Math 204C Homework 1

Spring 2019

This homework is due on in class Friday April 19th.

Question 1 (Riemann Hypothesis). The Riemann Hypothesis states that all of the nontrivial zeroes of $\zeta(s)$ lie on the line $\Re(s) = 1/2$. Show that this would imply the much stronger bounds for the prime number theorem of

$$\vartheta(x) = x + O(\sqrt{x}\log^2(x)).$$

Question 2 (Even Zeta Values). Recall that we have

$$\zeta(s)\Gamma(s) = \int_0^\infty t^s \left(\frac{e^{-t}}{1-e^{-t}}\right) \frac{dt}{t}.$$

Show how this integral can be related to an integral of the same quantity over a contour from $\infty$ to itself looping around $0$ (Hint: for non-real $s$ consider the difference of the two branches). Use this to relate the values of $\zeta$ at negative integers to the coefficients of $1/(1 - e^{-t})$. Use this to get a formula for $\zeta(2n)$ for positive integers $n$. You may want to use the fact that $\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$ (this can be proved by noting that the ratio is periodic mod 1 and has limit 1 as $s \to \infty$).

Question 3 (Sums of Sigma). Let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of $n$. Show that

$$\sum_{n \leq x} \sigma(n) \sim \frac{x^2\pi^2}{12}.$$

You might want to note that

$$\sum_n \sigma(n)n^{-s} = \zeta(s)\zeta(s-1).$$

Additionally, show that if instead you want to approximate

$$\sum_{n \leq x} \sigma(n)((x-n)/x)^k$$

for some integer $k$ that there is a formula with error $O(x^{-c_k})$ where $c_k \to \infty$ with $k$. 