A partition of the positive reals into algebraically closed subsets

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Abstract

The positive reals can be partitioned into non-empty subsets closed under addition and multiplication.

Theorem 1. *There exist non-empty sets A and B so that $A \cup B = \mathbb{R}^+$, $A \cap B = \emptyset$ and so that both A and B are closed under both addition and multiplication.*

We will prove Theorem 1 by associating each positive real number with a differentiable function based on its algebraic relation to the transcendental constant π. We will divide the reals into sets based on the sign of the derivate of this function.

Let $M \cup \{\pi\}$ be a transcendence basis for $\mathbb{R}$. Let R and R' be the rings $\mathbb{Z}[M, \pi, x]$ and $\mathbb{Z}[M, x, y]$ respectively. Notice that these rings are isomorphic under the mapping J witch maps π to y and fixes M and x. Also note that both rings are Unique Prime Factorization Domains. For $r \in \mathbb{R}$, let $f_r(x)$ be the irreducible element of R with root r. Let $J(f_r(x)) = F_r(x, y)$. Notice that $F_r(r, \pi) = 0$.

We now define r as a "function" of π, using the fact that r is a root of $F_r(x, \pi)$.

Lemma 2. *For every $r \in \mathbb{R}$ there exists an $\epsilon > 0$ and a differentiable function $g_r : \mathbb{R} \to \mathbb{R}$ so that for all (x, y) where $|(x, y) - (r, \pi)| < \epsilon$*

$$f_r(x, y) = 0 \Leftrightarrow x = g_r(y).$$

Proof of Lemma 2. We note that $\frac{\partial}{\partial x} f_r(x)$ has lower x degree than $f_r(x)$, and therefore does not have x as a root. Hence $\frac{\partial}{\partial x} F_r(r, \pi) \neq 0$, and the result follows by the implicit function theorem. □

Notice that $g_r(\pi) = r$, allowing us to thing of r as equal to the function g at π. Next we will show that this association is compatible with the arithmetic operations.

Lemma 3. *For every $a, b \in \mathbb{R}$, there exists an $\epsilon > 0$ so that for $|y - \pi| < \epsilon$, $g_a(y) + g_b(y) = g_{a+b}(y)$.*

Proof of Lemma 3. Let $f_a(x)$ have roots $a_1, \ldots, a_n \in \mathbb{R}$, and $f_b(x)$ have roots $b_1, \ldots, b_m \in \mathbb{R}$. Notice that for $|y - \pi|$ sufficiently small, $x = g_{a_i}(y)$ are the roots of $F_a(x, y)$, and $x = g_{b_j}(y)$ are the roots of $F_b(x, y)$. Let

$$G(x, y) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - a_i(y) - b_j(y)).$$

be a function from $\mathbb{R}^2 \to \mathbb{R}$. Notice that G is a polynomial in $a_i(y), b_j(y)$ with coefficients in R' that is symmetric with respect to a_i and with respect to b_j. Therefore by the theorem on symmetric polynomials, G is a polynomial in symmetric polynomials in a_i and symmetric polynomials in b_j. Since symmetric polynomials in a_i and b_j are all elements of $\mathbb{Z}[M, y]$, for y sufficiently close to π, G is an element of R'. Let $h(x, y) = \gcd(G(x, y), F_{a+b}(x, y))$. Since $F_{a+b}(x, y)$ is irreducible, $h(x, y) = 1$ or $h(x, y) = F_{a,b}(x, y)$. Since at $x = a + b$, $y = \pi$ we have that $G(x, y) = F_{a+b}(x, y) = 0$, we have that $h(x, y) \neq 1$. Therefore, $G(x, y)$ is a multiple of $F_{a+b}(x, y)$. Therefore, all roots of $F_{a+b}(x, y)$ are the sum of a root of $F_a(x, y)$ and a root of $F_b(x, y)$ for any given y. Furthermore, for some n, $G(x, y) = (F_{a,b}(x, y))^n H(x, y)$ for some $H(x, y) \in R'$, relatively prime to $F_{a,b}(x, y)$ and hence satisfying $H(a + b, \pi) \neq 0$. Hence, we can find some $\epsilon_1 > 0$ so that for $|(x, y) - (a + b, \pi)| < \epsilon_1$ we have that

$$G(x, y) = 0 \Leftrightarrow F_{a+b}(x, y) = 0.$$

Also, by Lemma 2, there exists an $\epsilon_2 > 0$ so that in $|(x, y) - (a + b, \pi)| < \epsilon_2$ there is at most one root of $F_{a+b}(x, y)$ for every y. We let ϵ_3 and ϵ_4 be the ϵ's used in defining $g_a(y)$ and $g_b(y)$ respectively, and let $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. For $|(x, y) - (a + b, \pi)| < \epsilon$, we have that $F_{a+b}(x, y) = 0$ if and only if $G(x, y) = 0$. Note that for fixed y, $F_{a+b}(x, y)$ has at most one solution in this range. Since $g_a(y) + g_b(y)$, and $g_{a+b}(y)$ are both solutions, they must be equal. □

Lemma 4. *For every $a, b \in \mathbb{R}$, there exists an $\epsilon > 0$ so that for $|y - \pi| < \epsilon$, $g_a(y) g_b(y) = g_{ab}(y)$.*

Proof of Lemma 4. This proof is analogous to the proof of Lemma 3. □

We need a couple of other facts about the g_r is order to finish up.

Let $m \in M$. WLOG $m > 0$ (otherwise work with $-m$). If $r = m$ then $f_r(x, y) = x - m$, so $g_r(y) = m$, so $g'_r(\pi) = 0$. If N is a natural number so that $Nm > \pi$ and $r = Nm - \pi$, then $f_r(x, y) = x - Nm + \pi$ and $g_r(y) = Nm - y$, so $g'_r(\pi) = -1 < 0$.

To summarize, the functions $g_r(y)$ have the following properties:

- $g_r(\pi) = r$
- $\exists r > 0 : g'_r(\pi) \geq 0$
- $\exists r > 0 : g'_r(\pi) < 0$

- $g_a(y) + g_b(y) = g_{a+b}(y)$
- $g_a(y)g_b(y) = g_{ab}(y)$.

Proof of Theorem. Let $A = \{r \in \mathbb{R}^+ : g'_r(\pi) \geq 0\}$. Let $B = \{r \in \mathbb{R}^+ : g'_r(\pi) < 0\}$. It is clear that A and B form a partition of $\mathbb{R}^+$. It is also clear from the above that A and B are non-empty. A and B are closed under the arithmetic operations, since:

- If $a, b \in A$, then $g'_{a+b}(\pi) = g'_a(\pi) + g'_b(\pi) \geq 0$, so $a + b \in A$.
- If $a, b \in B$, then $g'_{a+b}(\pi) = g'_a(\pi) + g'_b(\pi) < 0$, so $a + b \in B$.
- If $a, b \in A$, then $g'_{ab}(\pi) = g'_a(\pi)g_b(\pi) + g_a(\pi)g'_b(\pi) = g'_a(\pi)b + g'_b(\pi)a \geq 0$, so $ab \in A$.
- If $a, b \in B$, then $g'_{ab}(\pi) = g'_a(\pi)g_b(\pi) + g_a(\pi)g'_b(\pi) = g'_a(\pi)b + g'_b(\pi)a < 0$, so $ab \in B$.

□

Note: This proof depends on constructing a non-trivial derivation from $\mathbb{R} \to \mathbb{R}$. This could have been done directly using the axiom of choice, thus proving the analogous statement for any ordered field transcendental over $\mathbb{Q}$. This proof does require the axiom of choice, since assuming the axiom that all sets in $\mathbb{R}$ are Lebesgue measureable, there can be no such sets A and B closed under even just addition.