Math 184A Homework 4

Spring 2018

This homework is due on gradescope by Friday May 11th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommend though not required.

Question 1 (Permutation Parity, 20 points). Let \( n > 1 \) be an integer and let \( S \) be a set of pairs of numbers \((i,j)\) with \( i, j \in [n] \). Say that a permutation \( \pi \) of \([n]\) avoids \( S \) if \( \pi(i) \neq j \) for all \((i,j) \in S\). So, for example, a derangement is a permutation that avoids \( \{(1,1), (2,2), (3,3), \ldots, (n,n)\} \). Suppose that for any \( n-1 \) elements of \( S \) that either some two share a first coordinate or some two share a second coordinate. Prove that the number of permutations that avoid \( S \) is even. [Hint: Count the number using Inclusion-Exclusion.]

Question 2 (Size of Central Binomial Coefficients, 20 points). Show that for any \( n \geq 1 \)

\[ 4^n \geq \binom{2n}{n} \geq \frac{4^n}{2n+1}. \]

[Hint: for the lower bound show that \( \binom{2n}{n} \geq \binom{2n}{k} \) for any \( k \).] [Note: For those who know some number theory, it is not hard to see that \( \binom{2n}{n} \) is divisible by the product of all primes \( n \leq p \leq 2n \). This allows one to prove rough upper bounds on the number of primes.]

Question 3 (Sums of Binomial Coefficients, 30 points).

(a) Give a formula for \( \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{\lfloor n/2 \rfloor} \) as a function of \( n \). [Hint: use the binomial theorem. You’ll need a way to make the odd terms go away.][10 points]

(b) Give a formula for \( \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{\lfloor n/3 \rfloor} \) as a function of \( n \). [Hint: same idea, but you might need to use complex numbers.][20 points]

Question 4 (Linear Homogeneous Recurrence Relations, 30 points). Suppose that a sequence \( A_n \) satisfies a linear homogeneous recurrence relation with constant coefficients. Namely, suppose that there are constants \( C_1, C_2, \ldots, C_k \) so that

\[ A_n = C_1 A_{n-1} + C_2 A_{n-2} + \cdots + C_k A_{n-k} \]

for all \( n \geq k \).

(a) Show that the generating function \( F(x) = \sum_{n=0}^{\infty} A_n x^n \) is given by a rational function in \( x \) (namely a ratio of polynomials in \( x \)). [15 points]

(b) Given that partial fraction decompositions, allow you to write any rational function as a polynomial plus a linear combination of terms of the form \( 1/(1-bx)^m \), show that there’s a formula expressing \( A_n \) as some linear combination of terms of the form \( n^k b_i^n \) for all sufficiently large \( n \). [15 points]

Question 5 (Extra credit, 1 point). Approximately how much time did you spend on this homework?