Math 184A Homework 2

Spring 2018

This homework is due on gradescope by Friday April 20th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommend though not required.

**Question 1** (Partition Recurrence, 15 points). We didn’t mention in class any method to compute partition numbers, but there is a relatively simple recurrence relation that can be used for them. Prove that for all \( n \geq k \geq 1 \) that

\[
p_k(n) = \sum_{i=0}^{k} p_i(n - k).
\]

**Question 2** (Partitions with Sequential Part Sizes, 15 points). Show that the number of partitions of \( n \) into parts of distinct sizes is the same as the number of partitions of \( n \) so that the adjacent parts have sizes differing by at most 1 (so in particular \( a_i \geq a_{i+1} \geq a_i - 1 \)) and the smallest part has size 1.

**Question 3** (Compositions and Fibonacci Numbers, 30 points).

(a) Show that the number of compositions of \( n \) into odd parts is the same as the number of compositions of \( n - 1 \) into parts of size 1 and 2 for all \( n \geq 1 \). [15 points]

(b) Define the Fibonacci numbers by the recurrence relation \( F_0 = F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for all \( n \geq 2 \). Show that the number of compositions of \( n \) into odd parts is \( F_n \) for all \( n \geq 0 \). [15 points]

**Question 4** (Summation Polynomials, 40 points).

(a) Show that the number of compositions of \( n \) into \( k \) parts is the sum of \( m \) going from 0 to \( n - 1 \) of the number of compositions of \( m \) into \( k - 1 \) parts. [10 points]

(b) Show that for any \( n \) and \( k \) that

\[
\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.
\]

[10 points]

(c) Recall that

\[
x^m = \sum_{k=0}^{m} k! S(m, k) \binom{x}{k}.
\]

We would like to come up with a formula for

\[
\sum_{i=0}^{n} i^m = P_m(n).
\]

In particular, we claim that for each \( m \), we claim that \( P_m(n) \) is a polynomial in \( m \). For example,

\[
\sum_{i=0}^{n} i = \frac{n(n + 1)}{2},
\]

so \( P_2(n) = n(n + 1)/2 \). Using the above formula and the result in part (b), give a formula for \( P_m(n) \) in terms of Stirling numbers, and binomial coefficients. [20 points]
Question 5 (Extra credit, 1 point). Approximately how much time did you spend on this homework?