This homework is due in class on Friday, November 2nd. Please complete one of the following problems.

1973 B2 Let $z = x + iy$ be a complex number with $x$ and $y$ rational numbers and $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every integer $n$.

2008 A3 Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that $a_j$ does not divide $a_k$, and replace $a_j$ and $a_k$ by $\gcd(a_j, a_k)$ and $\text{lcm}(a_j, a_k)$ respectively. Prove that if this process is repeated, it will eventually stop and that the final sequence does not depend on the choices being made.

2008 B4 Let $p$ be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2 - 1)$ are distinct modulo $p^2$. Show that $h(0), h(1), \ldots, h(p^3 - 1)$ are distinct modulo $p^3$. 