Asymptotics of McKay numbers for S_n

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Abstract

For a partition Λ of n, let $H(\Lambda)$ denote its hook product. If ℓ is prime and $a \geq 0$ an integer, then define

$$p_\ell(a; n) = |\{\Lambda : |\Lambda| = n, a = \mathrm{ord}_\ell(H(\Lambda))\}|.$$

These numbers are simply related to the McKay numbers in the representation theory of the symmetric group. Using a generating function of Nakamura and the "circle method", we determine asymptotic properties of $p_\ell(a; n)$ and $\sum_a (-1)^a p_\ell(a; n)$, resolving questions of Ono. In particular we show that for fixed ℓ and n, $p_\ell(a; n)$ roughly fits a given distribution that is dependent on ℓ, is centered near $n - c_1\sqrt{n}\log n$ and has width $c_2\sqrt{n}$. We also give an asymptotic formula for $\sum_a (-1)^a p_\ell(a; n)$ that is valid whenever $\sqrt{n}$ is not, for any k, within a multiplicative factor of $c \log \ell$ of ℓ^k. This formula is of the form $\pm c(n)/n \exp(\kappa(n)\sqrt{n})$ where c and κ are specific functions of n and the sign is determined by n.

1 Introduction

For a prime number ℓ and a finite group G, let $m_\ell(k; G)$ denote the ℓ^k-power *McKay number* for G. This is the number of irreducible characters Ψ of G so that ℓ^k is the largest power of ℓ dividing $\deg(\Psi)$. Here we investigate the asymptotic properties of these numbers for symmetric groups. The McKay numbers of symmetric groups relate to certain partition numbers.

A *partition* is a non-increasing sequence of natural numbers, $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$. The size of the partition is $|\Lambda| = \sum_{i=1}^m \lambda_i$. The Ferrers-Young diagram for a partition Λ is an array of nodes with λ_k nodes in the k^{th} row. For each node d, we let its hook number, $H(d)$ be the number of nodes in the Ferrers-Young diagram to the right or below d (including d itself). We let $H(\Lambda) = \prod_d H(d)$, and we define, (letting $\mathrm{ord}_\ell(m)$ be the largest k such that $\ell^k | m$)

$$p_\ell(a; n) = |\{\Lambda : |\Lambda| = n, a = \mathrm{ord}_\ell(H(\Lambda))\}|.$$

By the representation theory of the symmetric group (see [4]) we have that

$$p_\ell(a; n) = m_\ell(\mathrm{ord}_\ell(n!) - a; S_n).$$

For a more detailed discussion of the above see [6].

In [6], Ono proves a number of congruence relations satisfied by the numbers $p_\ell(a;n)$. Among other things, he showed that the Ramanujan congruences (see [1]) for the partition numbers generalize to $p_\ell(a;n)$. At the end of his paper, Ono asks about the order of magnitude of $p_\ell(a;n)$ and about the properties of $\sum_a (-1)^a p_\ell(a;n)$ for odd ℓ.

In this paper we will attempt to answer these questions. We will use a generating function of Nakamura and some techniques from analytic number theory. Our overriding approach will be the "circle method". In order to perform the requisite computations we will use techniques such as the inverse mellon transform (as used in [1]), the functional equations for modular forms ([2]), facts about power series with non-negative coefficients, the saddle point method, and approximation of functions by normal distributions.

We will show that for n and ℓ fixed and a a varying, $p_\ell(a;n)$ has distribution that can be normalized to

$$g(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\prod_{k=0}^{\infty}\left(1+\frac{y}{\ell^k}\right)^{\ell^k} e^{-y}\right)^{-(\ell-1)/2} e^{ty}dy$$

near its peak. The distribution of $p_\ell(a;n)$ is centered at roughly $n - c\sqrt{n}\log n$, and has width proportional to $\sqrt{n}$. The distribution is normalized so that its integral is approximately $(4n\sqrt{3})^{-1}\exp(\pi\sqrt{2n/3})$, the asymptotic for the partition number $p(n)$. In particular, we show that if $w_0 = \pi\sqrt{\frac{1}{6n}}$, and

$$\Delta_0 = w_0\left(\left(\left(\frac{\log w_0}{w_0}\right)\left(\frac{\ell-1}{2\log\ell}\right) + \left(\frac{K(w_0)}{w_0}\right)\right) + n - a(\ell-1)\right),$$

where

$$K(w) = \left(\frac{\ell-1}{\log\ell}\right)\left(\frac{-\log(2\pi)}{2} + \frac{\ell\log\ell}{2(\ell-1)} - \frac{\log\ell}{4} + \sum_{t_k=1+2\pi ik/\log\ell}^{k\neq 0}\Gamma(t_k)\zeta(t_k)\zeta(t_k-1)w^{-t_k-1}\right),$$

then if $-O(\log n) < \Delta_0 < O(n^{1/24})$,

$$p_\ell(a;n) = \frac{\sqrt{2}\pi(\ell-1)g(\Delta_0)n^{-3/2}}{24}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1+O(n^{-1/8})).$$

Notice that Δ_0 depends importantly on ℓ, linearly on a, and for fixed a and ℓ it depends on n as $\sqrt{n}$ times a periodic function in $\log_\ell n$.

Below we have a table of supporting evidence for this claim with $\ell = 3$ and $n = 1000$. The table below approximates its values to three significant figures. We let $A_\ell(a;n)$ be the main term of the above approximation. Notice that for this n that $n^{(-1/8)} = .422\ldots$, $\log(n) = 6.91\ldots$ and $n^{1/24} = 1.333\ldots$.

a	$p_\ell(a;n)$	$A_\ell(a;n)$	$\frac{A_\ell(a;n)}{p_\ell(a;n)}$	Δ_0
400	$8.67 \cdot 10^{27}$	$1.07 \cdot 10^{28}$	1.23	4.52
405	$1.30 \cdot 10^{28}$	$1.60 \cdot 10^{28}$	1.23	4.12
410	$1.96 \cdot 10^{28}$	$2.40 \cdot 10^{28}$	1.23	3.71
415	$3.08 \cdot 10^{28}$	$3.60 \cdot 10^{28}$	1.17	3.31
420	$4.90 \cdot 10^{28}$	$5.39 \cdot 10^{28}$	1.10	2.90
425	$7.65 \cdot 10^{28}$	$8.04 \cdot 10^{28}$	1.05	2.49
430	$1.13 \cdot 10^{29}$	$1.19 \cdot 10^{29}$	1.06	2.09
435	$1.76 \cdot 10^{29}$	$1.67 \cdot 10^{29}$	1.05	1.68
440	$2.56 \cdot 10^{29}$	$2.57 \cdot 10^{29}$	1.00	1.28
445	$3.84 \cdot 10^{29}$	$3.65 \cdot 10^{29}$	0.95	0.87
450	$5.39 \cdot 10^{29}$	$5.01 \cdot 10^{29}$	0.93	0.47
455	$7.06 \cdot 10^{29}$	$6.48 \cdot 10^{29}$	0.92	0.06
460	$8.45 \cdot 10^{29}$	$7.61 \cdot 10^{29}$	0.90	-0.34
465	$8.22 \cdot 10^{29}$	$7.67 \cdot 10^{29}$	0.93	-0.75
470	$5.49 \cdot 10^{29}$	$6.08 \cdot 10^{29}$	1.11	-1.16
475	$1.95 \cdot 10^{29}$	$3.28 \cdot 10^{29}$	1.68	-1.56
480	$2.03 \cdot 10^{28}$	$9.64 \cdot 10^{28}$	4.75	-1.97
485	$1.79 \cdot 10^{26}$	$1.06 \cdot 10^{28}$	59.4	-2.37
490	$2.01 \cdot 10^{22}$	$2.42 \cdot 10^{26}$	$1.20 \cdot 10^{4}$	-2.78

We also determine asymptotic information about $\sum_a (-1)^a p_\ell(a;n)$ for odd ℓ. In particular we show that as long as $\sqrt{n}$ is not within a multiplicative factor of $O(\log \ell)$ of a power of ℓ, that

$$\sum_a (-1)^a p_\ell(a;n) = (-1)^{n\lfloor \log_\ell n \rfloor} \frac{c(n)}{n} \exp(\kappa(n)\sqrt{n})(1 + O(n^{-1/4})),$$

where $c(n)$ and $\kappa(n)$ are continuous functions periodic in $\log_\ell(n)$.

Our results are interesting both in that they provide information about the size of the McKay numbers and in that they provide information about the coefficients of infinite products of modular forms, yielding results that are qualitatively different from the results for finite products of modular forms (for example for the sizes of the partition numbers, see [1], [2] or [7]), in that they tend to have some terms periodic in $\log n$. Section 2 will cover some background material that will be used later. In Section 3, we prove the asymptotic results for $p_\ell(a;n)$. In Section 4 we, prove some asymptotic results for $\sum_a (-1)^a p_\ell(a;n)$. Section 5 will contain a summary of the results and a discussion of further questions.

2 Background Information

This section contains a number of results that we will need in this paper. Nakamura proved the existence of following generating function for $p_\ell(a;n)$ (see [5]):

Theorem 1 (Nakamura)**.**

$$F_\ell(x;q) = \sum_{a,n=0}^{\infty} p_\ell(a;n)x^a q^n = \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{(1-x^{\ell n[k]_\ell} q^{\ell^{k+1}n})^{\ell^{k+1}}}{(1-x^{n[k]_\ell} q^{\ell^k n})^{\ell^k}},$$

where $[k]_\ell = \frac{\ell^k-1}{\ell-1}$. *The above holds as a formal power series. Notice also that we have absolute convergence as long as* $|q|, |xq^{\ell-1}| < 1$.

This generating function is related to the function

$$F(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

because $F_\ell(x;q)$ is an infinite product of series related to $F(q)$. $F(q)$ is related to the Dedekind Eta function, $\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$, by $\frac{e^{\pi i\tau/12}}{F(\exp(2\pi i\tau))} = \eta(\tau)$, thus allowing us to produce a number of functional equations that it satisfies. In particular if h, H, k are integers with $k > 0$, $hH \equiv -1 \pmod k$, and z is a complex number with $\Re(z) > 0$, then we have that

$$F\left(\exp\left\{\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right\}\right) = e^{\pi is(h,k)}\left(\frac{z}{k}\right)^{1/2}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F\left(\exp\left\{\frac{2\pi iH}{k} - \frac{2\pi}{z}\right\}\right)$$

where

$$s(h,k) = \sum_{i=1}^{k}\left(\frac{i}{k} - \left\lfloor\frac{i}{k}\right\rfloor - \frac{1}{2}\right)\left(\frac{ih}{k} - \left\lfloor\frac{ih}{k}\right\rfloor - \frac{1}{2}\right)$$

is the Dedekind sum (see [2] Theorem 5.1). Setting $y = \frac{2\pi z}{k^2}$ in the above we get that

Proposition 2.

$$F\left(\exp\left\{\frac{2\pi ih}{k} - y\right\}\right) = e^{\pi is(h,k)}\left(\frac{yk}{2\pi}\right)^{1/2}\exp\left(\frac{\pi^2}{6yk^2} - \frac{y}{24}\right) F\left(\exp\left\{\frac{2\pi iH}{k} - \frac{4\pi^2}{k^2y}\right\}\right). \tag{1}$$

We will also be interested in the generating function

$$F_\ell(q) := \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})^\ell}{(1-q^n)} = \frac{F(q)}{F(q^\ell)^\ell} = F_\ell(0;q), \tag{2}$$

which is relevant for reasons similar to why F is. We notice that the last of the equalities in (2) implies that the coefficients of $F_\ell(q)$ are all non-negative. Hence

$$F_\ell(q) = 1 + q + \sum_{n=2}^{\infty} a_n q^n$$

with $a_n \geq 0$. Therefore we have that if $q \neq |q|$, then

$$|F_\ell(q)| = |1 + q + \sum_{n=2}^{\infty} a_n q^n| \leq |1 + q| + \sum_{n=2}^{\infty} a_n |q|^n < F_\ell(|q|). \tag{3}$$

Remark It was proved in [3] that for $\ell \geq 5$, that the coefficients of $F_\ell(q)$ are all strictly positive.

Also the relationship between $F_\ell(q)$ and $F(q)$ allows us to get a number of functional equations for F_ℓ. In particular using the $k = 1, 2, \ell, 2\ell$ cases of (1), we get that

Lemma 3. *If $hH \equiv -1 \pmod{\ell}$ in (6) and $hH \equiv -1 \pmod{2\ell}$ in (7) and if ℓ is odd in (5), then*

$$F_\ell(\exp(-y)) = \ell^{-\ell/2} \left(\tfrac{y}{2\pi}\right)^{-(\ell-1)/2} \exp\left(\tfrac{(\ell^2-1)y}{24}\right) \frac{F\left(\exp\left\{\frac{-4\pi^2}{y}\right\}\right)}{F\left(\exp\left\{\frac{-4\pi^2}{\ell y}\right\}\right)^{\ell}}. \tag{4}$$

$$F_\ell(-\exp(-y)) = \ell^{-\ell/2} \left(\tfrac{y}{\pi}\right)^{-(\ell-1)/2} \exp\left(\tfrac{(\ell^2-1)y}{24}\right) \frac{F\left(-\exp\left\{\frac{-\pi^2}{y}\right\}\right)}{F\left(-\exp\left\{\frac{-\pi^2}{\ell y}\right\}\right)^{\ell}}. \tag{5}$$

$$F_\ell\left(\exp\left\{\frac{2\pi i h}{\ell} - y\right\}\right) = \\ e^{\pi i s(h,\ell)} \left(\frac{y\ell}{2\pi}\right)^{-(\ell-1)/2} \exp\left(-\frac{(1-\ell^{-2})\pi^2}{6y} + \frac{(\ell^2-1)y}{24}\right) \cdot F_\ell\left(\exp\left\{\frac{2\pi i H}{\ell} - \frac{4\pi^2}{\ell^2 y}\right\}\right). \tag{6}$$

$$F_\ell\left(\exp\left\{\frac{2\pi i h}{2\ell} - y\right\}\right) = \\ e^{\pi i s(h,2\ell)} \left(\frac{y\ell}{\pi}\right)^{-(\ell-1)/2} \exp\left(-\frac{(1-\ell^{-2})\pi^2}{24y} + \frac{(\ell^2-1)y}{24}\right) \cdot F_\ell\left(\exp\left\{\frac{\pi i H}{\ell} - \frac{\pi^2}{\ell^2 y}\right\}\right). \tag{7}$$

We will also make use of the special functions $\Gamma(s)$ and $\zeta(s)$, the gamma function and the Riemann zeta function respectively, and will need some approximations of them. In particular, we will need the approximations of them on a vertical strip given in [7]. In particular if $s = \sigma + it$ where σ and t are real and $\sigma < -1$, then we have that

$$\Gamma(s) = O(e^{-(\pi/2)|t|}|t|^{\sigma-(1/2)})$$

and

$$\zeta(s) = O(|t|^{1/2-\sigma})$$

for large $|t|$, where the implied constants depend on σ. We will also need below the fact that if $\sigma < 0$, and s is not within $\frac{1}{2}$ of an integer, then

$$\Gamma(s) = O\left(e^{-(\pi/2)|t|}\left(\frac{|s|}{e}\right)^{-\sigma}\sqrt{|s|}\right),$$

which comes from Sterling's approximation (see [7]).

We also use a technique from [1] to more directly approximate the value of some of these products of hypergeometric functions. It employs the fact that if $c > 0$ and if q is a complex number with $|\arg(q)| < \frac{\pi}{2}$, (arg taking values in $(-\pi, \pi]$) then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}\Gamma(s)q^{-s}ds = \exp(-qn).$$

This result is proven by pushing the line of integration to the line $\Re(s) = c$ as $c \to -\infty$, and picking up residues of size $\frac{(-nq)^m}{m!}$ at $-m$. The bound on $|\arg(q)|$ guarantees that the integrand goes to 0 as s approaches $c \pm i\infty$, and the $\Gamma(s)$ term guarantees that the integral goes to 0 as $c \to -\infty$. In particular, if $s = c + it$, where $c - 1/2$ is and integer then the integrand is $O\left((e^{-(\pi/2-|\arg(q)|)|t|}\left(\frac{qe}{c}\right)^c)\sqrt{|s|}\right)$. Hence the integral equals

$$\lim_{N\to\infty}\sum_{m=0}^{N}\frac{(-nq)^m}{m!} + O\left(\left(\frac{qe}{N}\right)^N\right) = \exp(-qn).$$

Therefore, we have:

Lemma 4. *If $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for $s = c$, and if $|\arg(q)| < \pi/2$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D(s)\Gamma(s)q^{-s}ds = \sum_{n=1}^{\infty} a_n(\exp(-q))^n. \tag{8}$$

This lemma will be useful because it will allow us to approximate $\sum_{n=1}^{\infty} a_n(\exp(-q))^n$ using properties of $D(s)$.

3 Asymptotics of $p_\ell(a; n)$

3.1 Theorem Statement and Preliminaries

In this section we will prove the following theorem:

Theorem 5. *Let ℓ be a prime. If $-\Omega(\log n) < \Delta_0 < O(n^{1/24})$,*

$$p_\ell(a;n) = \frac{\sqrt{2}\pi(\ell-1)g(\Delta_0)n^{-3/2}}{24}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1 + O(n^{-1/8}))$$

where we used the following definitions:

$$
\begin{aligned}
w_0 &:= \pi\sqrt{\frac{1}{6n}},\\
\Delta_0 &:= w_0\left(\left(\left(\frac{\log w_0}{w_0}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(w_0)}{w_0}\right)\right)+n-a(\ell-1)\right),\\
K(w) &:=
\end{aligned}
$$

$$
\left(\frac{\ell-1}{\log\ell}\right)\left(\frac{-\log(2\pi)}{2}+\frac{\ell\log\ell}{2(\ell-1)}-\frac{\log\ell}{4}+\sum_{t_k=1+2\pi ik/\log\ell}^{k\neq 0}\Gamma(t_k)\zeta(t_k)\zeta(t_k-1)w^{-t_k+1}\right),
$$

and

$$
g(t) := \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\prod_{k=0}^{\infty}\left(1+\frac{y}{\ell^k}\right)^{\ell^k}e^{-y}\right)^{-(\ell-1)/2}e^{ty}dy.
$$

There are a few points to note about this theorem. Consider the distribution of $p_\ell(a;n)$ for fixed ℓ and n, as a varies. The distribution can be normalized to g, is centered at roughly $n-c\sqrt{n}\log n$, and has width proportional to $\sqrt{n}$. Furthermore, noticing that g is the inverse Laplace transform of a function allows us to tell that the sum of $p_\ell(a;n)$ in the range covered by Theorem 5 is asymptotic to $(4n\sqrt{3})^{-1}\exp(\pi\sqrt{2n/3})$, which is an asymptotic formula for the n^{th} partition number (see [2]). We will also consider the range of $m=n-(\ell-1)a$ for which our proof is valid. The lower end is roughly $\log n$ standard deviations below the mean value of m, and the high end is about $n^{1/24}$ standard deviations above the center.

Our general approach will be the "circle method". We begin by considering a change of variables for the generating function $F_\ell(x;q)$ obtaining the function

$$
G_\ell(x;q) := \prod_{k=0}^{\infty}\prod_{n=1}^{\infty}\frac{(1-x^{n\ell}q^{n\ell^{k+1}})^{\ell^{k+1}}}{(1-x^nq^{n\ell^k})^{\ell^k}}.
$$

Next, we approximate the value of $G_\ell(x;q)$ with (x,q) near $(1,1)$. Then we bound the size of $G_\ell(x;q)$ from above when (x,q) is not near $(1,1)$. Notice that these last two steps differ both in that they are interested in looking at different ranges of x and q, but also in that for the former we want an asymptotic formula for $G_\ell(x;q)$, while in the later, we only want an upper bound on its absolute value. Lastly we perform an integral to calculate $p_\ell(a;n)$. This will be done in two steps, integration with respect the each of the variables in turn.

In the rest of this section we will consider ℓ to be a constant. This will mean in particular that our asymptotic notation will contain constants that may depend on ℓ in them. This will also mean that some of the functions we define will have a non-explicit dependence on ℓ in them.

3.2 Definition of $G_\ell(x;q)$, and the Integral Formula for $p_\ell(a;n)$

We will simplify our computations by considering the change of variables

$$\begin{aligned} G_\ell(x;q) &:= F_\ell(x^{-(\ell-1)};xq) \\ &= \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{(1-x^{n\ell}q^{n\ell^{k+1}})^{\ell^{k+1}}}{(1-x^n q^{n\ell^k})^{\ell^k}} \\ &= \sum_{a,n} p_\ell(a;n) x^{n-a(\ell-1)} q^n. \end{aligned}$$

Where the last equality makes use of Theorem 1. Notice that $G_\ell(x;q)$ is invariant under replacement of x and q by $xe^{2\pi i/(\ell-1)}$ and $qe^{-2\pi i/(\ell-1)}$. Therefore, letting $m = n - a(\ell-1)$, we have that,

$$p_\ell(a;n) = \frac{(\ell-1)}{4\pi^2}\int_{-\frac{\pi}{\ell-1}}^{\frac{\pi}{\ell-1}}\int_{-\pi}^{\pi} G_\ell(e^{-z_0+iz};e^{-w_0+iw})e^{m(z_0-iz)}e^{n(w_0-iw)}dwdz \quad (9)$$

where z_0 and w_0 are real constants with $w_0, z_0 + (\ell-1)w_0 > 0$. Notice that the above integral converges absolutely, and hence we may freely change order of integration.

3.3 Approximation of $G_\ell(x;q)$ for (x,q) Near $(1,1)$

Let

$$f(y) := \left(\prod_{k=0}^{\infty}\left(1+\frac{y}{\ell^k}\right)^{\ell^k} e^{-y}\right)^{-(\ell-1)/2}.$$

In this section we will prove the following Lemma:

Lemma 6. *If $w = O(1)$, $|\arg w| < 1$, $\Re\left(\frac{1}{z+w}\right) > 1$, then $G_\ell(e^{-z};e^{-w})$ is*

$$\sqrt{\frac{w}{2\pi}}\exp\left\{\frac{\pi^2}{6w}\right\}\exp\left\{\left(\left(\frac{\log w}{w}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(w)}{w}\right)\right)z\right\}$$
$$f\left(\frac{z}{w}\right)\exp\left(O\left(\frac{z^2}{w}+w\right)\right).$$

The proof of Lemma 6 uses a few ideas. First we use the fact that G_ℓ is a product of $F_\ell(q)$ evaluated for various values of q, along with (4) to show that $G_\ell(e^{-z};e^{-w})$ equals $f(z/w)$ times some error term. Unfortunately, the error term is not small. On the other hand, the error term does have a small second logarithmic derivative, so we approximate our error based on the first two terms of the Taylor expansion of its logarithm about $z = 0$. Lastly we make use of Lemma 4 to approximate these terms of the Taylor series.

Proof. We note that

$$\begin{aligned}
&\log(G_\ell(e^{-z};e^{-w}))\\
=&\sum_{k=0}^{\infty}\ell^k\log(F_\ell(\exp(-z-\ell^k w)))\\
=&\sum_{k=0}^{k_0}\ell^k\log(F_\ell(\exp(-z-\ell^k w)))+\sum_{k=k_0+1}^{\infty}\ell^k\log(F_\ell(\exp(-z-\ell^k w)))\\
=&\sum_{k=0}^{k_0}\ell^k\left(v_k+zd_k-\left(\frac{\ell-1}{2}\right)\log(z+\ell^k w)+\log\left(\frac{F\left(\exp\left\{\frac{-4\pi^2}{z+\ell^k w}\right\}\right)}{\left(F\left(\exp\left\{\frac{-4\pi^2}{\ell(z+\ell^k w)}\right\}\right)\right)^\ell}\right)\right)\\
&\qquad+\sum_{k=k_0+1}^{\infty}\ell^k\log(F_\ell(\exp(-z-\ell^k w)))
\end{aligned}$$

where v_k and d_k depend on w but not z. We set k_0 so that $\ell^{k_0}=\Theta(\Re(w)^{-1})$. Our assumption that $\Re\left(\frac{1}{z+w}\right)>1$ implies that all of the $\Re\left(\frac{1}{z+\ell^k w}\right)$ with $k\le k_0$ are bounded below by some constant. Therefore, since the second derivative of $\log(F(\exp(-y^{-1})))$ with respect to y goes to 0 as y approaches 0 with $|\arg(y)|<1$, the second derivative with respect to z of

$$\sum_{k=0}^{k_0}\ell^k\log\left(\frac{F\left(\exp\left\{\frac{-4\pi^2}{z+\ell^k w}\right\}\right)}{\left(F\left(\exp\left\{\frac{-4\pi^2}{\ell(z+\ell^k w)}\right\}\right)\right)^\ell}\right)$$

is $O(\ell^{k_0})=O(\Re(w)^{-1})$. Also we have that the second derivative with respect to z of

$$\sum_{k=k_0+1}^{\infty}\ell^k\log(F_\ell(\exp(-z-\ell^k w)))$$

is

$$O\left(\sum_{k=k_0}^{\infty}\ell^k\exp(-\ell^k w)\right)=O\left(\ell^{k_0}\sum_{n=0}^{\infty}n|\exp(-n\ell^{k_0}w)|\right)=O(\ell^{k_0})=O(\Re(w)^{-1}).$$

Therefore,

$$\begin{aligned}
\log(G_\ell(e^{-z};e^{-w}))&=\log(G_\ell(1,w))+z\frac{\partial}{\partial z}\left[\log(G_\ell(e^{-z};e^{-w}))\right]_{z=0}\\
&\quad-\left(\frac{\ell-1}{2}\right)\sum_{k=0}^{k_0}\left(\ell^k\log\left(1+\frac{z}{\ell^k w}\right)-\frac{z}{w}\right)+O\left(\frac{z^2}{w}\right).
\end{aligned}\tag{10}$$

Notice that the sum over k in (10) can be extended to infinity since this only introduces an error of order

$$O\left(\sum_{k=k_0}^{\infty}\frac{z^2}{\ell^k w^2}\right)=O\left(\frac{z^2}{w}\right).$$

To approximate the first term in (10), note that

$$\log(G_\ell(1,e^{-w})) = \log(F(e^{-w})) = \frac{\pi^2}{6w} + \frac{1}{2}\log w - \frac{1}{2}\log(2\pi) + O(w).$$

The next term is more difficult. We prepare to use Lemma 4 to approximate it. We begin with the observation that

$$\begin{aligned}
&\log(G_\ell(e^{-z};e^{-w}))\\
=&\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\left(\ell^{k+1}\log(1-\exp(-\ell nz)\exp(-\ell^{k+1}nw)) - \ell^k\log(1-\exp(-nz)\exp(-\ell^k nw))\right)\\
=&\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\ell^k}{m}\right)\left(\exp(-mnz)\exp(-mn\ell^k w) - \ell\exp(-mn\ell z)\exp(-mn\ell^{k+1}w)\right).
\end{aligned}$$

Now notice that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\ell^k}{m}\right)\left((mn)^{-s}(mn\ell^k)^{-t} - \ell(mn\ell)^{-s}(mn\ell^{k+1})^{-t}\right)\\
=&(1-\ell^{1-s-t})\left(\sum_{k=0}^{\infty}\left(\ell^{1-t}\right)^k\right)\left(\sum_{n=1}^{\infty}n^{-s-t}\right)\left(\sum_{m=1}^{\infty}m^{-1-s-t}\right)\\
=&\zeta(s+t)\zeta(s+t+1)\left(\frac{1-\ell^{1-s-t}}{1-\ell^{1-t}}\right).
\end{aligned}$$

Hence by the iterated use of Lemma 4 we have that

$$\begin{aligned}
&\log(G_\ell(e^{-z};e^{-w})) =\\
&\qquad\frac{1}{(2\pi i)^2}\iint \Gamma(s)\Gamma(t)\zeta(s+t)\zeta(s+t+1)\left(\frac{1-\ell^{1-s-t}}{1-\ell^{1-t}}\right)z^{-s}w^{-t}ds dt
\end{aligned} \tag{11}$$

where the integral is an iterated contour integral with both contours vertical lines from $2-i\infty$ to $2+i\infty$.

We now wish to compute the derivative of (11) as $z\to 0$ along the real line and w fixed. We get

$$\frac{-1}{(2\pi i)^2}\iint \Gamma(s+1)\Gamma(t)\zeta(s+t)\zeta(s+t+1)\left(\frac{1-\ell^{1-s-t}}{1-\ell^{1-t}}\right)z^{-s-1}w^{-t}ds dt \tag{12}$$

along the same contour. If we move the line of integration of s to the line from $-3/2-i\infty$ to $-3/2+i\infty$ our resulting integral is $O(z^{1/2}w^{-2})$, which goes to 0 as $z\to 0$. We pick up a residue at $s=-1$ equal to

$$\frac{-1}{2\pi i}\int \Gamma(t)\zeta(t-1)\zeta(t)\left(\frac{1-\ell^{2-t}}{1-\ell^{1-t}}\right)w^{-t}dt$$

where the integral is along the line from $2-i\infty$ to $2+i\infty$. If we then move the line of integration of t to the line $\Re(t)=-1/2$ we are left with a term of order

$w^{1/2}$ and pick up values from the residues at $0, 1$ and $1 + 2\pi ik/\log \ell$ for $k \in \mathbb{Z}$. The residue at 0 is

$$-\zeta(-1)\zeta(0)(1+\ell) = -\frac{\ell+1}{24}.$$

The residue at $t_k = 1 + 2\pi ik/\log \ell$ for $k \neq 0$ is

$$\left(\frac{\ell-1}{\log \ell}\right) \Gamma(t_k)\zeta(t_k)\zeta(t_k-1)w^{-t_k}.$$

The Laurent series of the integrand in (11) at $t = 1$ is

$$\begin{aligned}
&-(1-\gamma(t-1)+\dots)\left(\frac{-1}{2}-(t-1)\frac{1}{2}\log(2\pi)+\dots\right)\left(\frac{1}{t-1}+\gamma+\dots\right)\cdot \\
&(1-\ell+(t-1)\ell\log\ell+\dots)\left(\frac{1}{(t-1)\log\ell}+\frac{1}{2}+\dots\right)(w^{-1}-(t-1)w^{-1}\log w) \\
&=\left(\frac{1-\ell}{2\log\ell w(t-1)}\right)\left(\frac{1}{t-1}+\log(2\pi)+\frac{\ell\log\ell}{1-\ell}+\frac{\log\ell}{2}-\log w+\dots\right).
\end{aligned}$$

Hence the residue at $t = 1$ is

$$\left(\frac{1-\ell}{2\log \ell w}\right)\left(\log(2\pi)+\frac{\ell\log\ell}{1-\ell}+\frac{\log\ell}{2}-\log w\right).$$

Therefore, the sum of these residues at $t = 0$, $t = 1$ and $t = t_k$ is

$$\left(\frac{\log w}{w}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(w)}{w}\right)-\left(\frac{\ell+1}{24}\right) \tag{13}$$

where $K(w)$ is, as in the statement of Theorem 5:

$$\left(\frac{\ell-1}{\log\ell}\right)\left(\frac{-\log(2\pi)}{2}+\frac{\ell\log\ell}{2(\ell-1)}-\frac{\log\ell}{4}+\sum_{t_k=1+2\pi ik/\log\ell}^{k\neq 0}\Gamma(t_k)\zeta(t_k)\zeta(t_k-1)w^{-t_k-1}\right).$$

Note that $K(w) = K(\ell w)$.

Combining (10), (12) and (13) we find that if $w = O(1)$, $|\arg w| < 1$, $\Re\left(\frac{1}{z+w}\right) > 1$, then $G_\ell(e^{-z}; e^{-w})$ is

$$\begin{aligned}
\sqrt{\frac{w}{2\pi}}\exp\left\{\frac{\pi^2}{6w}\right\}\exp\left\{\left(\left(\frac{\log w}{w}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(w)}{w}\right)\right)z\right\} \\
f\left(\frac{z}{w}\right)\exp\left(O\left(\frac{z^2}{w}+w\right)\right),
\end{aligned}$$

as desired. $\square$

3.4 Bounding $|G_\ell(x;q)|$ When (x,q) is Far From $(1,1)$

In this section we will prove two Lemmas that bound $|G_\ell(x;q)|$ for different regions of x and q. They will both compare $|G_\ell(x;q)|$ to $G_\ell(|x|;|q|)$. The first of the Lemmas will be valid when $|\arg(x)| < \pi/\ell$ and bound G_ℓ solely based on the sizes of $|\arg(x)|$ and $|\arg(q)|$. The second bound will pick up where Lemma 6 stops, and will tend to work when w is close to 0, but will be significantly more complicated.

Lemma 7. *Suppose that* $|\arg(x)| < \frac{\pi}{\ell-1}$, $x = \exp(-z)$, $q = \exp(-w)$ *(where* $|\Im(z)|, |\Im(w)| < \pi$*), and* $\Re(z) = O(\Re(w)^{1/3})$. *If* $|\Im(z)| \geq \Re(w)^{1/3}$ *or* $|\Im(w)| \geq \Re(w)^{4/3}$, *then*

$$\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|) = \Omega(\Re(w)^{-1/3}).$$

This Lemma shows that if $|\arg(x)|$ or $|\arg(q)|$ are too large relative to $|q|$, that $|G_\ell(x;q)|$ differs from $G_\ell(|x|;|q|)$ by a significant factor. The basic idea in proving this Lemma will be to use (3). In particular, we shall write G_ℓ as a product of $F_\ell(q)$ for various values of q, and considering the correct terms in this expansion prove the appropriate bounds on G_ℓ.

Proof. We can rewrite G_ℓ as

$$G_\ell(x;q) = \prod_{k=0}^{\infty} F_\ell(xq^{\ell^k})^{\ell^k}.$$

Therefore, by (3) we have that

$$\frac{G_\ell(x;q)}{G_\ell(|x|;|q|)} \leq \left(\frac{F_\ell(xq^{\ell^{k_0}})}{F_\ell(|xq^{\ell^{k_0}}|)}\right)^{\ell^{k_0}} \left(\frac{F_\ell(xq^{\ell^{k_0+1}})}{F_\ell(|xq^{\ell^{k_0+1}}|)}\right)^{\ell^{k_0+1}}.$$

If we choose k_0 so that $\ell^{k_0} = \Theta\left(\frac{1}{1-|q|}\right)$, then $|q|^{\ell^{k_0}} = \Theta(1)$. Since $|x|$ is bounded below by a positive constant, then $|xq^{\ell^{k_0}}| = \Theta(1)$ as well. Therefore, by (3) and a compactness argument, we have that for some constant $c_0 > 0$,

$$\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|) \geq \frac{c_0}{1-|q|}((\arg(xq^{\ell^{k_0}}))^2 + (\arg(xq^{\ell^{k_0+1}}))^2).$$

(where the arg's are taken to be between $-\pi$ and π, and $c_0 > 0$). Note that $\frac{(xq^{\ell^{k_0}})^\ell}{xq^{\ell^{k_0+1}}} = x^{\ell-1}$. Also since $|\arg(x)| < \frac{\pi}{\ell-1}$, we can infer that $\arg(x^{\ell-1}) = (\ell-1)\arg(x)$. Lastly, since $\frac{xq^{\ell^{k_0}}}{x} = q^{\ell^{k_0}}$, we have for some constant $c_1 > 0$,

$$\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|) \geq \frac{c_1}{1-|q|}((\arg(x))^2 + (\arg(q^{\ell^{k_0}}))^2). \qquad (14)$$

Suppose $q^{\ell^{k_0}} = \exp(-\ell^{k_0}w')$ where the imaginary part of w' is at most $\Re(w)^{2/3}$. Suppose that for some k, $q^{\ell^k} \neq \exp(-\ell^k w')$. Let k_1 be the largest such k. Clearly

$k_1 < k_0$. Also $q^{\ell^{k_1}} = \exp(2\pi i h/\ell - \ell^{k_1} w')$ for some integer h not divisible by ℓ. Then we have by (3),(4) and (6) that

$$\begin{aligned}
&\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|)\\
&\geq \ell^{k_1}[\log(F_\ell(\exp(-\Re(z-\ell^{k_1}w')))) - \log(F_\ell(\exp(2\pi i h/\ell - z - \ell^{k_1}w')))]\\
&= \ell^{k_1}\left(\frac{(1-\ell^{-2})\pi^2}{6\Re(z+3^{k_1}w')} + O(\log(z+3^{k_1}w)+1)\right).
\end{aligned}$$

Therefore, since $z = O(w^{1/3})$, if $\ell^{k_0}\Re(w)$ was chosen to be below a sufficiently small constant, then $\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|)$ is at least some constant times $|w|^{-1/3}$. Combining this result with (14), we find that for some constant $C > 0$,

$$\log(G_\ell(|x|;|q|)) - \log(|G_\ell(x;q)|) \geq C\Re(w)^{-1/3}$$

unless $|\Im(z)| < \Re(w)^{1/3}$ and $|\Im(w)| < \Re(w)^{4/3}$, as desired. □

Next, we would like to be able to bound the size of $G_\ell(e^{-z};e^{-w})$ when $\Re\left(\frac{1}{z+w}\right) > 1$. In particular, we will prove:

Lemma 8. *For some constant $C > 0$, If $|\arg(w)| < \pi/8$, $|\arg(z)| > 3\pi/8$, $|z| < C$, $\Im(z+w) > \Re(z+w)$, and $\Re(w), \Re(z+w) > 0$, then*

$$\log(G_\ell(e^{-\Re(z)};e^{-\Re(w)})) - \log(|G_\ell(e^{-z};e^{-w})|) = \Omega\left(\frac{|z|}{|w|}\right).$$

The idea of the proof is to find a single F_ℓ term that accounts for the difference.

Proof. By Equation 3 we have that

$$\begin{aligned}
\log(G_\ell(e^{-\Re(z)};e^{-\Re(w)})) - \log(|G_\ell(e^{-z};e^{-w})|) >&\\
\ell^a[\log(F(\exp(-\Re(z+\ell^a w)))) - \log(|F(\exp(-(z+\ell^a w)))|)]&
\end{aligned}$$

For any positive integer a. We pick a to be the largest possible number so that

$$\Im(z) - \Re(z) > \ell^a(\Re(w) - \Im(w)).$$

Note that $a = 0$ suffices. If this holds, we have that

$$\Im(z+\ell^a w) > \Re(z+\ell^a w).$$

Furthermore since $\Re(w) > \Im(w)$ and $\Im(z) > \Re(z)$, we have that a is as large as possible so that

$$\ell^a < \frac{\Im(z)-\Re(z)}{\Re(w)-\Im(w)} = \Theta\left(\frac{|z|}{|w|}\right).$$

Hence $\ell^a = \Theta\left(\frac{|z|}{|w|}\right)$. This means that $|z + \ell^a w| = O(|z|) = O(C)$. Therefore, by Equation 4 we have that

$$\log(F(\exp(-\Re(z + \ell^a w)))) - \log(|F(\exp(-(z + \ell^a w)))|) = \\ -\frac{\ell - 1}{2} \log\left(\frac{\Re(z + \ell^a w)}{|z + \ell^a w|}\right) + O(1)$$

where the constant for the $O(1)$ term can be made as small as we like by making C small. Hence since $\Im(z + \ell^a w) > \Re(z + \ell^a w)$, $|z + \ell^a w| > \sqrt{2}\Re(z + \ell^a w)$, and hence we have that

$$\log(F(\exp(-\Re(z + \ell^a w)))) - \log(|F(\exp(-(z + \ell^a w)))|) = \Omega(1)$$

if C is sufficiently small. Therefore, for C sufficiently small,

$$\log(G_\ell(e^{-\Re(z)}; e^{-\Re(w)})) - \log(|G_\ell(e^{-z}; e^{-w})|) = \Omega\left(\frac{|z|}{|w|}\right).$$

□

Using similar techniques we prove the following lemma:

Lemma 9. *If $\Re(y) > -1$ and $|\Im(y)| > \Re(y + 1)$, then*

$$|f(y)| = |f(\Re(y))| \exp(-\Omega(|\Im(y)|))$$

Proof. We find the largest integer $a \geq 0$ so that $|\Im(y)| > \ell^a$. We note that $\ell^a = \Omega(|\Im(y)|)$. Then

$$\frac{|f(y)|}{f(\Re(y))} = \prod_{k=0}^{\infty} \left(\frac{|\ell^k + y|}{\ell^k + \Re(y)}\right)^{-\ell^k(\ell-1)/2} \geq \left(\frac{5}{4}\right)^{(-\ell^a(\ell-1)/4)} = \exp\left(-\Omega\left(|\Im(y)|\right)\right).$$

□

Note that we can also get that if $\Re(y) > -\ell^k$ and $|\Im(y)| > \Re(y + \ell^k)$, then $|f(y)| = |f(\Re(y) + i)| \exp(-\Omega(|\Im(y)|))$.

We also need some bounds on $f(y)$ as $\Re(y)$ changes.

Lemma 10. *If $|\Im(y)| > |\Re(y)|$ and $|y| > 1$, then the real part of the logarithmic derivative of f at y is $\Theta(\log(|y|))$ and the imaginary part is $\Theta(1)$ and has the same sign as $\Im(y)$.*

Proof. The logarithmic derivative is

$$-(\ell - 1)/2 \sum_{k=0}^{\infty} \left[(1 + y\ell^{-k})^{-1} - 1\right] = (\ell - 1)/2 \sum_{k=0}^{\infty} \frac{y}{\ell^k + y}.$$

Note that our conditions imply that $|\ell^k + y| = \Theta(\max(\ell^k, |y|))$.

The real part is

$$(\ell-1)/2\sum_{k=0}^{\infty}\frac{\Re(y)\ell^k+|y|^2}{|y+\ell^k|^2}.$$

The summand is $\Theta(1)$ for the $\Theta(\log(|y|))$ terms where $|y|>\ell^k$ and drops off exponentially after that. Hence the result is $\Theta(\log|y|)$.

The imaginary part is

$$(\ell-1)/2\sum_{k=0}^{\infty}\frac{\Im(y)\ell^k}{|\ell^k+y|^2}.$$

The summand is $\Theta(1)$ when $\ell^k=\Theta(|y|)$ and otherwise drops off exponentially. Therefore, this is $\Theta(1)$. □

Note that our results about the real part also hold if $\Re(y)>0$ and $|\Im(y)|$ potentially small.

3.5 The Integral dz for Small $\arg(w)$

We now wish to compute, for fixed w (with $|\arg(w)|$ small), the value of

$$\frac{\ell-1}{2\pi i}\int_{z_0-\frac{\pi i}{\ell-1}}^{z_0+\frac{\pi i}{\ell-1}} G_\ell(e^{-z};e^{-w})e^{mz}dz. \tag{15}$$

Let

$$\Delta_0:=\Re(w)\left(\left(\left(\frac{\log\Re(w)}{\Re(w)}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(\Re(w))}{\Re(w)}\right)\right)+m\right).$$

We will integrate over the line with $\Re(z)=z_0$, where z_0 is chosen to be the real part of the z with $\frac{z}{w}$ real, that minimizes the absolute value of

$$\exp\left(\Delta_0\frac{z}{w}\right)\left(\prod_{k=0}^{\infty}\left(1+\frac{z}{\ell^k w}\right)^{\ell^k}e^{-z/w}\right)^{-(\ell-1)/2}.$$

Our integral can be written as the sum of the integrals over two regions. The first will be the range of z where $\Re\left(\frac{1}{z+w}\right)>1$ and the other will be the rest of the interval.

Lemma 11. *If* $-O(\log(w^{-1}))<\Delta_0<O(|w|^{-1/12})$ *and* $|\Im(w)|<\Re(w)^{4/3}$*, then the integral in (15) over the range where* $\Re\left(\frac{1}{z+w}\right)>1$ *is*

$$Cw(\ell-1)g(\Delta_0)(1+O(w^{1/4}))$$

where

$$C=\sqrt{\frac{w}{2\pi}}\exp\left\{\frac{\pi^2}{6w}\right\}.$$

We prove this Lemma by approximating the integrand using Lemma 6. After that we just have to make a careful analysis of the errors introduced by the error term in Lemma 6, and by making the integral go to infinity.

Proof. In the range mentioned in Lemma 11 the integral from (15) is

$$C\frac{\ell-1}{2\pi i}\int \exp\left(\Delta\frac{z}{w}\right)\left(\prod_{k=0}^{\infty}\left(1+\frac{z}{\ell^k w}\right)^{\ell^k} e^{-z/w}\right)^{-(\ell-1)/2} \exp\left(O\left(\frac{z^2}{w}+w\right)\right)dz$$

where

$$\Delta = w\left(\left(\left(\frac{\log(w)}{w}\right)\left(\frac{\ell-1}{2\log\ell}\right)+\left(\frac{K(w)}{w}\right)\right)+m\right).$$

Since $|\Delta_0-\Delta| = O(\Re(w)^{1/3})$, our expression from (15) simplifies to

$$C\frac{\ell-1}{2\pi i}\int \exp\left(\Delta_0\frac{z}{w}\right)\left(\prod_{k=0}^{\infty}\left(1+\frac{z}{\ell^k w}\right)^{\ell^k} e^{-z/w}\right)^{-(\ell-1)/2} \exp\left(O\left(\frac{z^2}{w}+w+zw^{-2/3}\right)\right)dz.$$

Letting $y = z/w$ we have

$$\frac{Cw(\ell-1)}{2\pi i}\int \exp(\Delta_0 y)\left(\prod_{k=0}^{\infty}\left(1+\frac{y}{\ell^k}\right)^{\ell^k} e^{-y}\right)^{-(\ell-1)/2} \exp\left(O\left(y^2w+w+yw^{1/3}\right)\right)dy. \tag{16}$$

Recall that

$$f(y) = \left(\prod_{k=0}^{\infty}\left(1+\frac{y}{\ell^k}\right)^{\ell^k} e^{-y}\right)^{-(\ell-1)/2}.$$

Let y_0 be the real value greater than -1 so that

$$f(y_0)\exp(\Delta_0 y_0)$$

is minimal. We wish to integrate $\exp(\Delta_0 y)f(y)$ over a line through y_0 with slope $\Omega(w^{-1/3})$. This line extends until $\Re\left(\frac{1}{1+y}\right) < \Re(w)$.

Note that g is the inverse Laplace transform of f. f is a product of functions of the form $\left(1+\frac{y}{\ell^k}\right)^{\ell^k} e^{-y}$, whose inverse Laplace transform is positive. Hence g is a convolution of these inverse Laplace transforms and hence is everywhere positive.

We split into cases based on the sign of Δ_0.

If Δ_0 is positive, We wish to show that

$$\frac{1}{2\pi i}\int_{y_0-i\infty}^{y_0+i\infty} f(y)\exp(\Delta_0 y)dy = \Omega((y_0+1)(f(y_0)\exp(\Delta_0 y_0))). \tag{17}$$

For large Δ_0, and $\ell > 2$, we can push the line of integration past the pole at $y=-1$ and get that the left-hand side is

$$\Theta\left(\exp(-\Delta_0)\Delta_0^{(\ell-3)/2}\right).$$

For $\ell = 2$ we can move the line of integration so that it wraps around the singularity at $y = -1$ and again show that the integral equals

$$\Theta\left(\exp(-\Delta_0)\Delta_0^{(\ell-3)/2}\right).$$

On the other hand for $y < 0$, we have

$$f(y)\exp(\Delta_0 y) \geq \exp(-\Delta_0)(y+1)^{-(\ell-1)/2}\exp(\Delta_0(y+1))$$

which is $\Theta\left(\exp(-\Delta_0)\Delta_0^{(\ell-1)/2}\right)$ with $y_0 + 1 = \Theta(\Delta_0^{-1})$.

Noting that the integrand falls off like $\exp(-|\Im(y)|)$, we get that ignore the $|\Im(y)| > \Omega(\Delta_0 + \log(|w|^{-1}))$ introduces a relative error of $O(|w|)$.

Our range of integration is that where $\Re\left(\frac{1}{w(y+1)}\right) > 1$. Or, $\frac{\Re(w(y+1))}{|w|^2|(y+1)|^2} > 1$. Since the real part of wy is constant, this says that the integration ends when $\frac{|w|\Delta_0^{-1}}{|w|^2\Im(y)^2} = \Theta(1)$, or when $|\Im(y)| = \Theta\left(|w|^{-1/2}\Delta_0^{-1/2}\right)$.

Consider now the part of the integral where $|\Im(y)| = O(\Delta_0 - \log(|w|)$. The integrand is equal to $f(y')$ where $\Im(y') = \Im(y)$ and $\Re(y') = \Re(y_0)$ times an error of $\exp(O(y^2w + w + yw^{1/3} + yw^{2/3}\log|y|))$, where the last term comes from changing the real part. Since the error term is $1 + O(w^{1/4})$, the derivative of y' with respect to y is $1 + O(w^{2/3})$, and since $g(\Delta_0)$ equals the y' integral times $1 + O(w)$, the integral over this range is $g(\Delta_0)(1 + O(w^{1/4}))$.

The integrand over the rest of our range is

$$\exp\left(O\left(y^2w + w + yw^{1/3} + yw^{2/3}\log|y|\right) - \Omega(y))\right) = \exp(-\Omega(y))$$

and hence produces a small contribution.

For small Δ_0 negative, y_0 becomes large. The logarithmic second derivative of f at y_0 is

$$(\ell-1)/2\sum_{k=1}^{\infty}\ell^{-k}(1+y_0\ell^{-k})^{-2} = \Omega(y_0^{-1})$$

by considering the term where $\ell^k = \Theta(y_0)$. On the other hand the logarithmic third derivative at y with real part y_0 has absolute value

$$(\ell-1)\sum_{k=1}^{\infty}\ell^{-2k}(1+y\ell^{-k})^{-3} \leq 2\ell\sum_{k=0}^{\infty}\ell^{-2k}\min(1, y^3\ell^{-3k}) = O(y_0^{-2}).$$

Therefore, since the first logarithmic derivative of $f(y)\exp(\Delta_0 y)$ is 0 at y_0, we have that $f(y)\exp(\Delta_0 y)$ is well approximated by its first two logarithmic derivatives for $|\Im(y)| = O(y_0^{2/3})$. In particular, since at $|\Im(y)| = O(y_0^{2/3})$, $|f(y)| = f(y_0)\exp(-\Omega(y_0^{1/3}))$, and since $|f(y)|$ decreases with $|\Im(y)|$, and exponentially so for $|\Im(y)| > y_0$, the integral of $f(y)$ with $\Im(y) = \Im(y_0)$ and $|\Im(y)| > y_0^{2/3}$ is $f(y_0)\exp(-\Omega(y_0^{2/3})$. Therefore, using the normal approximation, $g(\Delta_0) = \Theta(f(y_0)\exp(\Delta_0 y_0)y_0^{1/2})$.

Now note that by our approximations of the logarithmic derivative of $f(y)$, that we must have $y_0 = \exp(\Theta(-\Delta_0))$.

Again, our range of integration is that where $\Re\left(\frac{1}{w(y+1)}\right) > 1$. Or, $\frac{\Re(w(y+1))}{|w|^2|(y+1)|^2} > 1$. The endpoints of this are where $\frac{y_0}{|w||y|^2} = \Theta(1)$, or where $|y| = \Theta(w^{-1/2}y_0^{1/2})$.

We may assume that $y_0 = O(w^{-1/12})$. Again if we replace our integral by the one where $\Re(y) = \Re(y_0)$ maintaining the imaginary parts of y at the endpoints, we introduce a multiplicative error of $1 + O(w^{2/3})$ from the change of variables, and are integrating

$$f(y)\exp(\Delta_0 y)\exp\left(O\left(y^2 w + w + yw^{1/3} + yw^{2/3}\log|y|\right)\right).$$

For $|\Im(y)| = O(y_0 - \log(w))$ this is $1 + O(w^{1/4})$. Giving in this range $g(\Delta_0)(1 + O(w^{1/4}))$. For larger y, we have an integrand of size

$$g(\Delta_0)y_0^{-1/2}\exp(\Delta_0 y)\exp\left(O\left(y^2 w + w + yw^{1/3} + yw^{2/3}\log|y|\right) - \Omega(y)\right).$$

The $\exp(-\Omega(y))$ term dominates giving us a small error. Hence our integral is $g(\Delta_0)(1 + O(w^{1/4}))$ as desired. □

We return to the problem of approximating

$$\frac{\ell - 1}{2\pi i}\int G_\ell(e^{-z}; e^{-w})e^{mz}dz.$$

Let C and Δ_0 be as before.

Lemma 12. *If* $\Re(w) + z_0 > w^{4/3}$, $z_0 = O(1)$, $z_0 = O(w^{-1/3}m^{-1})$, $|\Im(w)| < \Re(w)^{4/3}$ *the integral in (15) in the region where* $\Re\left(\frac{1}{z+w}\right) \leq 1$ *is*

$$\exp\left(\frac{\pi^2}{6\Re(w)} - \Omega\left(w^{-1/3}\right)\right)$$

Proof. This follows immediately from Lemma 7 and the fact that $|\Im(z)| > w^{2/3}$. □

The conditions for Lemma 12 are satisfied if

$$\Omega(w^{1/3}) < y_0 + 1 < O(w^{-1}), O(w^{-4/3}m^{-1}). \tag{18}$$

The lower bound holds if $\Delta_0 < O(w^{-1/3})$. The upper bounds holds if $m < w^{-4/3+\epsilon}$ and $\Delta_0 > -O(\log(w^{-1}))$. Notice that these hold if $w = w_0$ and Δ satisfies the appropriate bounds.

3.6 The w Integral

We now evaluate $p_\ell(a;n)$ by performing the integral in Equation (9). We perform the integral dz first and then dw. We use Lemma 7 when $|\Im(w)|$ is too large, and the results in the previous section otherwise.

We now work on evaluating

$$\frac{\ell-1}{(2\pi i)^2}\iint G_\ell(e^{-z};e^{-w})e^{nw}e^{mz}dzdw$$

when $O(n^{1/6}) > \Delta_0 > -O(\log n)$. We will use the path $\Re(w) = \pi\sqrt{\frac{1}{6n}} = w_0$. We have by Lemma 7 that when $|\Im(w)| > w_0^{4/3}$, the value of the integral is at most

$$e^{-\Omega(w_0^{-1/3})}G_\ell(e^{-z_0};e^{-w_0})e^{mz_0}e^{nw_0} = O\left(\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}g(\Delta_0)\exp(-n^{1/6})\right). \tag{19}$$

Lemma 13. *Suppose that $O(n^{1/6}) > \Delta_0 > -O(\log n)$. If the integral in (9) is performed along the line $\Re(w) = \pi\sqrt{\frac{1}{6n}} = w_0$ in the region where $|\Im(w)| < \Re(w)^{4/3}$, the result is*

$$\frac{\sqrt{2}\pi(\ell-1)g(\Delta_0)n^{-3/2}}{24}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1+O(n^{-1/8})).$$

We prove this by using results from the previous section to approximate the nested z-integral. We then approximate our integrand by a normal distribution.

Proof. Using Lemmas (11) and (12) to perform the z integral we find that in the region of interest the integral in (9) equals

$$\frac{g(\Delta_0)(\ell-1)}{2\pi i}\int\sqrt{\frac{w^3}{2\pi}}\exp\left\{\frac{\pi^2}{6w}+nw\right\}(1+O(n^{1/8}))+\exp\left\{nw+\frac{\pi^2}{6w_0}-\Omega(n^{1/6})\right\}dw =$$

$$\exp\left\{\pi\sqrt{\frac{2n}{3}}-\Omega(n^{1/6})\right\}+\frac{g(\Delta_0)(\ell-1)}{2\pi i}\sqrt{\frac{w_0^3}{2\pi}}\int\exp\left\{\frac{\pi^2}{6w}+nw\right\}(1+O(n^{1/8}))dw.$$

We ignore the first term and proceed to approximate the second summand above by approximating the integrand as a normal distribution. Note that the first, second and third logarithmic derivatives at $w = w_0$ are 0, $\frac{\pi^2}{3w_0^3} = \frac{2\sqrt{6}n^{3/2}}{\pi}$ and $O(n^2)$ respectively. Notice that the third derivative is $O(n^2)$ for all w. Hence we have that for $|\Im(w)| = O(n^{-17/24})$, the integrand is

$$\exp\left\{\pi\sqrt{\frac{2n}{3}}-\frac{\sqrt{6}n^{3/2}}{\pi}\Im(w)^2+O(n^2\Im(w)^3)\right\}(1+O(n^{-1/8})) =$$

$$\exp\left\{\pi\sqrt{\frac{2n}{3}}-\frac{\sqrt{6}n^{3/2}}{\pi}\Im(w)^2\right\}(1+O(n^{-1/8})).$$

The above integral is the integral of $\Omega(n^{1/24})$ standard deviations of a normal distribution times an error of $(1 + O(n^{-1/8}))$, so it equals

$$i\sqrt{\frac{\pi^2}{\sqrt{6}n^{3/2}}}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1 + O(n^{-1/8})).$$

For $|\Im(w)| = \Omega(n^{-17/24})$, the size of the integrand of (9) is decreasing in $|\Im(w)|$, leaving us with

$$\exp\left\{\pi\sqrt{\frac{2n}{3}} - \Omega(n^{1/12})\right\}.$$

Hence the integral in (9) over the complete range $|\Im(w)| < \Re(w)^{4/3}$ is

$$\begin{aligned}
&\frac{g(\Delta_0)(\ell-1)}{2\pi}\sqrt{\frac{w_0^3}{2\pi}}\sqrt{\frac{\pi^2}{\sqrt{6}n^{3/2}}}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1 + O(n^{-1/8}))\\
=&\frac{g(\Delta_0)(\ell-1)}{2\pi}\sqrt{\frac{\sqrt{6}\pi^2}{72n^{3/2}}}\sqrt{\frac{\pi^2}{\sqrt{6}n^{3/2}}}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1 + O(n^{-1/8}))\\
=&\frac{\sqrt{2}\pi(\ell-1)g(\Delta_0)n^{-3/2}}{24}\exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1 + O(n^{-1/8})),
\end{aligned}$$

which is larger than our error term, proving our Lemma. □

Proof of Theorem 5. This follows from (9) and (19) and Lemma (13). □

4 The Asymptotics of $p_\ell^{even}(n) - p_\ell^{odd}(n)$

In this section we will consider only ℓ odd and sufficiently large.

4.1 Statement of the Theorem and a Generating Function

Define

$$p_\ell^{even}(n) := \sum_{a\ even} p_\ell(a;n), \qquad p_\ell^{odd}(n) := \sum_{a\ odd} p_\ell(a;n)$$

as done in [6]. Let

$$a_\ell(n) := p_\ell^{even}(n) - p_\ell^{odd}(n) = \sum_a (-1)^a p_\ell(a;n).$$

In this section we will prove the following theorem:

Theorem 14. *If for some k_0 we have that*

$$\Omega\left(\frac{\log \ell}{\ell}\right) < \ell^{k_0}\pi\sqrt{\frac{1}{6n}} < O(\log^{-1}\ell),$$

then

$$a_\ell(n) = \frac{2^{\frac{\ell}{2(\ell+1)}}}{2\pi}\sqrt{\frac{q_n}{\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n}}}\exp\left\{\frac{K_\ell^+(q_n)}{q_n}+nq_n\right\}(1+O(n^{-1/4})),$$

if k_0 *is even and*

$$a_\ell(n) = (-1)^n\frac{2^{\frac{\ell+2}{2(\ell+1)}}}{2\pi}\sqrt{\frac{q_n}{\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^-(w)}{w}\right]_{w=q_n}}}\exp\left\{\frac{K_\ell^-(q_n)}{q_n}+nq_n\right\}(1+O(n^{-1/4})),$$

if k_0 *is odd. Where we used the following definitions:*

$$K_\ell(w) := \sum_{s_k=1+2\pi i(2k+1)/\log\ell}\Gamma(s_k)\zeta(s_k)\zeta(s_k+1)\left(\frac{1}{\log\ell}\right)(2-2^{-s_k})(1-2^{-s_k})w^{-s_k+1},$$

$$K_\ell^+(w) := \frac{5\pi^2}{48}+K_\ell(w)$$
$$K_\ell^-(w) := \frac{5\pi^2}{48}-K_\ell(w).$$

and q_n *is the unique real number satisfying*

$$\Omega\left(\frac{\log\ell}{\ell}\right) < \ell^{k_0}q_n < O(\log^{-1}\ell)$$

and

$$\frac{\partial}{\partial w}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n} = -n, \qquad \text{if } k_0 \text{ is even},$$

and

$$\frac{\partial}{\partial w}\left[\frac{K_\ell^-(w)}{w}\right]_{w=q_n} = -n, \qquad \text{if } k_0 \text{ is odd}.$$

Note that $\ell^{k_0}/\sqrt{n}$ can always be picked to be between 1 and ℓ^{-1}, hence our theorem asks that the value of $\ell^{k_0}/\sqrt{n}$ in this range to not be within a factor of $\log_\ell$ of each of these endpoints. Hence for large ℓ most of the multiplicative range of n is covered by this theorem. Note that $K_\ell^\pm(w) = K_\ell^\pm(\ell w)$. Therefore, when n is in the appropriate range, this formula can be rewritten as

$$a_\ell(n) = (\pm 1)^n\frac{c_\ell^\pm(n)}{n}\exp\left\{\kappa_\ell^\pm(n)\sqrt{n}\right\}(1+O(n^{-1/4}))$$

where $c_\ell^\pm$ and $\kappa_\ell^\pm$ are both analytic functions that are periodic in $\log_\ell(n)$ with period 2, and the $\pm$ is determined by the sign of $\lfloor\log_\ell n\rfloor$.

We prove this theorem also with an application of the "circle method". We begin by bounding the size of $H_\ell(q)$ when q is far from ±1. Next we determine the size of $H_\ell(q)$ when q is near ± 1. We then use an analysis of H_ℓ to determine properties of $K(w)$. Lastly, we use this to perform an integration to calculate $a_\ell(n)$.

We start by noticing that $[k]_\ell \equiv k \pmod 2$. Let

$$\begin{aligned} H_\ell(q) &= \sum_n a_\ell(n) q^n = F_\ell(-1;q) \\ &= \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{(1-(-1)^{nk}q^{n\ell^{k+1}})^{\ell^{k+1}}}{(1-(-1)^{nk}q^{n\ell^k})^{\ell^k}} = \prod_{k=0}^{\infty} F_\ell((-1)^k q^{\ell^k})^{\ell^k}. \end{aligned}$$

4.2 Bounding the Size of $|H_\ell(q)|$

In this section, we will bound the size of $|H_\ell(q)|$ from above when q is not near ± 1. We will do this primarily by looking at the functional equations for the function $F(q)$.

For given $q = \exp(-w)$, let k_0 be the largest integer such that $\ell^{k_0}\Re(w) \le 1$. We prove that:

Lemma 15. *If $\ell^{k_0+1}\Re(w) \ge 3\log \ell$, $\ell^{k_0}\Re(w) \le c(\log^{-1}\ell)$ for a sufficiently small c, and $|\arg((-1)^{k_0}q)| > \pi/4 - \epsilon$ for sufficiently small $\epsilon > 0$, then*

$$|H_\ell(q)| \le \exp\left(\frac{7.2\pi^2}{48\Re(w)}\right).$$

Proof. Recalling that $|F_\ell(q)| \le F_\ell(|q|)$ we discover that

$$H_\ell(q) \le F(|q|)\left(\frac{|F_\ell((-1)^k q^{\ell^{k_1}})|}{F_\ell(|q|^{\ell^{k_1}})}\right)^{\ell^{k_1}}.$$

for any k_1. Hence if $k_1 = k_0$, under the conditions of Lemma 15,

$$\log\left(\frac{|F_\ell((-1)^{k_0} q^{\ell^{k_1}})|}{F_\ell(|q|^{\ell^{k_0}})}\right) = \log(|F((-1)^{k_0}q^{\ell^{k_0}})|) - \log(F(|q|^{\ell^{k_0}})) + O(\ell^{-2}).$$

By Dirichlet's approximation theorem (see [2]) we can pick a rational number h/k so that $k < \sqrt{\ell^{k_0}\Re(w)}$, and $\left|\frac{\Im(\ell^{k_1}w)}{2\pi} - \frac{k_0}{2} - \frac{h}{k}\right| < \frac{2}{\sqrt{\ell^{k_0}\Re(w)}k}$. Since $\Re\left(\frac{1}{k^2(\ell^{k_0}w - ih/k)}\right) = \Omega(1)$, we have by applying the functional equation for F with h and k that if $\ell^{k_0}w' = \ell^{k_0}w - \pi i k_0 - 2\pi i h/k$, then

$$\begin{aligned} \log(|H_\ell(\exp(w))|) \le& \frac{\pi^2}{6w'} + O(\log w) + \ell^{k_0}\left(\left|\frac{\pi^2}{6k^2\ell^{k_0}\Re(w)}\right| - \frac{\pi^2}{6\ell^{k_0}w'} + O(\log(\ell))\right) \\ =& \Re\left(\frac{\pi^2}{6k^2w'}\right) + O(\ell^{k_0}\log \ell) \\ =& \Re\left(\frac{\pi^2}{6k^2w'}\right) + O\left(\frac{1}{\Re(w)}\right). \end{aligned}$$

The constant in the last error term can be made as small as desired by making c as small as desired. Note that $\Re(w) = \Re(w')$. Therefore, for small enough c, if $k > 1$, or if $|\Im(w')| > \Re(w')(1-\epsilon')$, then

$$|H_\ell(\exp(-w))| < \exp\left(\frac{7\pi^2}{48\Re(w)}\right).$$

Therefore, the above holds unless $|\arg(\log((-1)^{k_0}q^{\ell^{k_0}}))| < \pi/4 - \epsilon$. The sequence of $(-1)^{k_0}q^{\ell^k}$ is obtained from $(-1)^{k_0}q^{\ell^{k_0}}$ by taking repeated ℓ^{th} roots. Suppose that $|\arg(\log((-1)^{k_0}q^{\ell^{k_0}}))| < \pi/4$ but that $|\arg(\log((-1)^{k_1}q^{\ell^{k_1}}))| > \pi/4$ for some largest k_1. Then noticing that

$$|H_\ell(q)| \le F(|q|)\left(\frac{|F_\ell((-1)^{k_1}q^{\ell^{k_1}})|}{F_\ell(|q|^{\ell^{k_1}})}\right)^{\ell^{k_1}},$$

and letting $\ell^{k_1}w' = \ell^{k_1}w - \pi i k_1 - 2\pi i h/\ell$, for the appropriate value of h (to make $\Im(w')$ small), and using (6) or (7), we find that

$$\log(|H_\ell(q)|) \le \Re\left(\frac{1}{w'}\right)\left(\frac{\pi^2}{6}\right)\left(1 - \frac{1-\ell^{-2}}{4} + O(1)\right).$$

Therefore,

$$|H_\ell(q)| < \exp\left(\frac{7.2\pi^2}{48\Re(w)}\right)$$

unless no such k_1 exists, which implies that $|\arg(\log((-1)^{k_0}q))| < \pi/4$, as desired. □

4.3 Approximating $H_\ell(q)$ for q Near ± 1

We would like to able to approximate $H_\ell(q)$ for q near ± 1. We use the following Lemma:

Lemma 16. *If $w = O(1)$ and $|\arg(w)| < \pi/4$, then*

$$H_\ell(e^{-w}) = 2^{\frac{\ell}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\left(\frac{5\pi^2}{48w}\right) + \left(\frac{K_\ell(w)}{w}\right)\right\}(1+O(w)) \quad (20)$$

$$H_\ell(-e^{-w}) = 2^{\frac{\ell+2}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\left(\frac{5\pi^2}{48w}\right) - \left(\frac{K_\ell(w)}{w}\right)\right\}(1+O(w)). \quad (21)$$

The basic idea of this proof is to approximate the logarithm of $H_\ell(e^{-w})/F(e^{-w})$ using Lemma 4.

Proof. Note that

$$
\begin{aligned}
&\log(H_\ell(e^{-w})) - \log(F(e^{-w}))\\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-\ell)^k(\log(1-e^{-n\ell^k w}) - \log(1-(-1)^n e^{-n\ell^k w}))\\
=&\sum_{k,n,m=1}^{\infty}\frac{(-\ell)^k}{m}((-1)+(-1)^{nm})e^{-nm\ell^k w}\\
=&2\sum_{k=1}^{\infty}\sum_{n\ odd}^{\infty}\sum_{m\ odd}^{\infty}\frac{(-\ell)^k}{m}e^{-nm\ell^k w}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&2\sum_{k=1}^{\infty}\sum_{n\ odd}^{\infty}\sum_{m\ odd}^{\infty}\frac{(-\ell)^k}{m}(nm\ell^k)^{-s}\\
=&2\left(\sum_{k=1}^{\infty}(-\ell^{1-s})^k\right)\left(\sum_{m\ odd}^{\infty}m^{-s-1}\right)\left(\sum_{n\ odd}^{\infty}n^{-s}\right)\\
=&2\left(\frac{-\ell^{1-s}}{1+\ell^{1-s}}\right)\left(\zeta(s+1)(1-2^{-s-1})\right)\left(\zeta(s)(1-2^{-s})\right)\\
=&\zeta(s)\zeta(s+1)\left(\frac{-\ell^{1-s}}{1+\ell^{1-s}}\right)(2-2^{-s})(1-2^{-s}).
\end{aligned}
$$

Therefore, we have, by Lemma 4, that

$$
\log(H_\ell(e^{-w}))-\log(F(e^{-w})) = \frac{1}{2\pi i}\int \Gamma(s)\zeta(s)\zeta(s+1)\left(\frac{-\ell^{1-s}}{1+\ell^{1-s}}\right)(2-2^{-s})(1-2^{-s})w^{-s}ds.
$$

We can move the line of integration back to $\Re(w) = -2$. We pick up poles at $s = 1, 0, -1$ and at $s = 1 + \pi i(2k+1)/\log\ell = s_k$. The remaining term is of order w^2. The residue at $s = -1$ is of order w. The residue at $s = 0$ is

$$
\zeta(0)\left(\frac{-\ell}{1+\ell}\right)\log 2 = \frac{\ell\log 2}{2(1+\ell)}.
$$

The residue at $s = 1$ is

$$
\zeta(2)\left(\frac{-1}{2}\right)\left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)w^{-1} = \frac{-3\pi^2}{48w}.
$$

The residue at $s = s_k$ is

$$
\Gamma(s_k)\zeta(s_k)\zeta(s_k+1)\left(\frac{1}{\log\ell}\right)(2-2^{-s_k})(1-2^{-s_k})w^{-s_k}.
$$

Therefore, we have that

$$
\log(H_\ell(e^{-w})) - \log(F(e^{-w})) = \left(\frac{\ell\log 2}{2(1+\ell)}\right) + \left(\frac{-3\pi^2}{48w}\right) + \left(\frac{K_\ell(w)}{w}\right) + O(w),
$$

where

$$K_\ell(w) = \sum_{s_k=1+2\pi i(2k+1)/\log \ell} \Gamma(s_k)\zeta(s_k)\zeta(s_k+1)\left(\frac{1}{\log \ell}\right)(2-2^{-s_k})(1-2^{-s_k})w^{-s_k+1}$$

as in the statement of Theorem 14. Hence since $w = O(1)$, and since we have that

$$\log(H_\ell(e^{-w})) = [\log(H_\ell(e^{-w})) - \log(F(e^{-w}))] + \log(F(e^{-w}))$$
$$\log(H_\ell(-e^{-w})) = \log(F(-e^{-w})) - [\log(H_\ell(e^{-w})) - \log(F(e^{-w}))].$$

By (1) we have that

$$H_\ell(e^{-w}) = 2^{\frac{\ell}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\left(\frac{5\pi^2}{48w}\right) + \left(\frac{K_\ell(w)}{w}\right)\right\}(1+O(w))$$
$$H_\ell(-e^{-w}) = 2^{\frac{\ell+2}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\left(\frac{5\pi^2}{48w}\right) - \left(\frac{K_\ell(w)}{w}\right)\right\}(1+O(w)),$$

as desired. □

4.4 Analysis of $K_\ell(w)$

Next, we would like to analyze $K_\ell(w)$ a little bit more. We will do this by looking at the size of $H_\ell(\exp(-w))$. In particular we prove that:

Lemma 17. *If* $|\arg(w)| < \pi/4$ *and* $(2\log \ell)/\ell < \ell^{k_0}\Re(w) < c(\log^{-1}\ell)$*, then*

$$\frac{\partial^n}{\partial w^n}\left(\frac{K_\ell(w)}{w}\right) = \frac{(-1)^n n! 3\pi^2(-1)^{\lfloor \log_\ell w\rfloor}}{48w^{n+1}} + O(\ell^{-1+\log_\ell w - \lfloor \log \ell w\rfloor}(\log \ell) w^{-n-1})$$

for $n = 0, 1, 2, 3$.

The basic idea of the proof is to use Lemma 16, and to obtain independent bounds on the sizes of the logarithmic derivatives of $H_\ell(q)$ by using (1), (4) and (5).

Proof. Given the assumptions of Lemma 17, we have, by (4) and (5), that

$$\begin{aligned}
&\log(H_\ell(\exp(-w))) \\
&= \log(F((-1)^{k_0}\exp(-\ell^{k_0}w))) + \sum_{k=0}^{k_0-1}\ell^k \log(F_\ell((-1)^k\exp(-\ell^k w))) + O(\ell^{-1}/\Re(w)) \\
&= \left(\frac{\pi^2}{6w}\right)\left(\frac{5+3(-1)^{k_0}}{8} + O(c)\right)
\end{aligned}$$

Since the error term is linear in c, and since we can also prove the above for smaller values of c, and since $k_0 = \lfloor \log_\ell \Re(w)^{-1}\rfloor$, if $\log_\ell \Re(w)^{-1} - \lfloor \log \ell\Re(w)^{-1}\rfloor = \Omega(\log \ell/\log\log \ell)$, then

$$K_\ell(w) = \frac{3\pi^2(-1)^{\lfloor \log_\ell w\rfloor}}{48} + O(\ell^{-1+\log_\ell w - \lfloor \log \ell w\rfloor}\log \ell).$$

Differentiating $\log(H_\ell(e^{-w}))$ n times, we may use this same technique, to find that

$$\frac{\partial^n}{\partial w^n}\left(\frac{K_\ell(w)}{w}\right) = \frac{(-1)^n n! 3\pi^2 (-1)^{\lfloor \log_\ell w \rfloor}}{48 w^{n+1}} + O(\ell^{-1+\log_\ell w - \lfloor \log \ell w \rfloor}(\log \ell) w^{-n-1})$$

for $n = 0, 1, 2, 3$, as desired. □

4.5 Integration of $H_\ell(q)$

We are now ready to prove our theorem. The idea is simply to use the approximations in the previous two section to approximate the value of an integral to extract the coefficients of $H_\ell(q)$. We will use the saddle point method to chose our contour of integration, and will use the normal approximation to evaluate the integral.

Proof of Theorem (14). Recall the definitions,

$$K_\ell^+(w) = \frac{5\pi^2}{48} + K_\ell(w)$$
$$K_\ell^-(w) = \frac{5\pi^2}{48} - K_\ell(w)$$

as they are in the statement of Theorem 14. Let $w_0 = \pi\sqrt{\frac{1}{6n}}$. Suppose that $\Omega((\log \ell)/\ell) < \ell^{k_0} w_0 < O(\log^{-1} \ell)$ for sufficiently tight constants (we shall later on puts some constraints on the size of these constants). Suppose that k_0 is even. Then by Lemma 17 there exists a unique real q_n such that

$$\frac{\partial}{\partial w}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n} = -n$$

and $\Omega((\log \ell)/\ell) < \ell^{k_0} q_n < O(\log^{-1} \ell)$. (with weaker constants than those required for w_0 in place of q_n) We pick our constants to be tight enough that

$$K_\ell^+(q_n) > \frac{7.5\pi^2}{48}.$$

Also notice that

$$\frac{1}{2}\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n} = \Theta(q_n^{-3}).$$

and

$$\frac{1}{6}\frac{\partial^3}{\partial w^3}\left[\frac{K_\ell^+(w)}{w}\right] = O(q_n^{-4}).$$

We fix the constants on our bounds for w_0 so that these constants are are close as we like.

We are now ready to approximate $a_\ell(n)$ under these assumptions. We have that

$$a_\ell(n) = \frac{1}{2\pi i}\int H_\ell(e^{-w})e^{nw}dw$$

where the integral is over one full period, namely some path from w_1 to $w_1+2\pi i$ for some w_1. We chose the line of integration defined by $\Re(w) = q_n$. We break the integral up into three regions, $|\Im(w)| > \Re(w)(1-\epsilon)$, $\Re(w)^{4/3} < |\Im(w)| < \Re(w)(1-\epsilon)$ and $|\Im(w)| < \Re(w)^{4/3}$. We already know by Lemma 15 that the integral over the first region is

$$O\left(\exp\left\{\frac{7.2\pi^2}{48q_n} + nq_n\right\}\right).$$

Also we know by Lemma 16 that in the second and third regions, the integrand is

$$2^{\frac{\ell}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\frac{K_\ell^+(w)}{w} + nw\right\}(1+O(n^{-1/2})) =$$

$$2^{\frac{\ell}{2(\ell+1)}}\sqrt{\frac{w}{2\pi}}\exp\left\{\frac{K_\ell^+(q_n)}{q_n} + nq_n - \frac{\Im(w)^2}{2}\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n} + \frac{\Im(w)^3}{q_n^4}E(w)\right\}.$$
$$(1+O(n^{-1/2}))$$

where $|E(w)| < \frac{1}{2}\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n}(1+\epsilon/2)$. Hence, in the second region the integral is at most

$$\exp\left\{\frac{K_\ell^+(q_n)}{q_n} + nq_n - \Omega(n^{1/6})\right\}.$$

In the last region the integrand is

$$2^{\frac{\ell}{2(\ell+1)}}\sqrt{\frac{q_n}{2\pi}}\exp\left\{\frac{K_\ell^+(q_n)}{q_n} + nq_n - \frac{\Im(w)^2}{2}\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n}\right\}(1+O(n^{-1/2}+\Im(w)^3n^2)).$$

Since this approximates a normal distribution, we have that the integral in this region equals

$$\frac{2^{\frac{\ell}{2(\ell+1)}}}{2\pi}\sqrt{\frac{q_n}{\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n}}}\exp\left\{\frac{K_\ell^+(q_n)}{q_n} + nq_n\right\}(1+O(n^{-1/4})).$$

Since the other terms fit into the error, we have that

$$a_\ell(n) = \frac{2^{\frac{\ell}{2(\ell+1)}}}{2\pi}\sqrt{\frac{q_n}{\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^+(w)}{w}\right]_{w=q_n}}}\exp\left\{\frac{K_\ell^+(q_n)}{q_n} + nq_n\right\}(1+O(n^{-1/4})).$$

Analogously, if k_0 is odd, and q_n is the unique real and near w_0 so that

$$\frac{\partial}{\partial w}\left[\frac{K_\ell^-(w)}{w}\right]_{w=q_n} = -n,$$

then

$$a_\ell(n) = (-1)^n \frac{2^{\frac{\ell+2}{2(\ell+1)}}}{2\pi}\sqrt{\frac{q_n}{\frac{\partial^2}{\partial w^2}\left[\frac{K_\ell^-(w)}{w}\right]_{w=q_n}}}\exp\left\{\frac{K_\ell^-(q_n)}{q_n}+nq_n\right\}(1+O(n^{-1/4})).$$

□

5 Conclusion

In Section 3, we came up with an asymptotic formula for $p_\ell(a;n)$ that was valid for a large range of a. There are still a number of open questions relating to this result though.

Question 1. *What happens to $p_\ell(a;n)$ when a becomes larger or smaller than the values covered by Theorem 5?*

As a becomes smaller than is allowed by Theorem 5, our formula should still hold except with weaker bound. As a gets even smaller, it is unclear what happens except that the function ceases to be smooth for sufficiently small a. For a larger than in our allowed region, some analysis can be done by using our integral expression to approximate $\log(G_\ell(e^{-z};e^{-w}))$ for $1 \gg z \gg w$. We get the terms $\log(C) + \Delta z$ plus

$$\frac{\ell-1}{2\log\ell}\frac{z\log z}{w} + \frac{\kappa_1(w)z}{w} + \frac{\kappa_2(w)z^2}{w} + \frac{\kappa_3(w)z^3}{w} + \dots$$

where κ_i is periodic in $\log_\ell(w)$. Using this we can see that if we define $m = n-(\ell-1)a$ as before that $p_\ell(a;n)$ should fall off very roughly like

$$\exp\left\{-c_1\sqrt{n}\exp\left(-\frac{c_2 m}{\sqrt{n}}\right)\right\}.$$

It seems unlikely that the above can be extended reasonably to an asymptotic formula in a much larger range of a than that in Theorem 5 because of all of the extra terms in the asymptotic expansion of the generating function.

There are also some other points of interest relating to this formula. For example,

Question 2. *What is the general behavior of g?*

In Section 4 we analyzed the asymptotics of $a_\ell(n)$ and came up with a formula that works when ℓ is odd and $\log_\ell(n)$ is far from an integer. In the range where $\log_\ell(n)$ is close to an integer, computer analysis suggests that $a_\ell(n)$ can look rather chaotic when the main terms cancel each other (although it is hard to get very many good data points). It seems likely that the asymptotics of $a_\ell(n)$ can also be demonstrated for small odd ℓ using a sufficient amount of computer computation in order to show the the generating function, $H_\ell(q)$ is small away from $q = \pm 1$. It would also be nice to know:

Question 3. *What is the behavior of $a_2(n)$?*

This case should be qualitatively different from the case of odd ℓ since here $H_\ell(q)$ is small for q near any root of unity. Techniques similar to those in this paper suggest that $a_2(n)$ is on the order of

$$\exp\left\{\sqrt{\frac{cn}{\log n}}\right\}.$$

It should also be possible to approximate the size of $\sum_a \omega^n a_\ell(n)$ if ω is any root of unity using similar techniques, although the integration formula may involve a complicated sum of Dirichlet L-functions.

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