Math 184 Homework 1

Spring 2021

This homework is due on gradescope Friday April 9th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommend though not required.

**Question 1** (Differential Equation, 35 points). Show that for every positive integer $m$ that there is a polynomial $P_m(x)$ satisfying the differential equation:

$$mP_m(x) = xP'_m(x) + P'_m(x) + P''_m(x).$$

*Hint:* Note that if $P(x) = x^m$ that $mP(x) - xP'(x) - P'(x) - P''(x) = -mx^{m-1} - m(m-1)x^{m-2}$. By adding an appropriate multiple of $x^{m-1}$ to $P$, we can remove the $x^{m-1}$ term, and then by adding lower degree terms, we can continue cleaning up the errors until they disappear entirely. You will likely need some kind of induction to formalize this argument.

**Question 2** (King Packing, 30 points). The game of chess is played on an 8 by 8 square board. A king can move from a given square to any adjacent square vertically, horizontally or diagonally. What is the maximum number of kings that can be placed on a chess board without any two of them attacking each other (i.e. being able to move to the others’ square)?

**Question 3** (Counting Permutations, 35 points). A permutation of the set $[10] = \{1, 2, 3, \ldots, 10\}$ is a way of listing the elements of $[10]$ in order so that each element appears exactly once. The following questions ask about counting the number of permutations of $[10]$ with certain properties. Remember to justify your answers.

(a) How many permutations of $[10]$ are there? [5 points]

(b) How many permutations of $[10]$ are there that put $1, 2, 3, 4, 5$ in the first five positions in some order? [5 points]

(c) We call a sequence unimodal if the elements of the sequence increase to some point, and then decrease from there on. So, for example $1, 2, 3, 4, 5, 10, 9, 8, 7, 6$ is unimodal. How many unimodal permutations of $[10]$ are there? [5 points]

(d) How many permutations of $[10]$ start with $k, k-1, k-2, \ldots, 1$ for some integer $k$? [5 points]

(e) How many permutations of $[10]$ have $1$ appear earlier in the sequence than $10$? [5 points]

(f) How many permutations of $[10]$ have all of the even numbers appearing in the first seven positions? [5 points]

(g) How many permutations of $[10]$ alternate even and odd numbers? [5 points]

**Question 4** (Extra credit, 1 point). Approximately how much time did you spend working on this homework?