Testing Properties of Linear Functions

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Abstract. The function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is *k-linear* if it returns the sum (over $\mathbb{F}_2$) of exactly k coordinates of its input. We introduce strong lower bounds on the query complexity for testing whether a function is k-linear. We show that for any $k \leq \frac{n}{2}$, at least $k - o(k)$ queries are required to test k-linearity, and we show that when $k \approx \frac{n}{2}$, this lower bound is nearly tight since $\frac{4}{3}k + o(k)$ queries are sufficient to test k-linearity. We also show that *non-adaptive* testers require $2k - o(k)$ queries to test k-linearity.
Our results improve known lower bounds on the query complexity of a number of property testing problems: juntas, sparse polynomials, small decision trees, and functions with low Fourier degree. Our lower bounds also give new results in testing function isomorphism and the characterization of the query complexity for testing properties of linear functions.

1 Introduction

What global properties of functions can we test with only a partial, local view of an unknown object? Property testing, a model introduced by Rubinfeld and Sudan [21], formalizes this question. In this model, a *property* of functions $\mathbb{F}_2^n \to \mathbb{F}_2$ is simply a subset of these functions. A function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is *ϵ-far* from a property $\mathcal{P}$ if for every $g \in \mathcal{P}$, the inequality $f(x) \neq g(x)$ holds for at least an ϵ fraction of the inputs $x \in \mathbb{F}_2^n$. A *q-query ϵ-tester* for $\mathcal{P}$ is a randomized algorithm that queries a function f on at most q inputs and distinguishes with probability at least $\frac{2}{3}$ between the cases where $f \in \mathcal{P}$ and where f is ϵ-far from $\mathcal{P}$ [21]. The aim of property testing is to identify the minimum number of queries required to test various properties. For more details on property testing, we recommend the recent surveys [18–20] and the collection [14].

Linearity testing is one of the earliest success stories in property testing. The function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is *linear* if it is of the form $f(x) = \sum_{i \in S} x_i$ for some set $S \subseteq [n]$, where the sum is taken over $\mathbb{F}_2$. Equivalently, f is linear if every pair $x, y \in \mathbb{F}_2^n$ satisfies the identity $f(x) + f(y) = f(x + y)$. Blum, Luby, and Rubinfeld [8] showed that, remarkably, linearity can be

ϵ-tested with only $O(1/\epsilon)$ queries. The exact query complexity of this problem has since been studied extensively [3, 4, 2, 16] and is well understood.

In this work, we examine the number of queries required to test properties that contain subsets of all linear functions. We refer to these properties as *properties of linear functions*. One particularly important property of linear functions is the set of *k-linear functions*—functions of the form $f(x) = \sum_{i \in S} x_i$ for some set $S \subseteq [n]$ of size $|S| = k$. The query complexity of the k-linearity testing problem provides a lower bound for the query complexity for testing juntas [12], testing low Fourier degree [10], testing computability by small-depth decision trees [10], and for a number of other fundamental problems in property testing.

Our goal is to determine the exact query complexity of the k-linearity testing problem. For any $0 \le k \le n$, the query complexity for the k-linearity and $(n-k)$-linearity testing problems are identical. (See Proposition 1 in the appendix.) This observation lets us concentrate on the range $0 \le k \le \frac{n}{2}$ from now on; all our results also apply to the range $\frac{n}{2} < k \le n$ by applying this identity.

The connection between property testing and learning theory, first established by Goldreich, Goldwasser, and Ron [15], yields a simple k-linearity tester with query complexity $n + O(1/\epsilon)$. For $i = 1, 2 \ldots, n$, define $e_i \in \mathbb{F}_2^n$ to be the vector with value 1 in the ith coordinate and value 0 elsewhere. The tester queries the function on the inputs $e_1, e_2, \ldots, e_n \in \mathbb{F}_2^n$. If $f(e_i) = 1$ for exactly k indices $i \in [n]$, then f is consistent with exactly one k-linear function h. We can query the function f on $O(1/\epsilon)$ additional inputs chosen uniformly and independently at random from $\mathbb{F}_2^n$ to verify that the rest of the function f is also consistent with h. This test always accepts k-linear functions, while the functions that are ϵ-far from k-linear functions fail at least one of the two steps of the test with high probability. We call this algorithm the *learning tester* for k-linearity.

Previous work. Fischer, Kindler, Ron, Safra, and Samorodnitsky [12] introduced an algorithm for testing k-linearity with roughly $O(k^2)$ queries. They also showed that for $k = o(\sqrt{n})$, non-adaptive testers—that is, testers that must fix all their queries before observing the value of the function on any of those queries—require roughly $\Omega(\sqrt{k})$ queries to test k-linearity. This implies a lower bound of $\Omega(\log k)$ queries for general (i.e., possibly adaptive) k-linearity testers for the same range of values of k.

The upper bound on the query complexity for testing k-linearity was improved implicitly by the introduction of a new algorithm for testing k-juntas—that is, testing whether a function depends on at most k variables—with only $O(k \log k)$ queries [5]. By combining this junta tester with the BLR linearity test [8], one can test k-linearity with roughly $O(k \log k)$ queries.

The first lower bound for testing k-linearity that applied to all values of k was discovered by Blais and O'Donnell [7], who, as a special case of a more general theorem on testing function isomorphism, showed that non-adaptive testers need at least $\Omega(\log k)$ queries to test k-linearity.

A much stronger bound was obtained by Goldreich [13], who showed that $\Omega(k)$ queries are required to test k-linearity non-adaptively, and that general testers require at least $\Omega(\sqrt{k})$ queries for the same task. He conjectured that this last bound could be strengthened to show that $\Omega(k)$ queries are required to test k-linearity for all $1 \leq k \leq \frac{n}{2}$.[3]

Consequences of Goldreich's Conjecture. The particularly intriguing case for k-linearity testing is when $k = \frac{n}{2}$. In this case, Goldreich's conjecture is that $\Omega(n)$ queries are required to test $\frac{n}{2}$-linearity. If true, this fact would imply the following two qualitative observations.

1. *Adaptivity does not help.* No tester for $\frac{n}{2}$-linearity can have asymptotically smaller query complexity than the best non-adaptive tester.
2. *Testing is as hard as learning.* No $\frac{n}{2}$-linearity tester can do asymptotically better than the learning tester.

All of the lower bounds described above are established by proving a lower bound for the more restricted problem of distinguishing k-linear and $(k+2)$-linear functions. If Goldreich's conjecture can be settled by the same approach, this would imply the following basic qualitative observation as well.

3. *Distinguishing is as hard as testing.* Testing $\frac{n}{2}$-linearity is not asymptotically harder than distinguishing $\frac{n}{2}$- and $(\frac{n}{2}+2)$-linear functions.

Our results. We confirm Goldreich's conjecture: $\Omega(k)$ queries are indeed required to ϵ-test k-linearity for any $k \leq \frac{n}{2}$ and for any $0 < \epsilon < \frac{1}{2}$. Furthermore, we prove Goldreich's conjecture by showing that the problem of distinguishing k-linear and $(k+2)$-linear functions also requires $\Omega(k)$ queries. As a result, all 3 qualitative observations mentioned above hold.

[3] Goldreich's results and conjecture are stated in terms of the slightly different problem of testing $\leq k$-linearity—the property of being a function that returns the sum over $\mathbb{F}_2^n$ of *at most* k variables. The $\leq k$-linearity and k-linearity problems are largely equivalent; see [6, 13] for more details.

Remark 1. In parallel and simultaneous work, Blais, Brody, and Matulef obtained a different proof of Goldreich's conjecture [6].

Interestingly, our results achieve a bit more than is required to establish Goldreich's conjecture: our general lower bound is not only linear in k, but makes some progress in determining the *exact* query complexity for testing k-linearity. Specifically, we show the following.

Theorem 1. *Fix $1 \le k \le \frac{n}{2}$. At least $k - O(k^{2/3})$ queries are required to test k-linearity.*

We obtain an even stronger lower bound for testing k-linearity with non-adaptive algorithms.

Theorem 2. *Fix $1 \le k \le \frac{n}{2}$. Non-adaptive testers for k-linearity need at least $2k - O(1)$ queries.*

As mentioned above, a particularly interesting case for k-linearity testing is when $k = \frac{n}{2}$. For this case, we introduce a new upper bound on the query complexity for testing $\frac{n}{2}$-linearity (adaptively).

Theorem 3. *It is possible to test $\frac{n}{2}$-linearity with $\frac{2}{3}n + O(\sqrt{n})$ queries.*

Remark 2. Theorem 3 is a special case of a more general upper bound that applies for all values of k close to $\frac{n}{2}$. See Appendix E for the details.

Our results show that the first two qualitative observations we made regarding the $\frac{n}{2}$-linearity testing problem no longer hold when we look at the exact query complexity of the problem:

1. *Adaptivity does help.* The best adaptive tester for $\frac{n}{2}$-linearity requires at most $\frac{2}{3}$ as many queries as the best non-adaptive tester.
2. *Testing is easier than learning.* There is an $\frac{n}{2}$-linearity tester that beats the query complexity of the learning tester by a constant multiplicative factor. No non-adaptive $\frac{n}{2}$-linearity tester, however, improves on the query complexity of the learning tester by more than an additive constant.

Both of our lower bounds, as all previous lower bounds, are obtained by looking at the more restricted problem of distinguishing k-linear and $(k+2)$-linear functions. Our final result shows that, for this more restricted problem, our lower bound is optimal, up to the lower order error term.

Theorem 4. *We can distinguish $\frac{n}{2}$-linear and $(\frac{n}{2}+2)$-linear functions with $\lceil \frac{n}{2} \rceil + 1$ queries. More generally, for $\ell \geq 1$, let b be the smallest positive integer for which 2^b does not divide ℓ. It is possible to distinguish $\frac{n}{2}$-linear and $(\frac{n}{2}+2\ell)$-linear functions with $\frac{2}{3}(1-2^{-2b})n + o(n)$ queries.*

We conjecture that the bounds in Theorem 4 are optimal and that $\frac{2}{3}n + o(n)$ queries are therefore required to test $\frac{n}{2}$-linearity. If so, this fact would imply that the third qualitative observation from the last section is also false in the setting of exact query complexity.

3. *Distinguishing may be easier than testing.* If the bounds in Theorem 4 are optimal, then testing $\frac{n}{2}$-linearity is harder than distinguishing $\frac{n}{2}$-linear and $(\frac{n}{2}+2)$-linear functions.

Implications. Lower bounds on the query complexity for testing k-linearity imply lower bounds for the query complexity of a number of other property testing problems. As a result, Theorem 1 sharpens several previous results. Due to space constraint, we only provide a short description of these results; the details are found in Appendix C.

Corollary 1. *Fix $1 \leq k \leq \frac{n}{2}$. At least $k - O(k^{2/3})$ queries are required to test (1) k-juntas, (2) k-sparse $\mathbb{F}_2$-polynomials, (3) functions of Fourier degree at most k, (4) functions computable by depth-k decision trees, and (5) isomorphism to the function $f : x \mapsto x_1 + \cdots + x_k$.*

A property of linear functions is called *symmetric* if it is invariant under relabeling of its variables. A symmetric property $\mathcal{P}$ of linear functions is completely characterized by the function $h_{\mathcal{P}} : \{0, 1, \ldots, n\} \to \{0, 1\}$ where $h_{\mathcal{P}}(k) = 1$ iff k-linear functions are included in $\mathcal{P}$. Define $\Gamma_{\mathcal{P}}$ to be the minimum value of $\ell \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ for which every value of k in the range $\ell \leq k \leq n - \ell$ satisfies $h_{\mathcal{P}}(k) = h_{\mathcal{P}}(k+2)$. This measure is closely related to the Paturi complexity of symmetric functions [17]. It also provides a lower bound on the query complexity for testing $\mathcal{P}$.

Corollary 2. *Let $\mathcal{P}$ be a symmetric property of linear functions. Then at least $\Gamma_{\mathcal{P}} - O(\Gamma_{\mathcal{P}}^{2/3})$ queries are required to test $\mathcal{P}$.*

Our techniques. We reduce the problem of testing k-linear functions to a purely geometric problem on the Hamming cube. Namely, we obtain our testing lower bound by showing that affine subspaces of large dimension intersect roughly the same fraction of the middle layers of the cube. More precisely, let $W_k \subseteq \mathbb{F}_2^n$ denote the set of vectors $x \in \mathbb{F}_2^n$ of Hamming weight k. Our main technical contribution is the following result.

Lemma 1. *There is a constant $c > 0$ such that for any affine subspace $V \subseteq \{0,1\}^n$ of codimension $d \le \frac{n}{2} - cn^{2/3}$,*

$$\left| \frac{|V \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} - \frac{|V \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|} \right| \le \frac{1}{36} 2^{-d}.$$

We prove the lemma with Fourier analysis and with the manipulation of Krawtchouk polynomials.

The proof of our lower bound for non-adaptive testers proceeds via a similar reduction to a geometric problem on the Hamming cube. Due to space constraints, the details of this lower bound are presented in Appendix D.

2 Preliminaries

Fourier analysis. For a finite dimensional vector space V over $\mathbb{F}_2$, the *inner product* of two functions $f, g : V \to \mathbb{R}$ is $\langle f, g\rangle = \mathbf{E}_{x\in V}[f(x) \cdot g(x)]$, where the expectation is over the uniform distribution on V. The L_2 norm of f is $\|f\|_2 := \sqrt{\langle f, f\rangle}$. A *character* of V is a group homomorphism $\chi : V \to \{1, -1\}^*$. Equivalently a character is a function $\chi : V \to \{1, -1\}$ so that for any $x, y \in V$, $\chi(x + y) = \chi(x)\chi(y)$. Define $\hat{V}$ to be the set of characters of V.

For a function $f : V \to \mathbb{R}$, the *Fourier transform* of f is the function $\hat{f} : \hat{V} \to \mathbb{R}$ given by $\hat{f}(\chi) := \langle f, \chi\rangle$. The *Fourier decomposition* of f is $f(x) = \sum_{\chi\in\hat{V}} \hat{f}(\chi)\chi(x)$. A fundamental property of the Fourier transform is that it preserves the squared L_2 norm.

Fact 5 (Parseval's Identity). *For any $f : V \to \mathbb{R}$, $\|f\|_2^2 = \sum_{\chi\in\hat{V}} \hat{f}(\chi)^2$.*

The *pushforward* of the function $f : V \to \mathbb{R}$ by the linear function $g : V \to W$ is defined by $(g_*(f))(x) := \frac{1}{|V|}\sum_{y\in g^{-1}(x)}[f(y)]$.

Fact 6. *For any linear function $g : V \to W$ and any function $f : V \to \mathbb{R}$, $\widehat{g_*(f)}(\chi) = \frac{1}{|W|}\hat{f}(\chi \circ g)$.*

Krawtchouk polynomials. For $n > 0$ and $k = 0, 1, \ldots, n$, the (binary) *Krawtchouk polynomial* $K_k^n : \{0, 1, \ldots, n\} \to \mathbb{Z}$ is defined by

$$K_k^n(m) = \sum_{j=0}^{k} (-1)^j \binom{m}{j}\binom{n-m}{k-j}.$$

The generating function representation of the Krawtchouk polynomial $K_k^n(m)$ is $K_k^n(m) = [x^k]\,(1-x)^m(1+x)^{n-m}$. Krawtchouk polynomials satisfy a number of useful properties [22]. We make use of the following identities in our proofs.

Fact 7. *Fix $n > 0$. Then*

i. *For every $2 \le k \le n$,* $K_k^n(m) - K_{k-2}^n(m) = K_k^{n+2}(m+1)$.
ii. $\sum_{k=0}^n K_k^n(m)^2 = (-1)^m K_n^{2n}(2m)$.
iii. *For every $0 \le d \le \frac{n}{2}$,* $\sum_{j=0}^d \binom{d}{j}(-1)^j K_{\frac{n}{2}}^n(2j+2) = 2^{2d} K_{\frac{n}{2}-d}^{n-2d}(2)$.
iv. $K_n^{2n}(2m+1) = 0$ *and* $(-1)^m K_n^{2n}(2m)$ *is positive and decreasing in* $\min\{m, n-m\}$.

Krawtchouk polynomials are widely used in coding theory [23] and appear in our proofs because of their close connection with the Fourier coefficients of the (Hamming weight indicator) function $I_{W_k} : \mathbb{F}_2^n \to \{0,1\}$ defined by $I_{W_k}(x) = \mathbf{1}[|x| = k]$.

Fact 8. *Fix $0 \le k \le n$, and $\alpha \in \{0,1\}^n$. Then* $\widehat{I}_{W_k}(\alpha) = 2^{-n} K_k^n(|\alpha|)$.

For completeness, we include the proofs of Facts 7–8 in Appendix A.

Property testing. The proof of Theorem 1 uses the following standard property testing lemma.

Lemma 2. *Let $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ be any two distributions over functions $\mathbb{F}_2^n \to \mathbb{F}_2$. If for every set $X \subseteq \mathbb{F}_2^n$ of size $|X| = q$ and any vector $r \in \mathbb{F}_2^q$ we have that $\left|\Pr_{f\sim\mathcal{D}_{\text{yes}}}[f(X) = r] - \Pr_{f\sim\mathcal{D}_{\text{no}}}[f(X) = r]\right| < \frac{1}{36}\,2^{-q}$, then any algorithm that distinguishes functions drawn from $\mathcal{D}_{\text{yes}}$ from those drawn from $\mathcal{D}_{\text{no}}$ with probability at least $\frac{2}{3}$ makes at least $q+1$ queries.*

Lemma 2 follows from Yao's Minimax Principle [24]. The proof of this result can be found in [11, 9] and, for the reader's convenience, in Appendix B.

3 Proof of the General Lower Bound

Proof (of Theorem 1). We first prove the special case where $k = \frac{n}{2} - 1$. There is a natural bijection between linear functions $\mathbb{F}_2^n \to \mathbb{F}_2$ and vectors in $\mathbb{F}_2^n$: associate $f(x) = \sum_{i\in S} x_i$ with the vector $\alpha \in \mathbb{F}_2^n$ whose coordinates satisfy $\alpha_i = 1$ iff $i \in S$. Note that $f(x) = \alpha \cdot x$.

For $0 \le \ell \le n$, let $W_\ell \subseteq \mathbb{F}_2^n$ denote the set of elements of Hamming weight ℓ. Fix any set $X \subseteq \mathbb{F}_2^n$ of $q < \frac{n}{2} - O(n^{2/3})$ queries and any response vector $r \in \mathbb{F}_2^q$. The set of linear functions that return the response vector r to the queries in X corresponds in our bijection to an affine subspace $V \subseteq \mathbb{F}_2^n$ of codimension q. This is because for each $x \in X$, the requirement that $f(x) = r_i$ imposes an affine linear relation on f. By Lemma 1, this subspace satisfies the inequality

$$\left| \frac{|V \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} - \frac{|V \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|} \right| \le \tfrac{1}{36} 2^{-q}. \tag{1}$$

Define $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ to be the uniform distributions over $(\frac{n}{2} - 1)$-linear and $(\frac{n}{2} + 1)$-linear functions, respectively. By our bijection, $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ correspond to the uniform distributions over $W_{\frac{n}{2}-1}$ and $W_{\frac{n}{2}+1}$. As a result, the probability that a function drawn from $\mathcal{D}_{\text{yes}}$ or from $\mathcal{D}_{\text{no}}$ returns the response r to the set of queries X is

$$\Pr_{f \sim \mathcal{D}_{\text{yes}}}[f(X) = r] = \frac{|V \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} \quad \text{and} \quad \Pr_{f \sim \mathcal{D}_{\text{no}}}[f(X) = r] = \frac{|V \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|}.$$

So (1) and Lemma 2 imply that at least $\frac{n}{2} - O(n^{2/3})$ queries are required to distinguish $(\frac{n}{2}-1)$-linear and $(\frac{n}{2}+1)$-linear functions. All $(\frac{n}{2}+1)$-linear functions are $\frac{1}{2}$-far from $(\frac{n}{2} - 1)$-linear functions, so this completes the proof of the theorem for $k = \frac{n}{2} - 1$.

For other values of k, we apply a simple padding argument. When $k < \frac{n}{2} - 1$, modify $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ to be uniform distributions over k-linear and $(k + 2)$-linear functions, respectively, under the restriction that all coordinates in the sum taken from the set $[2k+2]$. This modification with $k = \frac{n}{2} - 2$ shows that $\frac{n}{2} - O(n^{\frac{2}{3}})$ queries are required to distinguish $(\frac{n}{2}-2)$- and $\frac{n}{2}$-linear functions; this implies the lower bound in the theorem for the case $k = \frac{n}{2}$. $\square$

Proof (of Lemma 1). For any set $A \subseteq \mathbb{F}_2^n$, define $I_A : \mathbb{F}_2^n \to \{0,1\}$ to be the indicator function for A. For a given function $f : \mathbb{F}_2^n \to \{0,1\}$, let us write $\mathbf{E}[f]$ as shorthand for $\mathbf{E}_x[f(x)]$ where the expectation is over the uniform distribution of $x \in \mathbb{F}_2^n$. Similarly, for two functions f, g, we write $\mathbf{E}[f \cdot g]$ as short-hand for $\mathbf{E}_x[f(x) \cdot g(x)]$.

For any subsets $A, B \subseteq \mathbb{F}_2^n$, $|A \cap B| = 2^n \cdot \mathbf{E}[I_A \cdot I_B]$. Since $|W_{\frac{n}{2}-1}| = |W_{\frac{n}{2}+1}| = \binom{n}{\frac{n}{2}-1}$,

$$\left| \frac{|V \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} - \frac{|V \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|} \right| = \frac{2^n}{\binom{n}{\frac{n}{2}-1}} \cdot \mathbf{E}\big[I_V \cdot (I_{W_{\frac{n}{2}-1}} - I_{W_{\frac{n}{2}+1}})\big].$$

The subspace V can be defined by a set $S \subseteq [n]$ of size $|S| = d$ and an affine-linear function $f : \{0,1\}^{n-d} \to \{0,1\}^d$, where $x \in V$ iff $x_S = f(x_{\bar{S}})$. Define I_m^S and $I_m^{\bar{S}}$ to be indicator functions for $|x_S| = m$ and $|x_{\bar{S}}| = m$, respectively. Then

$$\mathbf{E}[I_V \cdot (I_{W_{\frac{n}{2}-1}} - I_{W_{\frac{n}{2}+1}})] = \sum_{m=0}^{d} \mathbf{E}\left[I_V \cdot I_m^S \cdot (I^{\bar{S}}_{\frac{n}{2}-m-1} - I^{\bar{S}}_{\frac{n}{2}-m+1})\right].$$

Let $U \subseteq \{0,1\}^S$ be the image of f. Let $d' = \dim(U)$. Define $h_m : \{0,1\}^S \to [-1,1]$ by setting $h_m(u) = \mathbf{E}_{x \in \{0,1\}^{\bar{S}}}[I_V(x,u) \cdot (I^{\bar{S}}_{\frac{n}{2}-m-1}(x) - I^{\bar{S}}_{\frac{n}{2}-m+1}(x))]$. Note that $h_m = f_*\left(I^{\bar{S}}_{\frac{n}{2}-m-1} - I^{\bar{S}}_{\frac{n}{2}-m+1}\right)$. Notice also that h_m is supported on U. We have

$$\mathbf{E}[I_V \cdot (I_{W_{\frac{n}{2}-1}} - I_{W_{\frac{n}{2}+1}})] = \sum_{m=0}^{d} \mathbf{E}\left[I_m^S \cdot h_m\right] = \sum_{m=0}^{d} \mathbf{E}\left[I_m^S \cdot \mathbf{1}_U \cdot h_m\right]. \quad (2)$$

Two applications of the Cauchy-Schwarz inequality yield

$$\sum_{m=0}^{d} \mathbf{E}\left[I_m^S \cdot \mathbf{1}_U \cdot h_m\right] \le \sum_{m=0}^{d} \|I_m^S \cdot \mathbf{1}_U\|_2 \cdot \|h_m\|_2 \le \sqrt{\sum_{m=0}^{d} \|I_m^S \cdot \mathbf{1}_U\|_2^2} \cdot \sqrt{\sum_{m=0}^{d} \|h_m\|_2^2}. \quad (3)$$

We bound the two terms on the right-hand side. The first term satisfies

$$\sum_{m=0}^{d} \|I_m^S \cdot \mathbf{1}_U\|_2^2 = \sum_m \mathbf{E}_x[I_m^S(x)^2 \cdot \mathbf{1}_U] = \mathbf{E}_x\left[\mathbf{1}_U \sum_m I_m^S(x)^2\right] = 2^{d'-d}, \quad (4)$$

where the last equality follows from the fact that for every $x \in \{0,1\}^n$, there is exactly one m for which $I_m^S(x) = 1$.

We now examine the second term. By Parseval's Identity, $\|h_m\|_2^2 = \sum_{\alpha \in \{0,1\}^S} \hat{h}_m(\chi_\alpha)^2$. Suppose that the image of f has dimension $d' \le d$. Then, since h_m is a pushforward,

$$\hat{h}_m(\chi) = 2^{-d}\left(\widehat{I}^{\bar{S}}_{\frac{n}{2}-m-1}(\chi \circ f) - \widehat{I}^{\bar{S}}_{\frac{n}{2}-m+1}(\chi \circ f)\right).$$

The characters $\chi \circ f$ depend only on the restriction of χ to $f(\{0,1\}^{\bar{S}})$. Thus these characters all lie in some subspace $W \subseteq \widehat{\{0,1\}^{\bar{S}}}$ of dimension d', with each character appearing $2^{d-d'}$ times. Thus, we have that

$$\|h_m\|_2^2 = 2^{-d-d'} \sum_{\chi \in W} \left(\widehat{I}^{\bar{S}}_{\frac{n}{2}-m-1}(\chi) - \widehat{I}^{\bar{S}}_{\frac{n}{2}-m+1}(\chi)\right)^2.$$

For any set $\chi \subseteq \bar{S}$, we can apply Facts 8 and 7(i) to obtain

$$\widehat{I}^{\bar{S}}_{\frac{n}{2}-m+1}(\chi) - \widehat{I}^{\bar{S}}_{\frac{n}{2}-m-1}(\chi) = 2^{-(n-d)} K^{n-d+2}_{\frac{n}{2}-m+1}(|\chi|+1).$$

Therefore, $\sum_{m=0}^{d} \|h_m\|_2^2 = 2^{-2n+d-d'} \sum_m \sum_{\chi\in W} K^{n-d+2}_{\frac{n}{2}-m+1}(|\chi|+1)^2$ and by Fact 7(ii),

$$\sum_{m=0}^{d} \|h_m\|_2^2 \leq 2^{-2n+d-d'} \sum_{\chi\in W} (-1)^{|\chi|+1} K^{2(n-d+1)}_{n-d+1}(2|\chi|+2). \tag{5}$$

There exist some d' coordinates such that the projection of W onto those coordinates is surjective. Therefore the number of elements of W with weight at most ℓ is at most $\sum_{j=1}^{\ell} \binom{d'}{j}$. We also have a similar bound on the number of elements of W of size at least $n-d-\ell$. Therefore, since by Fact 7(iv) the summand in (5) is decreasing in $\min(|\chi|, n-d-|\chi|)$, we have

$$\sum_{m=0}^{d} \|h_m\|_2^2 \leq 2^{-2n+d-d'+1} \sum_{j=0}^{d'} \binom{d'}{j} (-1)^{j+1} K^{2(n-d+1)}_{n-d+1}(2j+2).$$

By Fact 7(iii), the sum on the right-hand side evaluates to $-K^{2(n-d-d'+1)}_{n-d-d'+1}(2)$. We can then apply the generating function representation of Krawtchouk polynomials to obtain

$$\begin{aligned}\sum_{m=0}^{d} \|h_m\|_2^2 &\leq -2^{-2n+d+d'+1}[x^{n-d-d'+1}](1-x)^2(1+x)^{2(n-d-d')}\\ &= 2^{-2n+d+d'+2}\left(\binom{2(n-d-d')}{n-d-d'} - \binom{2(n-d-d')}{n-d-d'-1}\right)\\ &= 2^{-d-d'}\Theta(n-d-d')^{-3/2} = 2^{-d-d'}O\left((n-2d)^{-3/2}\right).\end{aligned}$$

Thus we have that

$$\mathbf{E}[I_V\cdot(I_{W_{\frac{n}{2}+1}} - I_{W_{\frac{n}{2}-1}})] \leq \sqrt{2^{d'-d}}\sqrt{2^{-d-d'}O\left((n-2d)^{-3/2}\right)} = 2^{-d}O\left((n-2d)^{-3/4}\right).$$

When $d = \frac{n}{2} - cn^{2/3}$ for some large enough constant $c > 0$, we therefore have $\mathbf{E}[I_V\cdot(I_{W_{\frac{n}{2}+1}} - I_{W_{\frac{n}{2}-1}})] < \frac{1}{36}\binom{n}{\frac{n}{2}-1}2^{-n-d}$ and the lemma follows. $\square$

4 Non-Adaptive Lower Bound

The strategy for the proof of Theorem 2 is similar to that of the proof of the general lower bound in the last section. Once again, we reduce the problem to a geometric problem on the Hamming cube. The main difference is that in this case we prove the following lemma.

Lemma 3. *There is a constant $d_0 > 0$ such that for any linear subspace $V \subseteq \{0,1\}^n$ of codimension $d \leq n - d_0$,*

$$\sum_{x \in \{0,1\}^n/V} \left(\frac{|(V+x) \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} - \frac{|(V+x) \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|} \right)^2 \leq \tfrac{1}{3} 2^{-d}.$$

The proof of Lemma 3 again proceeds through Fourier analysis and the manipulation of Krawtchouk polynomials. Due to space constraints, the details are deferred to Appendix D.

5 Upper Bounds

We provide a sketch of the proofs of Theorems 3 and 4 in this section.

Let us begin by describing the algorithm for distinguishing $\frac{n}{2}$-linear and $(\frac{n}{2}+2)$-linear functions. The starting point for this algorithm is an elementary observation: $\frac{n}{2} \not\equiv \frac{n}{2} + 2 \pmod 4$. For a set $S \subseteq [n]$, let $x_S \in \mathbb{F}_2^n$ be the vector with value 1 at each coordinate in S and 0 in the remaining coordinates. Query $f(x_{\{1,2\}}), f(x_{\{3,4\}}), \ldots, f(x_{\{n-1,n\}})$. Let m denote the number of queries that returned 1. Define the set $T = \bigcup_{i : f(x_{\{2i-1,2i\}})=0} \{2i\}$. Query $f(x_T)$; if $f(x_T) = 1$, increment m by 2. When f is k-linear, we have $m \equiv k \pmod 4$ and this algorithm completes the proof of the first claim in Theorem 4.

The algorithm that proves the more general claim in Theorem 4 is obtained by applying the same approach recursively. When $b > 0$ is the minimum integer for which $2^b \nmid \ell$ and f is k-linear, we can determine the value of k modulo 2^b in b rounds and thereby distinguish between the cases where $k = \frac{n}{2}$ and $k = \frac{n}{2} + 2\ell$.

Finally, to complete the proof of Theorem 3, we essentially combine the Blum–Luby–Rubinfeld (BLR) linearity test [8] with the algorithm described above. The BLR test rejects functions that are far from linear; after that, the problem of testing k-linearity is essentially equivalent to that of distinguishing k-linear from functions that are k'-linear for some $k' \neq k$. For the complete proofs of Theorems 3 and 4, see Appendix E.

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A Krawtchouk Polynomials

We include the proofs for the facts related to Krawtchouk polynomials that we introduced in Section 2. All these facts follow from elementary manipulations of the generating function representation of Krawtchouk polynomials. We include these proofs for the convenience of the reader; for a more complete reference on Krawtchouk polynomials, see [22, 23].

Fact 7. *Fix $n > 0$. Then*

i. *For every $2 \le k \le n$, $K_k^n(m) - K_{k-2}^n(m) = K_k^{n+2}(m+1)$.*

ii. $\sum_{k=0}^n K_k^n(m)^2 = (-1)^m K_n^{2n}(2m)$.

iii. *For every $0 \le d \le \frac{n}{2}$,* $\displaystyle\sum_{j=0}^{d} \binom{d}{j} (-1)^j K_{\frac{n}{2}}^n(2j+2) = 2^{2d} K_{\frac{n}{2}-d}^{n-2d}(2)$.

iv. $K_n^{2n}(2m+1) = 0$ *and* $(-1)^m K_n^{2n}(2m)$ *is positive and decreasing in* $\min\{m, n-m\}$.

Proof. We prove each statement individually.

i. The first statement follows directly from the generating function representation of Krawtchouk polynomials.

$$\begin{aligned} K_k^n(m) - K_{k-2}^n(m) &= \left([x^k]\ (1-x)^m(1+x)^{n-m}\right) - \left([x^{k-2}]\ (1-x)^m(1+x)^{n-m}\right) \\ &= [x^k]\ (1-x)^m(1+x)^{n-m}(1-x^2) \\ &= [x^k]\ (1-x)^{m+1}(1+x)^{n-m+1} = K_k^{n+2}(m+1). \end{aligned}$$

ii. By some more elementary manipulation of generating functions, we have

$$\begin{aligned} K_k^n(m) &= [x^k]\ (1-x)^m(1+x)^{n-m} \\ &= [x^{-k}]\ (1-\tfrac{1}{x})^m(1+\tfrac{1}{x})^{n-m} \\ &= [x^{n-k}]\ (x-1)^m(x+1)^{n-m} = (-1)^m K_{n-k}^n(m). \end{aligned}$$

Therefore,

$$\sum_{k=0}^{n} K_k^n(m)^2 = (-1)^m \sum_{k=0}^{n} K_k^n(m) K_{n-k}^n(m).$$

The Cauchy product of two sequences $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$ is

$$\Big(\sum_{n\geq 0} a_n\Big)\Big(\sum_{n\geq 0} b_n\Big) = \sum_{n\geq 0}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big).$$

Let $a_k = b_k = [x^k]\ (1-x)^m(1+x)^{n-m}$. Then $(\sum_{n\geq 0} a_n) = (1-x)^m(1+x)^{n-m}$ and

$$\sum_{k=0}^{n} K_k^n(m) K_{n-k}^n(m) = [x^n]\ (1-x)^{2m}(1+x)^{2(n-m)} = K_n^{2n}(2m).$$

iii. Considering generating functions and applying the binomial theorem, we get

$$\begin{aligned}\sum_{j=0}^{d}\binom{d}{j}(-1)^j K_n^{2n}(2j+2) &= [x^n]\ \sum_{j=0}^{d}\binom{d}{j}(-1)^j(1-x)^{2j+2}(1+x)^{2n-2j-2}\\ &= [x^n]\ (1-x)^2(1+x)^{2n-2d-2}\sum_{j=0}^{d}\binom{d}{j}\big(-(1-x)^2\big)^j\big((1+x)^2\big)^{d-j}\\ &= [x^n]\ (1-x)^2(1+x)^{2n-2d-2}(4x)^d = 2^{2d}K_{n-d}^{2(n-d)}(2).\end{aligned}$$

iv. By the last statement, $K_n^{2n}(2m+1)$ is pure imaginary. Since it is also real, it must be 0.
The last statement also yields

$$\begin{aligned}(-1)^m K_n^{2n}(2m) &= \frac{2^{2n-1}}{\pi}\int_0^{2\pi}\sin^{2m}(\theta)\cos^{2n-2m}(\theta)d\theta\\ &= \frac{2^{2n-2}}{\pi}\int_0^{2\pi}\sin^{2m}(\theta)\cos^{2n-2m}+\cos(\theta)^{2m}\sin(\theta)^{2n-2m}(\theta)d\theta.\end{aligned}$$

By AM-GM, for fixed n, the integrand is a decreasing function of $\min\{m, n-m\}$. $\square$

Fact 9. *Fix $n > 0$ and $-\frac{n}{2} \leq k \leq \frac{n}{2}$. Then*

$$K_{\frac{n}{2}+k}^n(m) = \frac{2^{n-1}}{\pi} i^m \int_0^{2\pi} \sin^m\theta\cos^{n-m}\theta e^{i2k\theta}\,\mathrm{d}\theta.$$

Proof. By elementary manipulation of generating functions, we obtain

$$\begin{aligned} K^n_{\frac{n}{2}+k}(m) &= [x^{\frac{n}{2}+k}]\ (1-x)^m(1+x)^{n-m} \\ &= [x^k]\ (\tfrac{1}{\sqrt{x}} - \sqrt{x})^m(\tfrac{1}{\sqrt{x}} + \sqrt{x})^{n-m} \\ &= [x^{-2k}]\ (x - \tfrac{1}{x})^m(x + \tfrac{1}{x})^{n-m}. \end{aligned}$$

Applying Cauchy's integral formula to this expression, we get

$$K^n_{\frac{n}{2}+k}(m) = \frac{1}{2\pi}\int_0^{2\pi} (e^{i\theta} - e^{-i\theta})^m (e^{i\theta} + e^{-i\theta})^{n-m} e^{i2k\theta}\, \mathrm{d}\theta.$$

From the trigonometric identities $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, we get

$$K^n_{\frac{n}{2}+k}(m) = \frac{2^n}{2\pi} i^m \int_0^{2\pi} \sin^m\theta \cos^{n-m}\theta e^{i2k\theta}\, \mathrm{d}\theta.$$

□

Fact 8. (Restated) *Fix $0 \le k \le n$ and $\alpha \in \{0,1\}^n$. Then*

$$\widehat{I}_{W_k}(\alpha) = 2^{-n} K_k(|\alpha|).$$

Proof. The Fourier coefficient of I_{W_k} at α is

$$\begin{aligned} \widehat{I}_{W_k}(\alpha) &= 2^{-n} \sum_{x \in \{0,1\}^n : |x| = k} (-1)^{\alpha \cdot x} \\ &= 2^{-n} \sum_{j=0}^{k} (-1)^j \binom{|\alpha|}{j}\binom{n-|\alpha|}{k-j} \\ &= 2^{-n} K^n_k(|\alpha|). \end{aligned}$$

□

B Property Testing Lemmas

We complete the proofs of Lemma 2 and a similar lemma for proving lower bounds on the query complexity for non-adaptive testers. We also provide the proof of the claim in the introduction that the k-linearity and $(n-k)$-linearity testing problems have the same query complexity.

Lemma 2. (Restated) *Let $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ be any two distributions over functions $\{0,1\}^n \to \{0,1\}$. If for every set $X \subseteq \{0,1\}^n$ of size $|X| = q$ and any vector $r \in \{0,1\}^q$ we have that*

$$\left| \Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right| < \tfrac{1}{36}\, 2^{-q},$$

then any algorithm that distinguishes functions drawn from $\mathcal{D}_{\text{yes}}$ from those drawn from $\mathcal{D}_{\text{no}}$ with probability at least $\frac{2}{3}$ makes at least $q+1$ queries.

Proof. Define $\mathcal{D}$ to be the distribution obtained by drawing a function from $\mathcal{D}_{\text{yes}}$ or from $\mathcal{D}_{\text{no}}$, each with probability 1/2. By Yao's Minimax Principle[24], to prove the lemma it suffices to show that any deterministic testing algorithm needs at least $q+1$ queries to distinguish functions drawn from $\mathcal{D}_{\text{yes}}$ or from $\mathcal{D}_{\text{no}}$ with probability at least 2/3.

A deterministic testing algorithm can be described by a decision tree with a query $x \in \{0,1\}$ at each internal node and a decision to accept or reject at every leaf. Each boolean function f defines a path through the tree according to the value of $f(x)$ at each internal node.

Consider a testing algorithm that makes at most q queries. Then it has depth at most q and at most 2^q leaves. Let us call a leaf ℓ *negligible* if the probability that a function $f \sim \mathcal{D}$ defines a path that terminates at ℓ is at most $\frac{1}{12}2^{-q}$. The total probability that $f \sim \mathcal{D}$ defines a path to a negligible leaf is at most $\frac{1}{12}$.

Fix ℓ to be some non-negligible leaf. This leaf corresponds to a set $X \subseteq \{0,1\}^n$ of q queries and a vector $r \in \{0,1\}^q$ of responses; a function f defines a path to the leaf ℓ iff $f(X) = r$. Since ℓ is non-negligible, $\Pr_{f\sim\mathcal{D}}[f(X) = r] > \frac{1}{12}2^{-q}$. So by the hypothesis of the lemma,

$$\left| \Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right| \le \tfrac{1}{36}\, 2^{-q} < \tfrac{1}{3} \Pr_{f\sim\mathcal{D}}[f(X) = r].$$

Then by Bayes' theorem

$$\begin{aligned} &\left| \Pr_{f\sim\mathcal{D}}[f \in \mathcal{P} \mid f(X) = r] - \Pr_{f\sim\mathcal{D}}[f \ \epsilon\text{-far from } \mathcal{P} \mid f(X) = r] \right| \\ &\qquad = \left| \frac{\Pr_{f\sim\mathcal{D}_{\text{yes}}}[f(X) = r] - \Pr_{f\sim\mathcal{D}_{\text{no}}}[f(X) = r]}{2\Pr_{f\sim\mathcal{D}}[f(X) = r]} \right| < \frac{1}{6}. \end{aligned}$$

Therefore, the probability that the testing algorithm correctly classifies a function $f \sim \mathcal{D}$ that lands at a non-negligible leaf ℓ is less than $\frac{7}{12}$.

So even if the algorithm correctly classifies all functions that land in negligible leaves, it still correctly classifies f with probability less than $\frac{11}{12} \cdot \frac{7}{12} + \frac{1}{12} < \frac{2}{3}$, so it is not a valid tester for $\mathcal{P}$. $\square$

Lemma 4. *Let $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ be any two distributions over functions $\mathbb{F}_2^n \to \mathbb{F}_2$. If for every set $X \subseteq \mathbb{F}_2^n$ of size $|X| = q$ we have that*

$$\sum_{r \in \{0,1\}^q} \left(\Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right)^2 < \tfrac{1}{9} 2^{-q},$$

then any non-adaptive algorithm that distinguishes functions drawn from $\mathcal{D}_{\text{yes}}$ from those drawn from $\mathcal{D}_{\text{no}}$ with probability at least $\frac{2}{3}$ makes at least $q+1$ queries.

Proof. As in the proof of Lemma 2, let $\mathcal{D}$ denote the distribution that obtained by drawing a function from $\mathcal{D}_{\text{yes}}$ or from $\mathcal{D}_{\text{no}}$, each with probability $\frac{1}{2}$. By Yao's Minimax Principle, the proof is completed by showing that any deterministic non-adaptive testing algorithm requires at least $q+1$ queries to distinguish functions drawn from $\mathcal{D}_{\text{yes}}$ or $\mathcal{D}_{\text{no}}$ with probability at least $\frac{2}{3}$.

A deterministic non-adaptive testing algorithm queries all functions on a fixed set X of queries, and must accept or reject strictly based on the values of $f(X)$. When $|X| = q$, the condition in the lemma and the Cauchy-Schwarz inequality imply that

$$\sum_{r \in \{0,1\}^q} \left| \Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right| < \frac{1}{3}. \tag{6}$$

This completes, the proof, since the maximum success probability of the algorithm is

$$\sum_{r \in \{0,1\}^q} \max \left\{ \Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r], \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right\} \le$$

$$\tfrac{1}{2} + \tfrac{1}{2} \sum_{r \in \{0,1\}^q} \left| \Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right| < \tfrac{2}{3}. \qquad \square$$

Proposition 1. *Fix $0 \le k \le n$. For any $0 < \epsilon < \frac{1}{2}$, the query complexities for testing k-linearity and $(n-k)$-linearity are identical.*

Proof. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be the function being tested for k-linearity. Let $g : \mathbb{F}_2^n \to \mathbb{F}_2$ be the function defined by setting $g(x) = f(x) + \chi_{[n]}(x)$, where $\chi_{[n]}$ is the parity function over all bits. Then if $f : x \mapsto \sum_{i \in S} x_i$, we have $g : x \mapsto \sum_{i \in [n] \setminus S} x_i$. In particular, if f is k-linear, then g is $(n-k)$-linear. Furthermore, if f is ϵ-far from k-linear, then g is also ϵ-far from $(n-k)$-linear. And, lastly, for any $x \in \mathbb{F}_2^n$, we can obtain the value of $g(x)$ by querying the value of f on a single input, namely, by querying $f(x)$. Therefore, we can use a $(n-k)$-linearity tester to test if f is k-linear without any loss in the query complexity.

The same argument obviously also shows that we can use a k-linearity tester to test $(n-k)$-linearity without any loss in the query complexity; the proposition follows. □

C Proofs of Corollaries 1 and 2

Definition 1 (Juntas). *The function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is a k-*junta *if there is a set $J \subseteq [n]$ of size $|J| \le k$ such that for each $x, y \in \mathbb{F}_2^n$ that satisfy $x_i = y_i$ for every $i \in J$, the identity $f(x) = f(y)$ holds.*

Definition 2 (Sparse polynomials). *The function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ has a unique representation as a multivariate polynomial over the variables $x_1, \ldots, x_n$. If this representation has at most k non-zero coefficients, we say that f is a k-*sparse $\mathbb{F}_2$-polynomial.

Definition 3 (Fourier degree). *The* Fourier degree *of a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is the maximum Hamming weight of any $\alpha \in \mathbb{F}_2$ such that the Fourier coefficient $\hat{f}(\chi_\alpha)$ is non-zero.*

Definition 4 (Decision trees). *A* decision tree *is a model of computation that can be represented as a rooted binary tree with each internal node of the tree labeled with an index $i \in [n]$ and the two edges going from a node to its children labeled with $0, 1$. The leaves of the tree are also labeled with $0, 1$. A decision tree* computes *the function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ if for every $x \in \mathbb{F}_2^n$, the path from the root to a leaf followed by taking the edge x_i at each node i leads to a leaf labeled with $f(x)$. The* depth *of a tree is the maximum length of any path from its root to one of its leaves.*

Definition 5 (Function isomorphism). *Given a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$, the f-*isomorphism *property includes all functions that are equal to f up to relabeling of the n variables. In other words, g is isomorphic to f if there exists a permutation $\pi \in \mathcal{S}_n$ such that for every $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$, $g(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.*

Corollary 1. (Restated) *Fix* $1 \le k \le \frac{n}{2}$. *At least* $k - O(k^{2/3})$ *queries are required to test*

(1) k*-juntas,*
(2) k*-sparse* $\mathbb{F}_2$*-polynomials,*
(3) functions of Fourier degree at most k*,*
(4) functions computable by depth-k *decision trees, and*
(5) isomorphism to the function $f : x \mapsto x_1 + \cdots + x_k$*.*

Proof. Recall that in our proof of Theorem 1, we showed that at least $k - O(k^{\frac{2}{3}})$ queries are required to distinguish k-linear and $(k+2)$-linear functions.

As we can easily verify, k-linear functions are k-juntas, they are k-sparse $\mathbb{F}_2$-polynomials, they have Fourier degree at most k, and they can be computed by a (complete) decision tree of depth k. To complete the proof of cases (1)–(4) of the corollary, it suffices to show that $(k+2)$-linear functions are $\frac{1}{2}$-far from those same properties. This is indeed the case, as Fischer et al. [12] showed for the k-junta property and Diakonikolas et al. [10] showed for the other three properties.

Finally, case (5) of the corollary follows immediately from the observation that the set of functions isomorphic to the function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ defined by $f(x) = x_1 + \cdots + x_k$ is exactly the set of k-linear functions. □

Corollary 2. (Restated) *Let* $\mathcal{P}$ *be a symmetric property of linear functions. Then at least* $\Gamma_{\mathcal{P}} - O(\Gamma_{\mathcal{P}}^{2/3})$ *queries are required to test* $\mathcal{P}$*.*

Proof. Once again, recall that the proof of Theorem 1 shows that at least $k - O(k^{\frac{2}{3}})$ queries are required to distinguish k-linear and $(k+2)$-linear functions. This also implies that the same number of queries are required to distinguish $(n-k)$-linear and $(n-k-2)$-linear functions.

By definition of $\Gamma_{\mathcal{P}}$, at least one of the inequalities $h_{\mathcal{P}}(\Gamma_{\mathcal{P}} - 1) \neq h_{\mathcal{P}}(\Gamma_{\mathcal{P}} + 1)$ or $h_{\mathcal{P}}(n - \Gamma_{\mathcal{P}} - 1) \neq h_{\mathcal{P}}(n - \Gamma_{\mathcal{P}} + 1)$ must hold. In either case, the corollary follows from the lower bounds above. □

D Proof of the Non-Adaptive Lower Bound

Proof (of Theorem 2). The proof of this theorem is very similar to the proof of Theorem 1. Let $k = \frac{n}{2} - 1$. Recall that linear functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$ can be represented as $f : x \mapsto \alpha \cdot x$ for some $\mathbb{F}_2^n$. This representation gives a natural bijection between the set of linear functions and $\mathbb{F}_2^n$. Let $W_\ell \subseteq \mathbb{F}_2^n$ denote the set of elements of Hamming weight ℓ. For any set $X \subseteq \mathbb{F}_2^n$ of

$q < n - O(1)$ queries and any response vector $r \in \mathbb{F}_2^q$, the set of linear functions that gives the response r to the queries X corresponds to an affine subspace $V + x \subseteq \mathbb{F}_2^n$ of codimension q. From Lemma 3,

$$\sum_{x \in \mathbb{F}_2^n / V} \left(\frac{|(V+x) \cap W_{\frac{n}{2}-1}|}{|W_{\frac{n}{2}-1}|} - \frac{|(V+x) \cap W_{\frac{n}{2}+1}|}{|W_{\frac{n}{2}+1}|} \right)^2 \leq \tfrac{1}{3} 2^{-d}. \tag{7}$$

Define $\mathcal{D}_{\text{yes}}$ and $\mathcal{D}_{\text{no}}$ to be uniform distributions over $(\frac{n}{2} - 1)$-linear and $(\frac{n}{2} + 1)$-linear functions, respectively. These distributions correspond by our bijection to the uniform distributions over $W_{\frac{n}{2}-1}$ and $W_{\frac{n}{2}+1}$, and so (7) implies that

$$\sum_{r \in \mathbb{F}_2^q} \left(\Pr_{f \sim \mathcal{D}_{\text{yes}}} [f(X) = r] - \Pr_{f \sim \mathcal{D}_{\text{no}}} [f(X) = r] \right)^2 \leq \tfrac{1}{3} 2^{-d}. \tag{8}$$

By Lemma 4, any non-adaptive algorithm that distinguishes $(\frac{n}{2}-1)$-linear from $(\frac{n}{2}+1)$-linear functions must therefore make at least $n - O(1)$ queries. This gives the desired lower bound for testing $(\frac{n}{2} - 1)$-linearity. We apply the same padding argument as in the proof of Theorem 1 to get the lower bound for the other values of k. □

Proof (of Lemma 3). As in the last section, define $I_A : \{0,1\}^n \to \{0,1\}$ to be the indicator function for the set $A \subseteq \{0,1\}^n$. To prove Lemma 3, we want to show that

$$\sum_{x \in \{0,1\}^n / V} \left(\frac{\mathbf{E}[I_{V+x} \cdot I_{W_{\frac{n}{2}-1}}]}{\mathbf{E}[I_{W_{\frac{n}{2}-1}}]} - \frac{\mathbf{E}[I_{V+x} \cdot I_{W_{\frac{n}{2}+1}}]}{\mathbf{E}[I_{W_{\frac{n}{2}+1}}]} \right)^2 \leq \tfrac{1}{3} 2^{-d}.$$

Let $D_{\frac{n}{2}} = I_{W_{\frac{n}{2}-1}} - I_{W_{\frac{n}{2}+1}}$, and note that $\mathbf{E}[I_{W_{\frac{n}{2}-1}}] = \mathbf{E}[I_{W_{\frac{n}{2}+1}}] = \binom{n}{\frac{n}{2}-1}/2^n$. Then the above inequality is equivalent to

$$\sum_{x \in \{0,1\}^n / V} \mathbf{E}[I_{V+x} \cdot D_{\frac{n}{2}}]^2 \leq \tfrac{1}{3} 2^{-d} \cdot \left(\frac{\binom{n}{\frac{n}{2}-1}}{2^n} \right)^2.$$

Let $\pi : \{0,1\}^n \to \{0,1\}^n / V$ be the projection map. Notice that $\mathbf{E}[I_{V+x} \cdot D_{\frac{n}{2}}] = \pi_* D_{\frac{n}{2}}(x)$. By Parseval's Theorem and Fact 6,

$$\mathbf{E}_{x \in \{0,1\}^n / V} [\pi_* D_{\frac{n}{2}}(x)^2] = |\pi_* D_{\frac{n}{2}}|_2^2 = \sum_{\chi \in \widehat{\{0,1\}^n / V}} \widehat{\pi_* D_{\frac{n}{2}}}(\chi) = 2^{d-n} \sum_{\chi \in V^\perp} \widehat{D_{\frac{n}{2}}}(\chi).$$

Where above $V^\perp$ is the set of pullbacks of $\widehat{\{0,1\}^n/V}$ to $\widehat{\{0,1\}^n}$, which is the space of characters of $\{0,1\}^n$ that are trivial on V.

By Fact 8, $\widehat{D}_{\frac{n}{2}}(\chi) = 2^{-n} K^{n+2}_{\frac{n}{2}+1}(|\chi|+1)$. By Fact 7(iv), the absolute value of this is 0 for $|\chi|$ even and otherwise decreasing in $\min(|\chi|, n-|\chi|)$. Since there are at most $2\sum_{j=0}^{\ell}\binom{d}{j}$ $\chi \in V^\perp$ with $\min(|\chi|, n-|\chi|) < \ell$, the above sum is less than it would be if there were 2 $\chi \in V^\perp$ with $|\chi| = 0$, $2\left(\binom{d}{2}+\binom{d}{1}\right)$ with $|\chi| = 2$, $2\left(\binom{d}{4}+\binom{d}{3}\right)$ with $|\chi| = 4$, and so on. Hence

$$\sum_{x\in\{0,1\}^n/V} \mathbf{E}[I_{V+x}\cdot D_{\frac{n}{2}}]^2 \le 2^{2-2n-d}\sum_{m=0}^{d}\binom{d}{m}K^{n+2}_{\frac{n}{2}+1}(m+1)^2.$$

By Fact 9, we can expand the sum on the right-hand side of the inequality into a double integral. Namely,

$$\begin{aligned}&\sum_{m=0}^{d}\binom{d}{m}K^{n+2}_{\frac{n}{2}+1}(m+1)^2\\ &\quad = \frac{2^{2n}}{\pi^2}\iint\sum_{m=0}^{d}\binom{d}{m}(-1)^m\sin^{m+1}\theta\sin^{m+1}\phi\cos^{n-m+1}\theta\cos^{n-m+1}\phi\,\mathrm{d}\theta\,\mathrm{d}\phi.\end{aligned}$$

Manipulating the trigonometric functions and applying the Cauchy-Schwarz inequality, we eventually obtain the bound

$$\sum_{m=0}^{d}\binom{d}{m}K^{n+2}_{\frac{n}{2}+1}(m+1)^2 \le O\big(2^{2n}d^{-\frac{1}{2}}(n-d+1)^{-\frac{3}{2}}\big). \tag{9}$$

The proof of this bound is presented at the end of this section. Using this bound, we obtain

$$\sum_{x\in\{0,1\}^n/V}\mathbf{E}[I_{V+x}\cdot D_{\frac{n}{2}}]^2 \le O\left(2^{-d}d^{-\frac{1}{2}}(n-d+1)^{-\frac{3}{2}}\right).$$

Note that $2^{-d}\cdot\left(\frac{\binom{n}{\frac{n}{2}-1}}{2^n}\right)^2 = \Theta(2^{-d}n^{-1/2})$. If $d < n/2$, $\sum_{x\in\{0,1\}^n/V}\mathbf{E}[I_{V+x}\cdot D_{\frac{n}{2}}]^2$ is $O(2^{-d}n^{-3/2})$, which is too small. Otherwise it is $O(2^{-d}n^{-1/2}(n-d+1)^{-3/2})$, which is too small as long as $n-d$ is bigger than a sufficiently large constant. $\square$

We now complete the proof of (9).

Claim. Fix $0 \le m \le d \le n$. Then

$$\sum_{m=0}^{d} \binom{d}{m} K^{n+2}_{\frac{n}{2}+1}(m+1)^2 \le O\big(2^{2n} d^{-\frac{1}{2}} (n-d+1)^{-\frac{3}{2}}\big).$$

Proof. By Fact 9 and manipulation of the integrand, we obtain

$$\begin{aligned}
&\sum_{m=0}^{d} \binom{d}{m} K^{n+2}_{\frac{n}{2}+1}(m+1)^2 \\
&\quad = \frac{2^{2n}}{\pi^2} \iint \sum_{m=0}^{d} \binom{d}{m} (-1)^m \sin^{m+1}\theta \sin^{m+1}\phi \cos^{n-m+1}\theta \cos^{n-m+1}\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&\quad = \frac{2^{2n}}{\pi^2} \iint \sin\theta \sin\phi \cos^{n-d+1}\theta \cos^{n-d+1}\phi (\cos\theta\cos\phi - \sin\theta\sin\phi)^d \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&\quad = \frac{2^{2n}}{\pi^2} \iint \sin\theta \sin\phi \cos^{n-d+1}\theta \cos^{n-d+1}\phi \cos^d(\theta - \phi) \, \mathrm{d}\theta \, \mathrm{d}\phi.
\end{aligned}$$

Letting $\psi = \theta - \phi$ this is

$$\frac{2^{2n}}{\pi^2} \iint \sin(\psi + \phi) \sin\phi \cos^{n-d+1}(\psi + \phi) \cos^{n-d+1}\phi \cos^d(\psi) \, \mathrm{d}\phi \, \mathrm{d}\psi.$$

Next we bound the inner integral using Cauchy-Schwarz to obtain the upper bound

$$\frac{2^{2n}}{\pi^2} \int |\cos^d(\psi)| \left(\int \sin^2(\psi+\phi) \cos^{2(n-d+1)}(\psi+\phi) \, \mathrm{d}\phi \right)^{1/2} \left(\int \sin^2\phi \cos^{2(n-d+1)}\phi \, \mathrm{d}\phi \right)^{1/2} \mathrm{d}\psi.$$

This is

$$\frac{2^{2n}}{\pi^2} \left(\int |\cos^d(\psi)| \, \mathrm{d}\psi \right) \left(\int \sin^2\phi \cos^{2(n-d+1)}\phi \, \mathrm{d}\phi \right).$$

Now

$$\begin{aligned}
\frac{1}{2\pi} \left(\int \sin^2\phi \cos^{2(n-d+1)}\phi \, \mathrm{d}\phi \right) &= 2^{-2(n-d+1)-1} \left(\binom{2(n-d+1)}{n-d+1} - \binom{2(n-d+1)}{n-d} \right) \\
&= \Theta\left((n-d+1)^{-3/2} \right).
\end{aligned}$$

If d is even, we have that

$$\frac{1}{2\pi} \int |\cos^d(\psi)| \, \mathrm{d}\psi = 2^{-d} \binom{d}{d/2} = \Theta\left(d^{-1/2} \right).$$

If d is odd, then $\int |\cos^d(\psi)| \, \mathrm{d}\psi$ is bounded between $\int |\cos^{d+1}(\psi)| \, \mathrm{d}\psi$ and $\int |\cos^{d-1}(\psi)| \, \mathrm{d}\psi$, so in this case also it is $\Theta\left(d^{-1/2}\right)$. $\square$

Algorithm 1 (k vs. $k + 2\ell$)-Linearity Tester

Initialize $S_0 = \{1, \dots, n\}$ and $m = 0$.
for $r = 1, \dots, b-1$ **do**
  Initialize $S_r = \emptyset$.
  if $|S_{r-1}|$ is odd **then**
    Choose $i \in S_{r-1}$ and update $S_{r-1} = S_{r-1} \setminus \{i\}$.
    Set $m = m + f(\{i\}) \cdot 2^{r-1}$.
  end if
  Let M be a random matching of S_{r-1}.
  for each pair $(i, j) \in M$ **do**
    if $f(\{i, j\}) = 1$ **then**
      Increment $m = m + 2^{r-1}$.
    else
      Update $S_r = S_r \cup \{i\}$.
    end if
  end for
end for
Output "f is k-linear" iff $m + 2^{b-1} f(S_{b-1}) \equiv k \pmod{2^b}$.

E Proof of Theorems 3 and 4

Theorem 4. (Restated) *We can distinguish $\frac{n}{2}$- and $(\frac{n}{2}+2)$-linear functions with $\lceil \frac{n}{2} \rceil + 1$ queries. More generally, for $\ell \geq 1$, let b be the smallest positive integer for which 2^b does not divide ℓ. It is possible to distinguish $\frac{n}{2}$- and $(\frac{n}{2} + 2\ell)$-linear functions with $\frac{2}{3}(1 - 2^{-2b})n + o(n)$ queries.*

Proof. As described briefly in Section 5, the general approach for the algorithm that obtains the desired bounds is to count the number of variables included in the parity, modulo 2^b, in order to distinguish k-linear from $(k+2\ell)$-linear functions. The details of a tester that implements this approach is described in Algorithm 1.

Let's first examine the correctness of the algorithm. To do so, we need to argue that the algorithm correctly counts the number of variables in the parity, modulo 2^b. This is easily verified by noting that every element $i \in S_r$ is the representative of a set of 2^r elements whose corresponding variables are either all included or all excluded from the parity.

To complete the analysis, we must also analyze the query complexity of the algorithm. In the worst-case, Algorithm 1 may query up to $\frac{n}{2^r}$ inputs in round r, for a total of $\frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots \approx n$ queries. But the expected number of queries is much smaller: since we pick our matching at random and f is nearly balanced, the expected number of queries that return 0 is $\frac{1}{2} \pm o(1)$. Therefore, the expected number of elements in S_r is $\frac{n}{2^{2r}} + o(n)$ and the expected number of queries is $\frac{n}{2} + \frac{n}{8} + \frac{n}{32} + \cdots + \frac{n}{2^{2(b-1)}} + o(n) =$

$\frac{2}{3}(1-2^{-2b})n+o(n)$. Furthermore, with high probability the number of queries required by the algorithm is within $\pm o(n)$ of this expected value; to complete the proof of the theorem, simply run Algorithm 1 with a query quota so that if the quota is reached, we terminate the algorithm and guess. Setting the quota large enough, this termination occurs only with probability $o(1)$, and so we have a valid (k vs. $k+2\ell$)-tester. $\square$

Remark 3. When $\ell > \omega(\sqrt{n})$, the result in Theorem 4 is not optimal. In fact, in this case it is possible to solve the ($\frac{n}{2}-\ell$ vs. $\frac{n}{2}+\ell$)-parity testing problem with $O(n/\ell^2)$ queries with a simple sampling approach. (Sample $O(n/\ell^2)$ elements $i_1,\ldots,i_s$ uniformly at random from $[n]$, query $f(e_{i_1}),\ldots,f(e_{i_s})$, and guess that f is a $\frac{n}{2}-\ell$-parity function iff at most $\frac{1}{2}$ of the queries returned the value 1. By a Chernoff bound argument, with high probably this approach correctly solves the testing problem.)

Theorem 10. *Let $n \geq k \geq 0$. Let $k' = n-k$. There is an adaptive k-linearity ϵ-tester that makes*

$$\sum_{i=0}^{\infty} 2^{-i-1}(k^{2^i}+k'^{2^i}) \prod_{j=0}^{i-1}(k^{2^j}+k'^{2^j})^{-1} + O(\sqrt{n}+1/\epsilon)$$

queries.

Remark 4. Theorem 3 is the special case of Theorem 10 where $k=\frac{n}{2}$.

Proof. We first define an algorithm that assumes the input is a linear function. We discuss how to handle non-linear functions at the end of the proof. We define the algorithm recursively. It tests if the parity of a linear function f on $\{0,1\}^S$ (for some set S) is equal to k with failure probability at most p. We assume for sake of simplicity that $k \leq |S|/2$. Were this not the case, we could test for the parity of the pointwise sum of f with the parity function on all of S. The full algorithm is presented in Algorithm 2.

We have left to verify that Algorithm 2 works in an appropriate runtime. Let

$$h(x,y) = \sum_{i=0}^{\infty} 2^{-i-1}(x^{2^i}+y^{2^i}) \prod_{j=0}^{i-1}(x^{2^j}+y^{2^j})^{-1}.$$

We first note that if f is a k-linear function on $\{0,1\}^S$ for $k \leq |S|/2$, then for a random $i \in S$, then the probability that $f(\{i\})=0$ will be at most $1/2$. Therefore, the probability that $\log_2(6/p)$ such i all have this property is at most $p/6$. Hence if $|S|$ is odd, the probability of failure is at most $p/6$ plus the probability of failure of the recursive call, which is at most $5p/6$.

We note that each of the pairs (i,j) for which $f(\{i,j\}) = 1$ have total weight of exactly one between them. The other pairs have $f(\{i\}) = f(\{j\})$. Therefore the weight of f on S is equal to k if and only if t plus twice the weight of f on T equals k. We note also that if f were weight k on S that the expected value of t would be $\frac{k(|S|-k)}{2|S|}$ with a variance of $O(|S|)$. Hence for C sufficiently large, we only report False in error due to t being too large or small with probability at most $p/6$. This verifies the correctness of the algorithm. We need to verify that it runs in at most $h(k,|S|-k) + O(\sqrt{|S|})$ queries.

We note that the algorithm makes $O(\log(|S|/p)) + |S|/2$ queries before making a recursive call on t,T. If $x = k$ and $y = |S| - k$, we note that we make recursive calls with new values of x and y given by either x and $y-1$ or by $\frac{x^2}{2(x+y)} + c$ and $\frac{y^2}{2(x+y)} + c$ with $c = O(\sqrt{(x+y)/p})$. For the first case, we note that for $x \le y$ that $h(x,y) \le h(x,y-1)$. Upon applying the latter recursion, we note that were c equal to 0 that

$$h(x,y) = (x+y)/2 + h\left(\frac{x^2}{2(x+y)}, \frac{y^2}{2(x+y)}\right).$$

We need to show that having a value of c not equal to 0 does not significantly effect the runtime of the recursive call to the algorithm. In particular we show that it changes the runtime by $O(c)$. We do this by showing that the directional derivative of $h(x,y)$ in the $(1,1)$ direction is $O(1)$. In order to do this we note that $h(x,y) = \sum_{i=0}^{\infty} h_i(x,y)$ where

$$h_i(x,y) = 2^{-i-1}(x^{2^i}+y^{2^i})\prod_{j=0}^{i-1}(x^{2^j}+y^{2^j})^{-1} = \frac{2^{-i-1}(x^{2^i}+y^{2^i})}{x^{2^i-1}+x^{2^i-2}y+\ldots+y^{2^i-1}}.$$

The derivative of h_i in the $(1,1)$ direction is

$$\frac{(x^{2^i-1}+y^{2^i-1})(x^{2^i-1}+x^{2^i-2}y+\ldots+y^{2^i-1}) - (x^{2^i}+y^{2^i})(x^{2^i-2}+x^{2^i-3}y+\ldots+y^{2^i-2})}{2(x^{2^i-1}+x^{2^i-2}y+\ldots+y^{2^i-1})^2}$$

$$= \frac{x^{2^i-1}y^{2^i-1}}{(x^{2^i-1}+x^{2^i-2}y+\ldots+y^{2^i-1})^2} \le 2^{-2i}.$$

Where the last step above is by AM-GM. Thus the directional derivative of $h(x,y)$ is $O(1)$. We prove inductively that for sufficiently large K that our algorithm runs in time at most $h(k,|S|-k) + K\sqrt{|S|/p}$. From the above discussion it is clear that upon this inductive hypothesis and for sufficiently large C' that our algorithm runs in time at most $h(k, |S|-k) + C'\sqrt{|S|/p} + K\sqrt{(|S|/2)/(2p/3)}$. Hence for K a sufficiently large multiple of C', we can complete our inductive step.

We now complete the proof by extending the algorithm to reject all functions—and not just linear functions—that are far from all k-linear functions. First, we add an extra step where we run the Blum–Luby–Rubinfeld linearity tester $O(1/\epsilon)$ times. This rejects all functions that are ϵ-far from linear.

At this point, we are almost, but not quite, done. We can still have functions that are very close to linear (so that they pass the linearity test), very far from k-linear (so that the overall test should reject), and yet that pass the test in Algorithm 2 with high probability. For example, a function f might be consistent with a k-linear function on all inputs of Hamming weight at most 2 and consistent with some $(k+2)$-linear function on the remaining inputs. Since the algorithm only queries the function on inputs of low Hamming weight, it will erroneously accept f.

To remove this last source of error, we add one last step. In this step, we choose uniformly at random one of the queries $\{i, j\}$ that was made by Algorithm 2. We then choose $x \in \mathbb{F}_2^n$ uniformly at random, and set $y \in \mathbb{F}_2^n$ to be identical to x except for y_i and y_j, who take the opposite values of x_i and x_j, respectively. We then verify that $f(x) = f(y)$ iff $f(\{i, j\}) = 0$. Running this test a constant number of times is sufficient to verify that the function is globally consistent with the answers returned by the queries and, in particular, functions that are close to linear but not are not k-linear and still passed the test in Algorithm 2 will fail this test with high probability. $\square$

We remark that the above testing algorithm tests k-linearity in time $cn + o(n)$ for some constant $c < 1$ except when $k = o(n)$. When k is small, however, there is another simple sampling algorithm that can be used to test k-linearity with $O(k \log(n/k))$ queries.

Algorithm 2 Linearity-Tester(k,S,p)

```
if |S| = 0 and k = 0 then
    Return True
end if
if |S| = 0 and k ≠ 0 then
    Return False
end if
if |S| is odd then
    Choose log_2(6/p) random i ∈ S
    for Each chosen i do
        query f({i})
    end for
    if All of these queries return 1 then
        Return False
    else
        For i so that f({i}) = 0, Return Linearty-Tester(k,S − {i},5p/6)
    end if
else
    Let M be a random matching of the elements of S
    Let T = ∅, t = 0
    for (i, j) ∈ M do
        if f({i, j}) = 1 then
            Set t = t + 1
        else
            Set T = T ∪ {i}
        end if
    end for
    if t ≢ k (mod 2) then
        Return False
    else if |t − k(|S|−k)/(2|S|)| > C √(|S|/p) for C a sufficiently large constant then
        Return False
    else
        Return Linearity-Tester((k − t)/2,T,5p/6)
    end if
end if
```