$$G_k(e^{-s}) \sim C_k \exp\left(\frac{\pi^2}{6s} \left(1 - \frac{2}{k(k+1)}\right)\right)$$

as $s \to 0^+$. 
Past Work

Anderws proved the $k = 2$ case of his conjecture using

$$G_2(q) = \prod_{n=1}^{\infty} \frac{1 + q^{3n}}{1 - q^{2n}} \chi(q),$$

where $\chi(q)$ is the mock theta function

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For $k > 2$, Holroyd, Liggett, and Romik show that

**Theorem**

*For fixed $k$, as $s \to 0^+$,*

$$\log(G_k(e^{-s})) \sim \frac{\pi^2}{6s} \left(1 - \frac{2}{k(k + 1)}\right).$$
Our Work

We prove:

**Theorem**

*For* $k \geq 2$, $s \geq 0$

$$G_k(e^{-s}) = \frac{1}{k} \exp \left( \frac{\pi^2}{6s} \left( 1 - \frac{2}{k(k+1)} \right) + O_k \left( s^{\frac{1}{2k+3}} \right) \right).$$
Recurrence Relation

- Unfortunately, it has proven difficult to get at the asymptotics of $G_k(q)$ by modular techniques.

Let $p_{k, r, L}(n)$ be the number of partitions of $n$ with:

- No $k$-sequences
- No parts of size bigger than $L$
- Parts of sizes $L, L-1, \ldots, L-r+1$, but no part of size $L-r$

Let $v_{k, r, L}(q) := \sum_{n=0}^{\infty} p_{k, r, L}(n) q^n$.

We have

$$v_{k, 0, L}(q) = \sum_{r=0}^{k-1} v_{k, r, L-1}(q)$$

$$v_{k, r, L}(q) = q^{L-1} - q^{L-r},$$
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- Let
  $$v_{r,L}^k(q) := \sum_{n=0}^{\infty} p_{k,r,L}(n) q^n.$$
- We have
  $$v_{0,L}^k(q) = \sum_{r=0}^{k-1} v_{r,L-1}^k(q)$$
  $$v_{r,L}^k(q) = \frac{q^L}{1 - q^L} v_{r-1,L-1}^k(q).$$
Recurrence Relation

Thus, letting \( z(x) := \frac{x}{1-x} \),

\[
v_L(q) := \begin{pmatrix}
v_0^k(q) \\
v_1^k(q) \\
\vdots \\
v_{k-1}^k(q)
\end{pmatrix}, \quad M(x) := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
z(x) & 0 & \cdots & 0 \\
0 & z(x) & \cdots & 0 \\
0 & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & z(x)
\end{pmatrix},
\]

then

\[v_L(q) = M(q^L)v_{L-1}(q).\]
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Thus, letting $z(x) := \frac{x}{1-x}$,

$$v_L(q) := \begin{pmatrix} v_{0,L}^k(q) \\ v_{1,L}^k(q) \\ \vdots \\ v_{k-1,L}(q) \end{pmatrix}, \quad M(x) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vspace{1em} \\ z(x) & 0 & \cdots & 0 \\ 0 & z(x) & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & z(x) \end{pmatrix},$$

then

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Since $v_0(q) = (1, 0, \cdots, 0)^T$, and $G_k(q) = \lim_{L \to \infty} (v_L(q))_1$ we have a recurrence relation that yields $G_k$. 
Eigenvalues

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Let

$$M(x) = A(x)D(x)A(x)^{-1}, \quad D(x) = \text{Diag}(\lambda_1(x), \ldots, \lambda_k(x))$$

for $|\lambda_1(x)| \geq |\lambda_2(x)| \geq \ldots \geq |\lambda_k(x)|$. 
Eigenvalues

We need to consider

\[ \cdots A(q^{L+1})D(q^{L+1})A(q^{L+1})^{-1}A(q^L)D(q^L)A(q^L)^{-1} \cdots A(q)^{-1}v_0. \]
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Since coefficients of $A$ are slowly varying, $T \approx I$. In particular, we can make it so that when $q = e^{-s}$,

$$T_L(q) = I + O(L^{-1} + s).$$
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\[ \cdots T_{L+1}(q)D(q^L)T_L(q)D(q^{L-1}) \cdots T_2(q)D(q)A(q)^{-1}v_0. \]
Primary Eigenvalue

- Starting with $A(q)^{-1}v_0$, we repeatedly multiply by $T_{L+1}(q)D(q^L)$.
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- After some point the vector is approximated by its first coordinate.
- This lets us ignore off-diagonal entries of $T$. 

Thus our final answer is roughly:

$$\prod_{L} \lambda_1(q^L) \prod_{L} [T_L(q)]_{1,1} \approx \exp(s - 1 \int \log(\lambda_1(e^{-x}))) + \sum_{L} ([T_L(q)]_{1,1} - 1)$$
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$$\approx \exp \left( s^{-1} \int \log(\lambda_1(e^{-x})) dx + \sum_L ([T_L(q)]_{1,1} - 1) \right).$$
The Run-Up

- The above argument only works once \( \nu_L \) is dominated by the primary eigenvector of \( M \).
- Need a different method for dealing with the early steps.
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- Consider partitions \( \lambda \) with no \( k \)-sequences consisting of parts of size \( \leq L \), weighted by \( q^{\left| \lambda \right|} \), with \( q^L \) near 1.

If we have parts of size \( s_1 < s_2 < \cdots < s_h \), contribution is

\[
\frac{q^{s_1}}{z} \frac{q^{s_2}}{z} \cdots \frac{q^{s_h}}{z}.
\]

Since \( z(q^{s_i}) \) is big, want \( h \) large.

For the most part, skip every \( k \)th size, leaving blocks of size \( k-1 \).

Only a few places \( t_1 < t_2 < \cdots \) where you have blocks of smaller size.

Compute contribution for given \( t \)'s and sum over choices.
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Putting it Together

- Use run-up to approximate $v_L$ for $L \approx s^{\frac{-3}{2k+2}}$.
- Use eigenvalue method for the rest.
- A somewhat detailed computation yields the result.
Concluding Remarks

From our asymptotic for $G_k(q)$, we obtain an asymptotic for $p_k(n)$:

**Corollary**

*For fixed $k$, as $n \to \infty$ we have*

\[
p_k(n) \sim \frac{1}{2k} \left( \frac{1}{6} \left( 1 - \frac{2}{k(k+1)} \right) \right)^{\frac{1}{4}} \frac{1}{n^{\frac{3}{4}}} \exp \left( \pi \sqrt{\frac{2}{3}} \left( 1 - \frac{2}{k(k+1)} \right) n \right).
\]
Concluding Remarks

We also suspect that these techniques can be used to obtain an asymptotic expansion for $G_k(e^{-s})$ with relative error $O(s^N)$ for any $N$. In fact we conjecture the following first correction term:

**Conjecture**

For $s \geq 0$

$$G_k(e^{-s}) = \frac{1}{k} \exp \left( \frac{\pi^2}{6s} \left( 1 - \frac{2}{k(k+1)} \right) + \sqrt{\frac{2}{9\pi}} s^{\frac{1}{k}} + O\left(s^{\frac{2}{k}}\right) \right)$$

This conjecture agrees well with numerical evidence. It is of particular interest because it would imply that $G_k(q)$ is not modular for any $k > 2$. 
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Acknowledgements

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