Announcements

• Homework 4 Due on Friday
• Exam 2 Instructions Out
  – Same format as Exam 1
  – Let me know by next Wednesday if you need an alternative time and didn’t for exam 1
  – Topics: Chapters 4, 6, 7, 8 (ordinary generating functions only)
Last Time

- Generating Function for Fibonacci numbers
  \[
  \frac{1}{1-x-x^2}
  \]
- \[(1-4x)^{1/2} = 1 - 2x - 2(2C1)/2x^2 - 2(4C2)/3x^3 - \ldots\]
- Products of Generating Functions
  \[
  A(x) = \sum_n a_n x^n,
  B(x) = \sum_n b_n x^n.
  \]
  \[
  A(x) \cdot B(x) = C(x) = \sum_n c_n x^n.
  \]
  \[
  c_k = (\sum_{n+m=k} a_n b_m).
  \]
Combinatorial Interpretation

Suppose that you have objects of type-A and objects of type-B. Each has a size which is a non-negative integer, and there are \( a_n \) objects of type-A of size \( n \), and \( b_m \) objects of type-B of size \( m \).

Then \( c_k \) is the number of ways to find a pair of an object of type-A and an object of type-B where the sum of the sizes is \( k \).
Today

• Applications
• Functions vs. Power Series
• Catalan Numbers
Playing Around

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\]

Summing over \(n\), we find:

\[\frac{1}{1 - x - y} = \sum_{n,k} \binom{n}{k} x^k y^{n-k} = \sum_{n,m} \binom{n + m}{m} x^n y^m.\]

Find the \(y^m\)-coefficient:

\[\sum_{n} \binom{n + m}{m} x^n.\]
Coefficient

\[
\frac{1}{1 - x - y} = \frac{1/(1 - x)}{1 - \frac{y}{(1 - x)}} = \sum_{m} \left( \frac{1}{1 - x} \right)^{m+1} y^m.
\]

Comparing \( y^m \) coefficients we find:

\[
(1 - x)^{-m-1} = \sum_{n} \binom{n + m}{m} x^n.
\]
More Playing Around.

\[
\frac{1}{1 - x - y} = \sum_{n,m} \binom{n + m}{m} x^n y^m.
\]

Substitute \( x = z \) and \( y = z^2 \).

\[
\frac{1}{1 - z - z^2} = \sum_{n,m} \binom{n + m}{m} z^{n+2m}
\]

\[
= \sum_k \left( \sum_{m=0}^{k/2} \binom{k}{m} \right) z^k.
\]

\[
f_n = \sum_m \binom{n - m}{m}.
\]

\( 1/(1-z-z^2) \) was the g.f. for the Fibonacci numbers!
What Actually is a Generating Function

When we write $F(x) = \sum_n a_n x^n$, what do we mean by this? How do we interpret $F(x)$?

- As an **actual function** of a (real or complex) variable $x$, that should converge for $x$ in some range.

- A **formal power series**- namely a set of symbols that can be manipulated in the same way a real function could, but that can’t necessarily be evaluated anywhere.
Tradeoffs

If we treat $F(x)$ as a function, we need to worry about issues of convergence. For which (if any) values of $x$ does $F(x)$ converge? When we do manipulations on infinite sums, are we allowed to?

Because of this, for most combinatorial applications, it is better to treat $F(x)$ as a formal power series.

But sometimes actually having a function is useful.
Partition Sizes

Recall:
\[ F(x) = \sum_n p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)} \ldots \]
It is not hard to show that \( F(x) \leq e^{\frac{\pi^2}{6(1-x)}} \) for \( 0 < x < 1 \).

On the other hand, \( F(x) > p(n)x^n \).

Setting \( x = 1-\pi/\sqrt{6n} \), we find that

Which is quite accurate. \( p(n) \leq e^{\pi \sqrt{2n/3}} \).
Catalan Numbers

**Question:** How many ways can you produce $n$ matching pairs of parentheses?

**Example:** For $n = 2$ you can have:

( ) ( )
( ( ) )
Matching Parentheses

• You must have a total of $n \times (\times$ and $n \times$).
• There are $(2n)C(n)$ ways to arrange these. Do all of them work?
  – NO!
  – ) ) ( ( is illegal.
  – Cannot at any point have more closing parentheses than opening parentheses.
Lattice Paths

For a parenthesis pattern create a path where you keep track of
(# of left parens so far, # right parens so far)

**Example:** ( ( ) ( ) )

**Notes:**
- Steps (0,1) or (1,0)
- From (0,0) to (n,n)
- Above line x = y.
Catalan Numbers

How many such paths are there?

**Definition:** The *nth* Catalan Number $C_n$ is the number of up-left lattice paths from $(0,0)$ to $(n,n)$ that stay on or above the line $x = y$.

**Note:** Catalan numbers count many other things including matching parentheses sequences.
Recursion

To get started on counting, which parentheses does the opening ‘(‘ match with?
Matches with the $k^{\text{th}}$ ‘)’ for some $k$.

\[
\begin{align*}
\left( \ldots \right) \ldots \\
\text{k-1 pairs} & \quad \text{n-k pairs}
\end{align*}
\]

- $C_{k-1}$ ways to match first group.
- $C_{n-k}$ ways to match second group.

\[
C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}.
\]
Generating Function

Define the generating function:
\[ H(x) = \sum_n C_n x^n. \]

\[ H^2(x) = \sum_n (\sum_k C_k C_{n-k}) x^n \]
\[ = \sum_n C_{n+1} x^n. \]

\[ xH^2(x) = \sum_n C_{n+1} x^{n+1} = H(x) - 1. \]
Quadratic Formula

We have
\[ xH^2(x) - H(x) + 1 = 0. \]

What is \( H(x) \)?

\[
H(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

**Note:** Only the − term makes sense, since \( H(0) \) needs to be finite.
Coefficients

Recall:

$$\sqrt{1 - 4x} = 1 - 2x - 2x^2 \binom{2}{1}/2 - 2x^3 \binom{4}{2}/3 - 2x^4 \binom{6}{3}/4 - \ldots$$

So,

$$\frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x \binom{2}{1}/2 + x^2 \binom{4}{2}/3 + x^3 \binom{6}{3}/4 + \ldots$$

Therefore,

$$C_n = \frac{1}{n + 1} \binom{2n}{n}.$$
Compositions Into Odd Parts

**Question:** How many compositions of \( n \) into (any number of) odd parts are there?

**Example:** If \( n = 5 \) we have:
5, 3+1+1, 1+3+1, 1+1+3, 1+1+1+1+1
So 5 in total.
What if we want compositions into exactly \( k \) odd parts?

This is the number of solutions to \( a_1 + a_2 + ... + a_k = n \) for odd integers \( a_i \).

**Generating Function:**

\[
(x + x^3 + x^5 + ...) \cdot (x + x^3 + x^5 + ...) \cdot ... \cdot (x + x^3 + x^5 + ...)
= \left[ \frac{x}{1-x^2} \right]^k.
\]