On Solving Games Constructed Using Both Shortened and Continued Conjunctive Sums

By Daniel Kane
Combinatorial Games

What makes up a combinatorial game?
• A set of positions
• An initial position
• A set of moves between positions
• Two players take turns moving
• The last to be able to move wins

Generally, it is assumed that there can be no infinite sequence of moves.
Conjunctive Sums

- Defined by Conway
- Have copies of each composite game played side-by-side. Positions of the form \((A,B)\)
- A move consists of moving in all unfinished components. Moves of the form \((A,B) \rightarrow (A',B')\)
- Short Rule: Game ends when first component does
- Long Rule: Game ends when last component does
- Short sum of \(A\) and \(B\) = \(A \land B\)
- Long sum of \(A\) and \(B\) = \(A \Delta B\)
Remoteness and Suspense

• For short sums, only matters who wins shortest component game
• Strategy: win games quickly, lose games slowly
• With this strategy, length of A is $R(A) = \text{remoteness of } A$
• Similarly define $S(A) = \text{suspense of } A$
• $R(A \land B) = \min(R(A), R(B))$
• $S(A \Delta B) = \max(S(A), S(B))$
• Contain all strategically relevant information about games under short/long conjunctive sums
Our Objective

We would like to find:
$I(G)$ contains all strategically relevant information about $G$ under either conjunctive sum.

Idea: Consider length of $G$
Problem: Depending on sums, length of $G$ may vary
Solution: Quantify *Control* over the length of $G$
Ordinal Length Game

Game that “takes \( \alpha \) moves to play”

First Try: \([\alpha] \rightarrow [\beta]\) for some \( \beta < \alpha \)

Problem: Can decrease too quickly.

Solution: Create second coordinate as “lower bound”

Game \([\alpha, \beta]\) for \( \alpha > \beta \) goes to \([\gamma, \delta]\)

for \( \alpha > \gamma \geq \beta, \gamma > \delta \)

Let \([\alpha] = [\alpha + 1, \alpha]\).

Heuristically \([\alpha]\) takes \( \alpha \) moves to play
Some Lemmas

This heuristic is born out in the following lemmas:

- \( R([\alpha]) = \alpha \)
- \( S([\alpha]) = \{ \alpha \text{ or } \alpha+1 \} \)
- \( [\alpha] \wedge [\beta] = [\min(\alpha,\beta)] \)
- \( [\alpha] \Delta [\beta] = [\max(\alpha,\beta)] \)
- \( (A \wedge B) \Delta [\gamma] =_w (A \Delta [\gamma]) \wedge (B \Delta [\gamma]) \)
- \( (A \Delta B) \wedge [\gamma] =_w (A \wedge [\gamma]) \Delta (B \wedge [\gamma]) \)
Our Information

We let $I(G)$ associate with $G$ the winners of all the games $(G \wedge [\alpha]) \Delta [\beta]$ for all ordinals $\alpha$ and $\beta$.

$I(G)$ can be shown to contain all strategically relevant information about $G$.

Can be used to classify the algebraic structure of games under this equivalence with the operations of short and long conjunctive sums.
Further Work

There appears to be a duality:

<table>
<thead>
<tr>
<th>Short Conjunctive Sums</th>
<th>Long Conjunctive Sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remoteness</td>
<td>Suspense</td>
</tr>
<tr>
<td>Minimum</td>
<td>Maximum</td>
</tr>
</tbody>
</table>

It would be nice to make this rigorous.

Also there’s the operation of concatenation - play one game and then play the other when it’s finished.