Practice Quiz

What is your favorite number?

(A) 0

(B) $e$

(C) $\pi$

(D) 17

(E) 184
Last Time

Proof by Induction

• Need to prove
  – Base Case (statement for n=1)
  – Inductive Step (statement for n implies statement for n+1)

• Implies statement for all n
Format for Inductive Proof

• Give statement you want to prove by induction.
• State that you will prove it by induction on <inductive variable>.
• State and prove the base case.
• Inductive Step
  – State you are starting inductive step
  – State the inductive hypothesis
  – Use to prove next step
• Conclude your original claim
Today

- Strong Induction
- Pigeonhole Principle
Define a sequence by:

\[ a_0 = 1 \]

\[ a_{n+1} = a_n + a_{n-1} + a_{n-2} + \ldots + a_0 \]

So,

\[ a_1 = a_0 = 1 \]

\[ a_2 = a_1 + a_0 = 1 + 1 = 2 \]

\[ a_3 = a_2 + a_1 + a_0 = 2 + 1 + 1 = 4 \]

\[ a_4 = a_3 + a_2 + a_1 + a_0 = 4 + 2 + 1 + 1 = 8 \]

Guess:

\[ a_n = 2^{n-1} \text{ for } n \geq 1 \]
Proof

Proof that $a_n = 2^{n-1}$ by induction on $n \geq 1$.

**Base Case:** $n = 1$

$a_1 = 1 = 2^{1-1}$. ✓

**Inductive Step:** Assume $a_n = 2^{n-1}$.

$a_{n+1} = a_n + a_{n-1} + a_{n-2} + ... + a_0 = ???$

Want to know not just $a_n$, but also all previous $a_i$. 
Strong Induction

For this we need strong induction. In order to prove $S_n$ for all $n$ we need:

1) $S_1$ is true.

2) If $S_m$ is true for all $m < n$, then $S_n$ is true.

This gives us $S_1$, which implies $S_2$. Together they imply $S_3$, and then $S_4$ and so on.
Format for **Strong** Inductive Proof

- Give statement you want to prove by strong induction.
- State that you will prove it by strong induction on <inductive variable>.
- State and prove the base case. (Or skip)
- Inductive Step
  - State you are starting inductive step
  - State the inductive hypothesis (for all smaller m)
  - Use to prove next step
- Conclude your original claim
Proof

Proof that $a_n = 2^{n-1}$ by strong induction on $n \geq 1$.

**Base Case:** $n = 1$

$a_1 = 1 = 2^{1-1}$. ✓

**Inductive Step:** Assume $a_m = 2^{m-1}$ for all $1 \leq m < n$.

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + ... + a_2 + a_1 + a_0$$

$$= 2^{n-2} + 2^{n-3} + 2^{n-4} + ... + 2 + 1 + 1$$

$$= (2^{n-1} - 1) + 1 = 2^{n-1}.$$

This completes our proof.
Bad Inductive Proof I

Proof by induction on n that \((2n+1)\) is even.

**Inductive Step:** Assume that \((2n+1)\) is even. Then \((2n+1) = 2m\) for some \(m\).

Therefore, \((2(n+1)+1) = 2m+2 = 2(m+1)\) is also even.

This completes our inductive step.

What went wrong here?
Bad Inductive Proof II

Proof that all horses are the same color.
We prove by induction on $n$ that any set of $n$ horses are all the same color.

**Base Case:** $n=1$
Given any one horse, it is the same color as itself.

**Inductive Step:** Assume any set of $n$ horses are the same color.
Take any $n+1$ horses. First $n$ are the same color. Last $n$ are same color. Therefore, all are the same color.
This completes our proof.

What went wrong?
Picking Correct Inductive Hypothesis

Usually proving a stronger statement is harder. When proving by induction this isn’t always the case. A stronger inductive hypothesis gives you more to work with when proving the inductive step. Often setting up your induction correctly makes proof much easier.
Example

Suppose we have a sequence
\[ a_1 = 1, \quad a_{n+1} \leq 2a_n + 1 \]
Prove that \( a_n \leq 2^n \) for all \( n \).
Use induction on \( n \).

**Base Case:** \( a_1 = 1 \leq 2^1 \).

**Inductive Step:** Assume \( a_n \leq 2^n \).
\[
a_{n+1} \leq 2a_n + 1 \leq 2 \cdot 2^n + 1 = 2^{n+1} + 1
\]
Not good enough!
Example

Suppose we have a sequence
\[ a_1 = 1, \ a_{n+1} \leq 2a_n + 1 \]
Prove that \( a_n \leq 2^{n-1} \) for all \( n \).
Use induction on \( n \).

**Base Case:** \( a_1 = 1 \leq 2^{1-1} \).

**Inductive Step:** Assume \( a_n \leq 2^{n-1} \).
\[
a_{n+1} \leq 2a_n + 1 \leq 2 \cdot (2^{n-1}) + 1 = 2^{n+1} - 1
\]
This completes the proof.
Pigeonhole Principle (Ch 1)

- Pigeonhole Principle
- Generalized Pigeonhole Principle
- Applications
**Pigeonhole Principle**

**Theorem:** Given $n$ pigeons each assigned to one of $m$ holes for some $m < n$, there must be some hole with at least two pigeons.

**Proof:** Assume for sake of contradiction that each hole has at most 1 pigeon.

Number of pigeons $= \sum_{\text{holes } h} \text{[Number of pigeons in hole } h\text{]}$

$n = \text{LHS} = \text{RHS} \leq m$

Contradiction!
Discussion

• Basic application of counting
• Seems obvious, but applications not always

**Corollary:** There is some pair of people in San Diego with exactly the same number of hairs on their heads.

**Proof:** There are about 1M people. Each person has ~100,000 hairs.

Use Pigeonhole Principle:

Pigeons = people
Holes = # of hairs ∈ \{0,1,2,...,100000\}
Example: Blue Moons

A blue moon is when you have two full moons in a month.

- A lunar cycle lasts 29.53 days.
- Every 3 years have (a bit more than) 37 full lunar cycles.
- At least 37 full moons in 3 years.
- Only 36 months in 3 years.
- PHP $\Rightarrow$ at least one blue moon in every 3-year period.
Things to Keep in Mind When Setting up Pigeonhole

• What are your pigeons?
• What are your holes?
• How are pigeons assigned to holes?
• How do you show that there are more pigeons than holes?
Dirichlet’s Theorem

**Theorem:** Let $\alpha$ be any real number and $q$ a positive integer. There exist integers $n$ and $m$ with $0 < m \leq q$ so that

$$|\alpha - n/m| \leq 1/mq.$$  

This is an important result about approximability by rationals.
Idea 1

Fix m and find n.

• Want $\alpha \approx n/m$.
• Equivalently, $n \approx \alpha m$.
• Can pick n to be nearest integer
  – $|n - \alpha m| \leq \frac{1}{2}$
  – $|\alpha - n/m| \leq 1/(2m)$
  – Not good enough
• Need an m so that $\alpha m$ is close to an integer.
Idea 2

Consider the fractional parts 
\{\alpha\},\{2\alpha\},\{3\alpha\},...,\{q\alpha\}.

Want one that’s within 1/q of either 0 or 1.

Actually, enough to find two close to each other. 
If |\{s\alpha\} – \{t\alpha\}| \leq 1/q, then (s-t)\alpha is within 1/q of an integer.
Packing

Take \( q+1 \) numbers: 0,\( \{\alpha\} \),\( \{2\alpha\} \),\( \{3\alpha\} \),...,\( \{q\alpha\} \)

Distribute them among intervals:
\([0,1/q],(1/q,2/q],...,((q-1)/q,1]\)

Pigeonhole Principle \( \Rightarrow \) two in same interval.

These differ by at most 1/q.
Finishing Up

Letting $m = |s-t|$, we have that $m\alpha$ is within $1/q$ of some integer $n$.

This means that

$|\alpha - n/m| \leq 1/mq$

as desired.
Packing Problems

The pigeonhole principle is actually useful for proving lower bounds for a bunch of packing problems.

**Theorem:** Given 1001 points in the unit cube, some pair have distance at most $\sqrt{3}/10$ from each other.

**Proof:** Split the cube into 1000, $0.1 \times 0.1 \times 0.1$ subcubes. Some two points from same subcube.