Math 184 Homework 2 Solutions

Spring 2021

Question 1 (Compositions with Large Parts, 10 points). How many compositions of \( n \) are there into \( k \) parts of size at least 3?

Solution 1. We want to count the number of ways to put \( n \) indistinguishable balls into \( k \) bins, such that each bin gets at least 3 balls. So first give each bin 3 balls, and then we just need to count the number of weak compositions of \( (n - 3k) \) into \( k \) parts: \( \binom{n-2k-1}{k-1} \)

Question 2 (Set Partitions and Integer Partitions, 25 points). Note that given any set partition of \([n]\), taking the sizes of the sets involved gives an integer partitions of \( n \). For example, the partition \([1, 5], [2, 3], [4]\) of \([5]\) corresponds to the integer partitions \(2 + 2 + 1\). Given an integer partition \(a_1 + a_2 + \ldots + a_k = n\), give a formula for the number of set partitions of \([n]\) that correspond to that integer partition. Conclude that for positive integers \(n\) that \(\binom{n^2}{2}!\) is an integer multiple of \((n!)^{n+1}\).

Solution 2. Let \(a_1 + a_2 + \cdots + a_k = n\) be an integer partition, and for \(i = 1, \ldots, n\) define \(b_i\) to be the number of times \(i\) appears in our integer partition. The number of set partitions corresponding to this integer partition is the number of ways to place \(n\) labeled balls in \(n\) unlabeled bins, such that the number of bins containing \(i\) balls is \(b_i\). To count such placements, we can think of first distributing unlabeled balls among the bins according to the integer partition, then labeling the balls. There are \(n!\) ways to label the balls, but distinct labelings may result in the same set partition, so we have to divide by the number of distinct labels that can be given to a single set partition. To count these, we note that the labels of balls within each part of size \(i\) can be permuted in \(i!\) ways. Additionally, we can swap all of the labels of balls in a bin of size \(i\) with all of the labels of balls in a different bin of size \(i\), with a total of \(b_i!\) ways to perform swaps of that type. Therefore the total number of distinct labels for a single set partition is

\[a_1!a_2!\cdots a_k!b_1!b_2!\cdots b_n!\].

So the number of set partitions corresponding to the integer partition \(a_1 + a_2 + \cdots + a_k = n\) is

\[\frac{n!}{a_1!a_2!\cdots a_k!b_1!b_2!\cdots b_n!}\].

If we consider the integer partition \(n + n + \cdots + n = n^2\) (so \(b_n = n\) and all other \(b_i = 0\)), the above formula says that there are

\[\frac{(n^2)!}{(n!)^{n+1}}\]

set partitions corresponding to this integer partition. Since we are counting the size of a set, this must be an integer and therefore \((n!)^2\) is an integer multiple of \((n!)^{n+1}\).

Question 3 (Even Bell Numbers, 30 points). Show that for the Bell numbers \(B(n)\) that \(B(n) - B(n-1) - B(n-2)\) is always an even number. Conclude that \(B(n)\) is even exactly when \(n\) is one less than a multiple of 3.

Hint: For a set partition \(\lambda\) of \([n]\) consider interchanging which sets \(n\) and \(n-1\) are in. This gives you a way of pairing up most of the set partitions of \([n]\).
Solution 3. We claim that $B(n) - B(n-1) - B(n-2)$ counts the number of set partitions of $[n]$ such that both

(a) $n$ and $n-1$ are not in the same set, and

(b) $n$ and $n-1$ are not both by themselves (i.e. we do not have $\{n\}, \{n-1\}$, . . . .

Indeed, $B(n-1)$ counts the number of partitions where (a) is violated: we can form all set partitions of $[n]$ such that $n$ and $n-1$ are in the same set by first forming a set partition of $[n-1]$ (there are $B(n-1)$ ways to do this) and then putting $n$ in the same set as $n-1$.

Also, $B(n-2)$ counts the number of partitions where (b) is violated: set partitions of $[n]$ where both $n$ and $n-1$ are by themselves arise from set partitions of $[n-2]$ and then adjoining $\{n\}, \{n-1\}$.

Now, any consider the set $S$ of set partitions of $[n]$ such that both (a) and (b) hold. We have shown that $|S| = B(n) - B(n-1) - B(n-2)$. On the other hand, partitions in $S$ arise in pairs - given a partition in $S$, we can switch $n$ and $n-1$ to get another, different partition in $S$. To see this, note that the only way to get the same partition back after switching $n$ and $n-1$ is if either (a) or (b) is violated. Thus, $|S|$ must be even, as needed.

Finally, if $B(n) - B(n-1) - B(n-2)$ is always even in other words, always $\equiv 0$ mod $2$, then

$$B(n) = B(n-1) + B(n-2) \mod 2$$

for all $n \geq 2$.

Since $B(0) = B(1) = 1$, the sequence $B(n)$ in mod $2$ goes $1, 1, 0, 1, 1, 0, 1, 1, 0, . . . .

Question 4 (Bounded Partitions, 35 points). (a) Show that the number of paths from $(0, 0)$ to $(n, m)$ taking steps of size $(0, 1)$ or $(1, 0)$ is \(^{n+m}_n\). [5 points]

(b) Show that the number of integer partitions (of any size) with at most $n$ parts and with largest part of size at most $m$ equals \(^{n+m}_n\). [30 points]

Solution 4. (a) To travel from $(0, 0)$ to $(m, n)$ using only steps of size $(1, 0)$ and $(0, 1)$, we must use exactly $m$ steps of size $(1, 0)$ and exactly $n$ steps of size $(0, 1)$. These steps can take place in any order. Therefore the number of paths is equal to the number of ways to select which $n$ of our $m+n$ total steps will be $(0, 1)$, so there are \(^{m+n}_n\) such paths.

(b) If an integer partition has at most $n$ parts and largest part of size at most $m$, then its Ferrers diagram will fit in a rectangle of width $m$ and height $n$, and vice versa. So the number of such integer partitions is equal to the number of Ferrers diagrams that fit within a rectangle of width $m$ and height $n$.

If a Ferrers diagram fits within a rectangle of width $m$ and height $n$, then we can associate a path of the sort described in (a) as follows: Place the diagram in the cartesian plane so that its upper left corner is at $(0, n)$. Take the path that travels from $(0, 0)$ upwards until it hits the Ferrers diagram, then follows the right boundary of the Ferrers diagram until it hits the top, then travels right to $(m, n)$. So, for example, if $m = n = 4$ and our Ferrers diagram corresponds to the partition $2 + 1 = 3$, the corresponding path would take the steps up, up, right, up, right, up, right, right. This association is a bijection, since it has an inverse function: to each path from $(0, 0)$ to $(m, n)$ we can associate the Ferrers diagram consisting of the unit squares of the cartesian plane that lie above the path, to the right of the line $x = 0$, and below the line $y = n$. Therefore the number of integer partitions with at most $n$ parts and largest part of size $m$ is equal to the number of paths from $(0, 0)$ to $(n, n)$, which we have found in part (a) to be \(^{m+n}_n\).