Math 184A Homework 6

Spring 2018

Question 1 (Limiting Exponent, 50 points).

(a) Show that for any permutation $q$ and any positive integers $n$ and $m$ that

$$S_n(q)S_m(q) \leq S_{n+m}(q).$$

[25 points]

(b) Use this to prove that for any permutation $q$ that

$$L(q) = \lim_{n \to \infty} \sqrt[n]{S_n(q)}$$

exists and is finite. [You may use the result mentioned in class that for each $q$ there is a constant $C_q$ so that $S_n(q) \leq C_q^n$] [25 points]

Solution.

(a) Left hand side is the number of pair $(p_n, p_m)$ such that $p_n$ is the permutation of $[n]$ avoiding $q$ and $p_m$ is the permutation of $[m]$ avoiding $q$. Right hand side is the number of permutation of $[n+m]$ that avoid $q$. The idea is to construct an injective map from left hand side to right hand side.

Suppose the permutation $q = a_1a_2\ldots a_n$. There are two cases: $a_1 > a_n$ or $a_1 < a_n$. If $a_1 < a_n$, for any pair $(p_n, p_m), p_n = b_1b_2\ldots b_n, p_m = c_1c_2\ldots c_m$, we map it to

$$(b_1 + m)(b_2 + m)\ldots (b_n + m)c_1c_2\ldots c_m$$

the first $n$ entries are always larger then the last $m$ entries, which implies that it avoid $q$. If $a_1 > a_n$, similar idea, we maps the pair to

$$b_1b_2\ldots b_n(c_1 + n)(c_2 + n)\ldots(c_m + n)$$

(b) Let $f_n = \log S_n(q)$, then we have $f_n \geq 0, f_n + f_m \leq f_{n+m}, \frac{f_n}{n} \leq C_q$. We want to prove that $\lim_{n \to \infty} \frac{f_n}{n}$ exists. Since $\frac{f_n}{n}$ is bounded above, the $\sup_{n \geq 1} \frac{f_n}{n}$ is finite. We can assume $\sup_{n \geq 1} \frac{f_n}{n} = a$. So now it remains to show that $\lim_{n \to \infty} \frac{f_n}{n} \geq a$.

For $\forall \epsilon > 0$, there exist a $m$ such that $\frac{f_m}{m} \geq a - \epsilon$. Hence for $n$ large enough, $n = km + r, 0 \leq r \leq m - 1$,

$$\frac{f_n}{n} \geq \frac{km + f_r}{km + r} \geq \frac{km}{km + m} (a - \epsilon) = \frac{k}{k + 1} (a - \epsilon) \to a - \epsilon as n \to \infty$$

So $\lim_{n \to \infty} \frac{f_n}{n} \geq a - \epsilon$ which implies that $\lim_{n \to \infty} \frac{f_n}{n} \geq a$

Question 2 (Avoiding 132 and 4321, 50 points). Let $S_n(132, 4321)$ be the number of permutations of $[n]$ that avoid both 132 and 4321. Show that

$$S_n(132, 4321) = 2 \left( \binom{n}{4} + \binom{n + 1}{3} \right) + 1.$$
Solution. We use induction here.

Base case: when \( n = 1, 2, 3, 4 \), we can check that the equation holds.

Inductive step: when \( n = 5 \), we consider the position of \( n \).

If its position is at \( i \)th place, where \( 1 \leq i \leq n - 1 \), then the right hand side is nonempty. Because it avoid 132, entries at left hand side are always larger than the entries at right hand side. Furthermore, at least 1 of them avoid pattern 21, that is, at least one of them is increasing. Otherwise, there will be 4 entries with pattern 4321. So here we have 2 cases: left hand side increasing or right hand side increasing. When the left hand side is increasing, right hand side avoid pattern 132 and 321. If there is 3 entries with pattern 321, \( n \) plus these 3 entries will have pattern 4321. Similarly, if the right hand side is increasing, then left hand side should avoid 132 and 321. If there is 3 entries with pattern 321, these 3 entries plus an entry at right hand side(which is nonempty) will have pattern 4321.

If \( n \)'s position is at \( n \)th place, then the left hand side should avoid 132 and 4321.

Therefore we have the following recursive formula.

\[
f_n(132, 4321) = \sum_{i=1}^{n-1} (f_{i-1}(132, 321) + f_{n-i}(132, 321) - 1) + f_{n-1}(132, 4321)
\]

Now we want to compute \( f_n(132, 321) \). Similar idea as above, consider the position of \( n \).

If its position is at \( i \)th place, where \( 1 \leq i \leq n - 1 \), then both side have to be increasing.

If its position is at \( n \)th place, then the left hand side avoid 132 and 321.

This give us a recursive formula

\[
f_n(132, 321) = n - 1 + f_{n-1}(132, 321)
\]

which give us

\[
f_n(132, 321) = \binom{n}{2} + 1
\]

Put this in the previous equation, and then use the inductive assumption, we have

\[
f_n(132, 4321) = \sum_{i=1}^{n-1} \left( i - 1 \cdot \binom{i-1}{2} + \binom{n-i}{2} + 1 \right) + 2 \binom{n-1}{4} + \binom{n}{3} + 1 = 2 \binom{n}{4} + \binom{n+1}{3} + 1
\]

So the equation holds for \( n \).

Question 3 (Extra credit, 1 point). Approximately how much time did you spend on this homework?