Combinatorics is a very broad area covering a lot of discrete mathematics, often with a focus on problems involving counting things. Here I’ll discuss some basic tools that are often useful in such problems.

1 The Pigeonhole Principle

The pigeonhole principle is a deceptively simple result, that turns out to be surprisingly powerful in a number of contexts. It says simply:

**Theorem.** If $n + 1$ pigeons are placed into $n$ holes (with each pigeon assigned to one hole), then some hole must contain at least two pigeons.

Or slightly more generally as:

**Theorem.** If more than $nk$ pigeons are placed into $n$ holes (with each pigeon assigned to one hole), then some hole must contain at least $k + 1$ pigeons.

The proof is quite simple: this is just saying that the maximum number of pigeons in a hole is at least the average number of pigeons in a hole. This might seem like it is too simple to be useful, but this is not the case. For example, you can use it to show:

**Corollary.** Let $S$ be a set of 20 integers each between 0 and 50,000. Then there exist two distinct subsets $A$ and $B$ of $S$ so that the sum of the elements of $A$ equals the sum of the elements of $B$.

**Proof.** The total number of such subsets is $2^{20} = 1,048,576$. However, the possible sums are only 0, 1, 2, \ldots, 20\cdot50,000 = 1,000,000. Since there are more subsets than sums, by the pigeonhole principle, some two subsets must have the same sum.

1978 A1 Let $A$ be a set of any 20 distinct integers from the arithmetic progression 1, 4, 7, \ldots, 100. Prove that there must be two distinct integers in $A$ whose sum is 104.
2 Inclusion-Exclusion

If two sets are disjoint, you can count the number of elements in either by taking the sum of their sizes. So \(|A \cup B| = |A| + |B|\). However, if they intersect, you have counted the members of the intersection twice and so you need to adjust your count by removing them. So more generally, \(|A \cup B| = |A| + |B| - |A \cap B|\). When dealing with more than two sets, the general result is called Inclusion-Exclusion and says for example that

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.
\]

In general, the size of a union of a bunch of sets can be written as a sum of the sizes of the intersections of all of the subcollections of these sets times appropriate signs.

This is often a useful counting technique as the sizes of the intersections of sets is often easier to get at than the sizes of their unions.

1983 A1 How many positive integers \(n\) are there so that \(n\) is an exact divisor of at least one of the numbers 10\(^4\), 20\(^3\)?

3 Binomial Coefficients

One of the more interesting basic counting objects are the binomial coefficients. Put simply, \(\binom{n}{k}\) \((n \text{ choose } k)\) counts the number of ways to pick \(k\) things from a set of size \(n\). It is given by the formula

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!}.
\]

Some other useful relations to keep in mind are:

\[
\binom{n}{k} = \binom{n}{n-k}, \quad \text{and} \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.
\]

Finally, binomial coefficients show up in the binomial theorem, which says that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

1983 A4 Let \(k\) be a positive integer and let \(m = 6k - 1\). Let

\[S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.
\]

For example, with \(k = 3\),

\[S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.
\]

Prove that \(S(m)\) is never zero.
4 Generating Functions

We saw in the last problem that sometimes we can gain insight into combinatorial sums by treating the terms as elements of a power series. It turns out that lots of combinatorial objects can be usefully studied by treating their elements as power series. One particular use of this is in computing the number of ways to write things as sums. In particular, if we consider the product of power series

\[
\left( \sum_{n \in A} x^n \right) \left( \sum_{m \in B} x^m \right) = \sum_{n \in A, m \in B} x^{n+m}.
\]

Collecting like terms, we find that the \( x^k \)-coefficient of the product is the number of ways to write \( k \) as a sum of an element from \( A \) and an element from \( B \).

1983 B2 For integers \( n \), let \( C(n) \) be the number of representations of \( n \) as the sum of a non-increasing sequence of powers of 2 where no power can be used more than three times. For example, \( C(8) = 5 \) since the representations of 8 are

\[
8 = 4 + 4 = 4 + 2 + 2 = 4 + 2 + 1 + 1 = 2 + 2 + 2 + 1 + 1.
\]

Prove or disprove that there is a polynomial \( P(x) \) so that \( C(n) = \lfloor P(n) \rfloor \) for all \( n \) (here \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \)).