Math 184 Homework 2

Spring 2021

This homework is due on gradescope Friday April 23rd at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \texttt{\LaTeX} is recommended though not required.

**Question 1** (Compositions with Large Parts, 10 points). How many compositions of \( n \) are there into \( k \) parts of size at least 3?

**Question 2** (Set Partitions and Integer Partitions, 25 points). Note that given any set partition of \([n]\), taking the sizes of the sets involved gives and integer partitions of \( n \). For example, the partition \( \{1, 5\}, \{2, 3\}, \{4\} \) of \([5]\) corresponds to the integer partitions \( 2 + 2 + 1 \). Given an integer partition \( a_1 + a_2 + \ldots + a_k = n \), give a formula for the number of set partitions of \([n]\) that correspond to that integer partition. Conclude that for positive integers \( n \) that \((n^2)!\) is an integer multiple of \((n!)^{n+1}\).

**Question 3** (Even Bell Numbers, 30 points). Show that for the Bell numbers \( B(n) \) that \( B(n) - B(n-1) - B(n-2) \) is always an even number. Conclude that \( B(n) \) is even exactly when \( n \) is one less than a multiple of 3.

Hint: For a set partition \( \lambda \) of \([n]\) consider interchanging which sets \( n \) and \( n-1 \) are in. This gives you a way of pairing up most of the set partitions of \([n]\).

**Question 4** (Bounded Partitions, 35 points). (a) Show that the number of paths from \((0, 0)\) to \((n, m)\) taking steps of size \((0, 1)\) or \((1, 0)\) is \( \binom{n+m}{n} \). [5 points]

(b) Show that the number of integer partitions (of any size) with at most \( n \) parts and with largest part of size at most \( m \) equals \( \binom{n+m}{n} \). [30 points]

**Question 5** (Extra credit, 1 point). Approximately how much time did you spend working on this homework?