Question 1 (Combinatorial Identity, 20 points). Come up with a combinatorial proof of the following identity for $n \geq 2m > 0$:

$$\sum_{k=m}^{n-m} \binom{n}{k} c(k, m) c(n-k, m) = \binom{2m}{m} c(n, 2m).$$

Solution. We count the same number twice to prove this identity. We color the numbers in $[n]$ with red or blue. We say a cycle is red (blue) if all numbers in the cycle are red (blue). We want to count the number of colored permutations with $2m$ cycles, $m$ of which are red while the rest $m$ cycles are blue. We count in the following two ways:

1. suppose $k$ of the numbers are red, then $n-k$ are blue. we have $\binom{n}{k}$ ways to pick such $k$ numbers out of $[n]$ and color them red (the rest are automatically blue). Then, in each coloring, we have $c(k, m)$ ways to distribute the $k$ red numbers into $m$ cycles and $c(n-k, m)$ ways to distribute the $n-k$ blue numbers into $m$ cycles. Note that we have at least $m$ red numbers and $m$ blue numbers, so totally we have $\sum_{k=m}^{n-m} \binom{n}{k} c(k, m) c(n-k, m)$ such colored permutations.

2. We first distribute the set $[n]$ into $2m$ cycles, in $\binom{n}{2m}$ possible ways. Then, in each way, we color $m$ of the cycles red, in $\binom{2m}{m}$ ways. Then, the rest cycles are automatically blue. Totally we have $\binom{2m}{m} c(n, 2m)$ such colored permutations.

We can say now the number of colored permutations of $[n]$ into $2m$ cycles with $m$ cycles red and $m$ cycles blue is

$$\sum_{k=m}^{n-m} \binom{n}{k} c(k, m) c(n-k, m) = \binom{2m}{m} c(n, 2m).$$

Question 2 (Generating Functions, 50 points).

(a) Consider the sequence defined by the recurrence, $a_0 = 0, a_1 = 3$ and

$$a_{n+2} = a_{n+1} + 2a_n - 6$$

for $n \geq 0$. Find a formula for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. [10 points]

(b) Using this generating function find a formula for $a_n$ (you will want to find a partial fractions decomposition). [10 points]

(c) Consider the sequence defined by the recurrence, $b_0 = 0$ and

$$b_n = n + \frac{2}{n} \sum_{i=0}^{n-1} b_i.$$  

Find a differential equation satisfied by the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ (you do not have to solve it). You may need to use the identity that

$$\sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1 + x}{(1-x)^3}.$$ 

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Note: For those of you interested in computer science, \( b_n \) is related to the runtime of the quicksort algorithm. [15 points]

(d) It turns out that the generating function above is given by

\[
B(x) = \frac{2 \log \left( \frac{1}{1-x} \right) - x}{(1-x)^2}.
\]

Use this to give a formula for \( b_n \). You may need to use the harmonic numbers \( H_k = \sum_{n=1}^{k} \frac{1}{n} \approx \log(k) \) to express your answer. Recall that \( \log(1/(1-x)) = \sum_{n=1}^{\infty} \frac{x^n}{n} \). [15 points]

Solution.

(a)

\[
\begin{align*}
A(x) &= \sum_{n=0}^{\infty} a_n x^n = 0 + 3x + \sum_{n=2}^{\infty} a_n x^n \\
&= 3x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2} - 6)x^n \\
&= 3x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2a_{n-2} x^n + \sum_{n=2}^{\infty} -6x^n \\
&= 3x + \sum_{n=1}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^{n+2} - \frac{6x^2}{1-x} \\
&= 3x + x \sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} 2a_n x^n - \frac{6x^2}{1-x} \\
&= 3x + (x + 2x^2) \sum_{n=0}^{\infty} a_n x^n - \frac{6x^2}{1-x} \\
&= 3x + (x + 2x^2)A(x) - \frac{6x^2}{1-x},
\end{align*}
\]

so we have

\[
A(x) = \frac{6x^2 - 3x}{2x^2 + x - 1} = \frac{3x - 9x^2}{(1-x)(1+x)(1-2x)}.
\]

(b) Writing \( A(x) \) in partial fraction expansion (recall your Math 20B), we have

\[
\begin{align*}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \frac{3}{1-x} - \frac{2}{x+1} - \frac{1}{1-2x} \\
&= 3 \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (2x)^n \\
&= \sum_{n=0}^{\infty} (3 - 2(-1)^n - 2^n) x^n,
\end{align*}
\]

so

\[a_n = 3 - 2(-1)^n - 2^n.\]
(c) The identity is equivalent to

\[ nb_n = n^2 + 2 \sum_{i=0}^{n-1} b_i. \]

We multiply both sides by \( x^{n-1} \) to obtain

\[ nb_n x^{n-1} = n^2 x^{n-1} + 2(\sum_{i=0}^{n-1} b_i)x^{n-1}. \]

Sum both sides for \( n \) from 1 to \( \infty \), we have

\[ \sum_{n=1}^{\infty} nb_n x^{n-1} = \sum_{n=1}^{\infty} n^2 x^{n-1} + 2(\sum_{i=0}^{n-1} b_i)x^{n-1}, \]

or

\[ \sum_{n=1}^{\infty} nb_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)^2 x^n + 2(\sum_{i=0}^{\infty} b_i)x^n. \]

We have two identities for generating function which help us to simplify the equation above. The first one is

\[ B'(x) = (\sum_{n=0}^{\infty} b_n x^n)' = \sum_{n=0}^{\infty} (b_n x^n)' = \sum_{n=1}^{\infty} nb_n x^{n-1}, \]

the second one is

\[ B(x) \frac{1}{1-x} = B(x) \cdot \frac{1}{1-x} = (\sum_{n=0}^{\infty} b_n x^n) \cdot (\sum_{n=0}^{\infty} x^n) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} b_i)x^n. \]

Based on these two identities, and also the identity given in the question that

\[ \sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3}, \]

our equation becomes a differential equation about \( B(x) \) that

\[ B'(x) = \frac{1+x}{(1-x)^3} + 2B(x). \]

(d) We need the following 2 identities:

\[ \frac{1}{(1-x)^2} = \left( \frac{1}{1-x} \right)' = (\sum_{i=0}^{\infty} x^n)' = \sum_{i=0}^{\infty} nx^{n-1} = \sum_{i=0}^{\infty} (n+1)x^n \]

and

\[ \log\left( \frac{1}{1-x} \right) = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]

Now, let’s write \( B(x) \) in power series:

\[ B(x) = (2 \log\left( \frac{1}{1-x} \right) - x) \cdot \frac{1}{(1-x)^2} \]

\[ = ((\sum_{n=1}^{\infty} \frac{x^n}{n}) - x)(\sum_{i=0}^{\infty} (n+1)x^n) \]

\[ = \sum_{n=1}^{\infty} ((\sum_{k=1}^{n} \frac{2}{k}(n-k+1)) - n)x^n \]

\[ = \sum_{n=1}^{\infty} ((2(n+1) \sum_{k=1}^{n} \frac{1}{k}) - 2n - n)x^n \]

\[ = \sum_{n=1}^{\infty} ((2(n+1)H_n - 3n)x^n. \]
So, $b_n = 2(n + 1)H_n - 3n.$

**Question 3** (Partition Generating Functions, 30 points). (a) Let $a_n$ be the number of integer partitions of $n$ into distinct parts. Show that this sequence has the generating function

$$
\sum a_n x^n = (1 + x)(1 + x^2)(1 + x^3) \cdots = \prod_{n=1}^{\infty} (1 + x^n).
$$

[10 points]

(b) Let $b_n$ be the number of integer partitions of $n$ into odd parts. Show that this sequence has the generating function

$$
\sum b_n x^n = \frac{1}{(1-x)(1-x^3)(1-x^5) \cdots} = \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}}.
$$

[10 points]

(c) Show directly that the above generating functions are equal. [10 points]

**Solution.**

(a) Like the solution of Example 8.9 in the text book, we solve this question by analyze the coefficient of $x^n$ of both sides. The RHS is a infinite product. We can take it as a sum of infinite products. In a infinite product term contributing to $x^n$, if $x^{j_i}$ is taken in the $j_i^{th}$ parenthesis for $i = 1, \ldots, k$ and 1 is taken in all the other parentheses, we will get $\sum_{i=1}^{k} j_i = n$, thus we get a partition, $(j_k, j_{k-1}, \ldots, j_1)$, of $n$ into $k$ distinct parts. Conversely, every partition of $n$ into distinct parts can be associated to a product on the RHS, meaning that the coefficient of $x^n$, $a_n$, is the number of partitions of $n$ into distinct parts.

(b) We want to show that

$$
\sum b_n x^n = \prod_{n=1}^{\infty} (1 + x^{2n-1} + (x^{2n-1})^2 + (x^{2n-1})^3 + \cdots).
$$

The RHS is also a sum of infinite products. In a infinite product term contributing to $x^n$, if $(x^{2i-1})^{\alpha_i}$ is taken in the $i^{th}$ parenthesis, we will get $\sum_{i=1}^{\infty} (2i - 1)\alpha_i = n$ and a partition $\left(\prod_{i=1}^{\infty} (2i - 1)^{\alpha_i}\right)$ of $n$ into odd parts. Conversely, every partition of $n$ into odd parts can be associated to a product on the RHS if the partition contains $\alpha_i$ parts of size $(2i - 1)$, we take $(x^{2i-1})^{\alpha_i}$ in the $i^{th}$ parenthesis, meaning that the coefficient of $x^n$, $a_n$, is the number of partitions of $n$ into odd parts.

(c) Using direct computation,

$$
\sum a_n x^n = \prod_{n=1}^{\infty} (1 + x^n)
= \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{1 - x^n}
= \prod_{n=0}^{\infty} \frac{1 - x^{2n}}{1 - x^n}
= \prod_{n>0 even} 1 - x^n
= \prod_{n>0 even and odd} 1 - x^n
= \sum b_n x^n,
$$

so that the two generating functions are equal.