Math 184A Homework 7

Spring 2018

This homework is due on gradescope by Friday June 8th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommend though not required.

Question 1 (Avoidance Bounds, 20 points). From the book we know that \(S_n(1432) \leq 9^n\). Find a constant \(C\) so that \(S_n(321456987) \leq C^n\) for all \(n\).

Solution. First let’s prove,

Claim 1. \(S_n(123) \leq 9^n\)

Proof. Since any permutation that avoids 321 must also avoid 1432, we have \(S_n(321) \leq S_n(1432) \leq 9^n\), and by reflection we have \(S_n(123) = S_n(321)\). Hence the claim holds.

Claim 2. \(S_n((31245) \oplus 1) = S_n(321456) \leq 36^n\)

Proof. Since 3214 is the reverse of 4123, which is the complement of 1432, we have

\[S_n((321) \oplus 1) = S_n(3214) = S_n(4123) = S_n(1432) \leq 9^n\]

By claim 1,

\[S_n(1 \oplus (12)) = S_n(123) \leq 9^n\]

By theorem 14.17,

\[S_n(321456) = S_n((321) \oplus 1 \oplus (12)) \leq (\sqrt{9} + \sqrt{9})^{2n} = 36^n\]

By theorem 14.17 again,

\[S_n(321456987) = S_n((32145) \oplus 1 \oplus (321)) \leq (\sqrt{36} + \sqrt{9})^{2n} = (6 + 3)^{2n} = 81^n\]

Hence \(S_n(321456987) \leq C^n\) holds for all \(n\) if we take \(C = 81\).

Question 2 (Hill Avoidance, 40 points). Let a \(k\)-hill in a permutation be a subsequence of \(2k - 1\) of the entries the first \(k\) of which are in increasing order and the last \(k\) of which are in decreasing order. Note that a \(k\)-hill is not a single pattern. For example, a 2-hill is either an instance of the pattern 132 or an instance of the pattern 231.

(a) Show that the number of permutations of \([n]\) with no 2-hill is \(2^{n-1}\). [15 points]

Solution. The number of permutations with no 2-hill = \(S_n(132, 231) = \sum_{m=1}^{n}(\text{the number of } n\text{-permutations that avoid (132, 231) where } n \text{ is located at the } m\text{-th position})\). Suppose \(n\) is located at the \(m\)-th position.

(1) If \(m = 1\), \(n\) cannot engage in any forbidden patterns with entries that appear after \(n\), hence for the whole permutation to avoid (132, 231), it suffices to have the last \(n - 1\) entries to avoid (132, 231). Hence the number of \(n\)-permutations that avoid (132, 231) where \(n\) is located at the \(m\)-th position = \(S_{n-1}(132, 231)\).
(2) If $1 < m < n$, then in order to avoid 132 any entries located at the right of $n$ must be greater than those at the left of $n$, otherwise there exists $(a,b)$ such that $a < b < n$ where $a$(resp. $b$) is at the right(resp. left) of $n$, then $(a,b,n)$ will form a 132 pattern. Similarly, to avoid 231 any entries located at the right of $n$ must be greater than those at the left of $n$. No permutation can satisfy both of these two requirements, hence the number of $n$-permutations that avoid (132, 231) where $n$ is located at the $m$-th position = 0.

(3) If $m = n$, $n$ cannot engage in any forbidden patterns with entries that appear before $n$, hence for the whole permutation to avoid (132, 231), it suffices to have the first $n - 1$ entries to avoid (132, 231). Hence the number of $n$-permutations that avoid (132, 231) where $n$ is located at the $m$-th position = $S_{n-1}(132, 231)$.

Therefore we have, $S_n(132, 231) = 2 * S_{n-1}(132, 231)$, solving the recurrence relation with the initial condition $S_1(132, 231) = 1 = 2^0$, we obtain $S_n(132, 231) = 2^{n-1}$ for all $n$.

(b) Show that the number of permutations of $[n]$ with no $k$-hill is at most $(4(k-1)^2)^n$. [Hint: try to find a decreasing sequence among elements that are the largest of a $k$-term increasing subsequence.] [25 points]

Solution. Let us say that an entry $x$ is of left order $i$ if $x$ is the top of an increasing subsequence of length $i$, but there is no increasing subsequence of length $i + 1$ whose top is $x$.

Let us say that an entry $x$ is of right order $i$ if $x$ is the top of a decreasing subsequence of length $i$, but there is no decreasing subsequence of length $i + 1$ whose top is $x$.

The for all $i$, elements of left order $i$ must form a decreasing subsequence and elements of right order $i$ must form an increasing subsequence.

For any permutation that avoids a $k$-hill, the minimum of the left order and the right order of $x$ is at most $k - 1$ for all $x \in [n]$ [otherwise there will be a $k$-hill].

We define $A^l_m$ to be the set of all elements contained in $[n]$ whose left order is $m$ and whose right order is greater than or equal to $m$ for $1 \leq m \leq n$.

And we define $A^r_m$ to be the set of all elements contained in $[n]$ whose right order is $m$ and whose left order is greater than $m$ for $1 \leq m \leq n$.

For any $k$-hill avoiding permutation, $\{A^l_m\}_{m=1}^{k-1} \cup \{A^r_m\}_{m=1}^{k-1}$ forms a partition of $[n]$. Therefore, any $k$-hill avoiding permutation can be decomposed into $k - 1$ classes of increasing subsequence and $k - 1$ classes of decreasing subsequence. There are $(2(k-1))^n$ ways to partition the elements into $2(k-1)$ classes and there are less than $(2(k-1))^n$ ways to assign each position to one of the subsequences, completing the proof.

**Question 3** (Marriage Lemma, 40 points). The Marriage Lemma states that if you are given two sets $S$ and $T$ of size $n$ and a set $E$ of pairs of one element of each set, then there is a matching between $S$ and $T$ (namely a set of $n$ pairs from $E$ using each element of $S$ and each element of $T$ exactly once) unless there is some subset $c$ so that the total number of elements of $T$ that pair with some element of $S'$ is less than $|S'|$.

Prove the Marriage Lemma using Dilworth’s Theorem.

**Solution.** For all $s \in S$, let $A_s$ be the set of elements $t \in T$ such that $(s, t)$ is contained in $E$.

(1) If for all subsets $S' \subset S$, we have $| \bigcup_{s \in S'} A_s | \geq |S'|$

Consider the poset $P$ whose elements are those of $S \cup T$, where $t \leq s$ if and only if $t \in T, s \in S$ such that $t \in A_s$, and no other comparisons hold.

**Claim.** Any chain of $P$ can consist of at most one element from $S$(resp. $T$)

**Proof.** Any two elements of $S$(resp. $T$) are not comparable, therefore they cannot be in the same chain.

**Claim.** The size of the largest antichain is $n$.

**Proof.** Assume for sake of contradiction that there exists an antichain of size $n + 1$, suppose this antichain contain $i$ elements from $S$ and $j$ elements from $T$, we call these elements $\{s_1, s_2 \ldots s_i\}$ and $\{t_1, t_2 \ldots t_j\}$.
Notice that \( i, j \geq 1 \) and \( i + j = n + 1 \Rightarrow j = n + 1 - i > n - i \).

For \( 1 \leq m \leq j \), \( t_m \) is not comparable with \( s_k \) for \( 1 \leq k \leq i \) by the definition of an antichain \( \Rightarrow t_m \notin \bigcup_{k=1}^{i} A_{s_k} \Rightarrow t_m \in T \setminus \bigcup_{k=1}^{i} A_{s_k} \Rightarrow \{t_1, t_2 \ldots t_j\} \subseteq T \setminus \bigcup_{k=1}^{i} A_{s_k} \Rightarrow j = |\{t_1, t_2 \ldots t_j\}| \leq |T \setminus \bigcup_{k=1}^{i} A_{s_k}| = |T| - |\bigcup_{k=1}^{i} A_{s_k}| \leq n - |\{s_1, s_2 \ldots s_i\}| \leq n - i \) by our previous assumption.

But \( j > n - i \) and \( j \leq n - i \) cannot both hold, contradiction! Hence the size of the largest antichain is at most \( n \).

We show that there exists an antichain of size \( n \).

Consider the set \( S \), notice that any two elements of \( S \) are not comparable, hence \( S \) is an antichain, and \( |S| = n \).

By Claim 1, any chain in \( P \) can consist of at most 2 elements. By Dilworth’s Theorem, the number of chains in the minimum chain cover of \( P \) is \( n \). To cover all \( 2n \) elements in \( P \), every chain in the minimum chain cover must consist of exactly 2 elements, namely, one from \( S \) and one from \( T \). Since the chains are disjoint, we have therefore proved that there exist a matching between \( S \) and \( T \).

(2) If there exists subset \( S' \subset S \) such that \( |\bigcup_{s \in S'} A_s| < |S'| \)

Suppose \( |S'| = k \leq n \), we can number the elements of \( S' = \{s_1, s_2 \ldots s_k\} \).

Assume for sake of contradiction that there exist a matching between \( S \) and \( T \), then for \( 1 \leq i \leq k \), there exist \( t_i \in T \) such that \( (s_i, t_i) \in E \). By our definition of \( A_s \) we must have

\[
\{t_1, t_2 \ldots t_k\} \subset \bigcup_{i=1}^{k} A_{s_i} = \bigcup_{s \in S'} A_s
\]

But,

\[
k = |\{t_1, t_2 \ldots t_k\}| \leq \bigcup_{i=1}^{k} A_{s_i} = \bigcup_{s \in S'} A_s < |S'| = k
\]

Contradiction! Therefore there does not exist a matching between \( S \) and \( T \).

**Question 4** (Extra credit, 1 point). **Free point!**