1 Changes of Variables

Change of variables is one of the most important techniques for evaluating integrals or sums. It is closely related to the combinatorial technique of counting things in two different ways. Some things to keep in mind when looking for a change of variable:

- What does my entire region of integration look like? Is there some nicer way to parameterize this region?
- What are the important terms in the integrand? Can I make them new variables of integration?
- What happens if I change the order of integration?

1982 B2 Let \( A(x, y) \) denote the number points \((n, m)\) in the plane with integer coordinates so that \( n^2 + m^2 \leq x^2 + y^2 \). Let \( g = \sum_{k=0}^{\infty} e^{-k^2} \). Express

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y)e^{-x^2-y^2} \, dx \, dy
\]
as a polynomial in \( g \).

2 Asymptotics and Finding the Main Term

A lot of calculus problems are about approximation. A lot of approximation is about finding out which terms you need to pay attention to and which terms can be ignored (and then the errors bounded). Things that you might want to ask yourself when trying to figure this out include:

- Where is the thing that I am trying to sum/integrate largest? How does it behave near the largest point?
- What things are approximately the same size as what I am looking for that might be easier to work with?
- Can I approximate a function by its Taylor series? How much error does this introduce?
- What is a rough bound for this error term? Is it small enough to ignore.

Useful notation: \( O(X) \) is used to denote a quantity that is always at most a constant multiple of \( X \). This often allows you to understand the size of error terms more easily.

1983 A6 Let \( \exp(t) \) denote \( e^t \) and

\[
F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3 + v^3) \, du \, dv.
\]

Find \( \lim_{x \to \infty} F(x) \) or show that the limit does not exist.
3 Approximating a Sum by an Integral

One important technique is that of approximating a sum by an integral. Namely for a nice function \( f \), we expect that

\[
\sum_{n=a}^{b} f(n) \approx \int_{a}^{b} f(x)dx.
\]

There are many versions of this. Perhaps the simplest is that for \( f \) a monotonic function, we have that

\[
\sum_{n=a}^{b} f(n)
\]

is between

\[
\int_{a-1}^{b} f(x)dx \quad \text{and} \quad \int_{a}^{b+1} f(x)dx.
\]

However, even for other \( f \) we can often say something useful. As long as \( f \) is reasonably continuous, we just need to show that

\[
f(n) \approx \int_{n}^{n+1} f(x)dx
\]

and sum over \( n \). This should hold so long as \( f \) does not vary too much over the interval \([n, n + 1]\).

1981 B1 Find

\[
\lim_{n \to \infty} \left[ \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} (5h^4 - 18h^2k^2 + 5k^4) \right].
\]

4 Optimization

When trying to solve optimization problems consider:

- What is the actual optimizer? Usually it is either highly symmetric or highly degenerate.
- What are the critical points?
- Can I simplify the equation with a change of variables?
- Can I think about the optimization in the opposite direction? Instead of optimizing \( x \) for fixed \( y \), optimize \( y \) for fixed \( x \).

1972 B2 A particle moving in a straight line starts from rest and attains a velocity \( v_0 \) after traversing a distance \( s_0 \). If the motion is such that the acceleration was never increasing, find the maximum time for the traverse.