Announcements

• Homework 6 Due on Friday (Minor Correction)
• Homework 5 Solutions online
Last Time

- Pattern Avoidance in Permutations
- Left-to-Right Minima

**Definition:** The Left-to-Right minima of a permutation $\pi$ are all of the indices $i$ so that $\pi(i) < \pi(j)$ for all $j < i$. 
Patterns

Given two permutations $\pi$ of $[n]$ and $\rho$ of $[m]$, we say there is a copy of $\rho$ in $\pi$ if there are $1 \leq x_1 < x_2 < \ldots < x_m \leq n$ so that $\pi(x_1), \pi(x_2),\ldots, \pi(x_m)$ have the same relative orders as $\rho(1), \rho(2),\ldots, \rho(m)$.

For a permutation $\rho$, let $S_n(\rho)$ be the set of permutations of $[n]$ that do not have a copy of $\rho$. 
Today

Pattern Avoidance Asymptotics

• General Idea
• Bounds on $|S_n(123...k)|$
• Erdos-Szekerez
• Combining Patterns
Asymptotics

A lot of the research on pattern avoidance has to do with figuring out the approximate size of these pattern avoiding sets.

**Conjecture (Proved by Marcus & Tardos in 2003):**
For every permutation \( \rho \), there exists a constant \( C_\rho \) so that

\[
|S_n(\rho)| \leq (C_\rho)^n
\]

for all \( n \).
Asymptotic Result

What if we take $\rho = 123\ldots k$?

**Theorem (14.12):**

$$|S_n(123\ldots k)| \leq (k-1)^2n.$$  

**Proof:**  

Split points into levels:  

- 1\textsuperscript{st} level is the left-to-right minima.  
- $k\textsuperscript{th}$ level is the left-to-right minima after remove first $k-1$ levels.
Example

2\textsuperscript{nd} level

3\textsuperscript{rd} level

4\textsuperscript{th} level

1\textsuperscript{st} level
Claim 1

**Claim:** The points of each level are sorted in up-left to down-right order.

**Proof:**
The points of each level are a set of left-to-right minima.
Claim 2

Claim: If the permutation avoids 123...k, it has at most k-1 levels.

Proof:
• Each level t point has a level t-1 point down-left of it
  – Otherwise that point would have been in level t-1
• Given level k point have level k-1 point down-left, has level k-2 point down-left, has level k-3 point... this gives a copy of 123...k.
Example

2^{nd} level

3^{rd} level

4^{th} level

1^{st} level
Summary

• A permutation that avoids 123...k has at most k-1 levels.

• The points in each level are sorted.

Claim: If for each x-coordinate and each y-coordinate you specify which level the corresponding point is on, that uniquely determines the permutation.
Example

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</table>

| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 1 |   |
Putting it Together

To specify a 123...k avoiding permutation, you just need to specify a number in \{1,2,3,...,k-1\} for each row/column.

There are only \((k-1)^{2n}\) ways to do this.

**Note:** Not all such assignments work.
The only way to avoid $123\ldots k$ is to have few levels. These levels must be decreasing.

**Theorem:** Any permutation of $[(n-1)(m-1)+1]$ contains a copy of $123\ldots n$ or a copy of $m\ldots 321$.

**Proof:**

- If $n$ or more levels have a $123\ldots n$.
- Otherwise, pigeonhole implies a level with $m$.
- This gives a copy of $m\ldots 321$. 
**Definition:** Given two permutations, p and q, define their sum $p \oplus q$ to be the permutation whose graph is given as follows:
Theorem (14.15): Let $p$ and $q$ be permutations with

\[ |S_n(p \oplus 1)| \leq C_p^n, \text{ and} \]
\[ |S_n(1 \oplus q)| \leq C_q^n. \]

Then

\[ |S_n(p \oplus 1 \oplus q)| \leq (\sqrt{C_p} + \sqrt{C_q})^{2n}. \]
Proof

• Take a permutation $\pi$ that avoids $p \oplus 1 \ominus q$.
• Color points blue if they are the 1 in a copy of $p \ominus 1$, and red otherwise.
• Red points are a permutation that avoids $p \ominus 1$ – otherwise the “1” should be red.
• Blue points are a permutation avoiding $1 \ominus q$ – otherwise the “1” is the end of $p \ominus 1$, giving a copy of $p \ominus 1 \ominus q$. 
Proof (Continued)

So you can specify a $p \ominus 1 \ominus q$-avoiding permutation by:

1. Specifying which x- and y- coordinates are blue and which are red
2. Specifying a $p \ominus 1$-avoiding permutation on the red points
3. Specifying a $1 \ominus q$-avoiding permutation on the blue points
Proof (continued)

How many ways can we do this?
If we want a permutation of \([n]\) with \(k\) red points:

• \((n\text{C}_k)\) ways to choose red \(x\)-coordinates
• \((n\text{C}_k)\) ways to choose red \(y\)-coordinates
• \(C_p^k\) ways to pick a \(p\oplus1\)-avoiding permutation
• \(C_q^{n-k}\) ways to pick a \(1\oplus q\)-avoiding permutation
Proof (Continued)

Therefore, we have that:

\[ |S_n(p \oplus 1 \oplus q)| \leq \sum_{k=0}^{n} \binom{n}{k}^2 C_p^k C_q^{n-k} \]

\[ \leq \sum_{k=0}^{n} \binom{2n}{2k} (\sqrt{C_p})^{2k} (\sqrt{C_q})^{2n-2k} \]

\[ \leq \sum_{m=0}^{2n} \binom{2n}{m} (\sqrt{C_p})^m (\sqrt{C_q})^{2n-m} \]

\[ = (\sqrt{C_p} + \sqrt{C_q})^{2n}. \]