Math 184

Final Exam Review
Office Hours

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https://ucsd.zoom.us/j/97552056959
Thursday 4:00-6:00
https://ucsd.zoom.us/j/97819295957

Bryan Hu: Friday 5:00-7:00
https://ucsd.zoom.us/j/97567361616

Or by appointment
Note

This review will cover course material that was not covered in Exams 1 and 2. However, the final exam will be comprehensive. If you want to review the earlier course material, please look up the review videos for the appropriate exams.
Exponential Generating Functions

Exponential generating functions
\[ \{a_n\} \leftrightarrow \sum_n a_n \frac{x^n}{n!} \]
Comparison

**Ordinary:**
- $F(x) = \sum_n a_n x^n$.
- $a_n = 1$, $F(x) = 1/(1-x)$.
- $a_n = n$, $F(x) = x/(1-x)^2$.
- $\sum_n a_{n-1} x^n = x F(x)$.
- Converges only if $a_n$ grows at most exponentially.

**Exponential:**
- $F(x) = \sum_n a_n x^n/n!$
- $a_n = 1$, $F(x) = e^x$.
- $a_n = n$, $F(x) = xe^x$.
- $\sum_n a_{n+1} x^n/n! = F'(x)$.
- Converges more generally.
Multiplication of Exponential Generating Functions

• Multiplication of ordinary generating functions important.
• Multiplication of exponential generating functions a bit different.
• This difference is one of the most important distinguishing features.
Multiplication of Exponential Generating Functions

\[ A(x) = \sum_n a_n \frac{x^n}{n!} \]
\[ B(x) = \sum_n b_n \frac{x^n}{n!} \]
\[ C(x) = A(x)B(x) = \sum_n c_n \frac{x^n}{n!} \]

What is \( c_n \)?
Multiplication Formula

\[ C(x) = A(x)B(x) = \left( \sum_{m=0}^{\infty} \frac{a_m x^m}{m!} \right) \left( \sum_{k=0}^{\infty} \frac{b_k x^k}{k!} \right) \]

\[ = \sum_{m,k=0}^{\infty} a_m b_k x^{m+k} / (m!k!) \]

\[ = \sum_{n=0}^{\infty} x^n \left( \sum_{m+k=n} a_m b_k / (m!k!) \right) \]

\[ = \sum_{n=0}^{\infty} x^n / n! \left( \sum_{m+k=n} \binom{n}{k} a_m b_k \right) . \]

\[ c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} . \]

Difference from ordinary formula
Define $A$-structure a thing so that there are $a_n$ $A$-structures on a set of size $n$.

Define $B$-structure a thing so that there are $b_n$ $B$-structures on a set of size $n$.

Ordinary generating function multiplication talks about the number of ways to find an $A$-structure and a $B$-structure of total size $n$.

Exponential generating function multiplication has $c_n = \text{number of ways to partition } [n] \text{ into two sets and put an } A\text{-structure on one and a } B\text{-structure on the other.}$

If $A$-structure of size $k$, $n\text{C}k$ ways to partition $[n]$, $a_k$ $A$-structures and $b_{n-k}$ $B$-structures.
Composition of Generating Functions

So if \( A(x) = a_1 x + a_2 x^2/2! + a_3 x^3/3! + \ldots \) and \( B(x) = b_0 + b_1 x + b_2 x^2/2! + b_3 x^3/3! + \ldots \)

What is \( B(A(x)) \)?

It equals \( b_0 + b_1 A(x) + b_2 A(x)^2/2! + b_3 A(x)^3/3! + \ldots \)

The \( x^n/n! \) coefficient is:

- \( b_1 \) times the number of partitions of \([n]\) into one part with an \( A \)-structure plus
- \( b_2 \) times the number of partitions of \([n]\) into two parts with an \( A \)-structure on each plus
- \( b_3 \) times the number of partitions of \([n]\) into three parts with an \( A \)-structure on each plus \( \ldots \)
Composition of Generating Functions

So the $x^n/n!$ coefficient of $B(A(x))$ counts the number of ways to partition $[n]$ into subsets, put an $A$-structure on each subset, and put a $B$-structure on the collection of subsets.
Permutations

A permutation of $[n]$ is equivalent to partitioning $[n]$ into subsets and then arranging each subset into a cycle.

Here an A-structure is a cycle. There are $(k-1)!$ cycles on a $k$-element subset so

$$a_k = (k-1)!$$

$$A(x) = \sum_{k \geq 1} (k-1)!\frac{x^k}{k!} = \sum_{k \geq 1} \frac{x^k}{k} = \log\left(\frac{1}{1-x}\right).$$
Sterling Numbers

What if we count permutations of \([n]\) weighted by \(y^{\text{# of cycles}}\)?

- A-structure is a cycle
- B-structure has \(b_n = y^n\).
  - \(B(x) = \sum_n x^n y^n / n! = e^{xy}\).

Generating function is

\[
B(A(x)) = e^{y \log(1/(1-x))} = (1/(1-x))^y.
\]

\(y^k x^n / n!\)-coefficient is \#perms of \([n]\) with \(k\) cycles.
Sterling Number Generating Function

**Theorem:**

\[
\sum_{n,k=0}^{\infty} c(n, k) x^n / n! y^k = \left( \frac{1}{1-x} \right)^y.
\]
Other Generating Functions

\[ \sum_{n,k=0}^{\infty} S(n, k) x^n / n! y^k = e^{y(e^x-1)}. \]

\[ \sum_{n=0}^{\infty} B(n) x^n / n! = e^{e^x-1}. \]

\[ \sum_{n=0}^{\infty} D_n x^n / n! = \frac{e^{-x}}{1-x}. \]
Pattern Avoidance in Permutations (Ch 14)

- Patterns in Permutation
- Some basic counts
- Asymptotic results
Stacks

A stack is an object that stores numbers. You can push more numbers onto it, or you can pop the most recently pushed number off.
Stack-Sortable Permutations

**Definition:** A permutation $\pi$ of $[n]$ is stack-sortable if there is a series of operations that involve $1, 2, ..., n$ being pushed into a stack in $\pi$ order, and popped off of the stack in sorted order.
Classification

**Theorem:** A permutation is stack sortable unless there are entries $k < i < j$ so that $i$ occurs before $j$ which occurs before $k$. 
Patterns

**Definition:** Given two permutations $\pi$ of $[n]$ and $\rho$ of $[m]$, we say there is a copy of $\rho$ in $\pi$ if there are $1 \leq x_1 < x_2 < \ldots < x_m \leq n$ so that $\pi(x_1)$, $\pi(x_2)$, $\ldots$, $\pi(x_m)$ have the same relative orders as $\rho(1)$, $\rho(2)$, $\ldots$, $\rho(m)$.

For example, a permutation $\pi$ is stack sortable if and only if it doesn’t have a copy of 231.
It is often useful to consider graphs of the permutations involved.

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<th>( \pi(n) )</th>
<th>( \rho(n) )</th>
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Pattern Avoidance

**Definition:** For a permutation $\rho$, let $S_n(\rho)$ be the set of permutations of $[n]$ that do not have a copy of $\rho$. 
Theorem: \( |S_n(231)| = C_n. \)

Proof:
Let \( A_n = |S_n(231)| \). We will show that \( A_n \) satisfies the same recurrence as \( C_n \).

Namely:

• \( A_0 = 1 \). [Counting the empty permutation]
• \( A_n = \sum_k A_{k-1} A_{n-k} \).
Proof

• Consider $n$ in $k^{th}$ location:
  • Entries on left must be smaller than entries on right.
    - Entries on left are $1, 2, \ldots, k-1$
    - Entries on right are $k+1, \ldots, n$
  • Entries on left/right must be $231$-avoiding.

Number of possibilities is $A_{k-1}A_{n-k}$. 
Rotations

**Lemma:**

\[ |S_n(231)| = |S_n(132)| = |S_n(312)| = |S_n(213)|. \]

**Proof:**

Note that the diagrams for these permutations are rotations of each other.

\[
\begin{array}{ccc}
3 & X & \\
2 & X & \\
1 & X & \\
1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
3 & X & \\
2 & X & \\
1 & X & \\
1 & 2 & 3 \\
\end{array}
\]

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\]

\[
\begin{array}{ccc}
3 & X & \\
2 & X & \\
1 & X & \\
1 & 2 & 3 \\
\end{array}
\]
Proof

For a permutation $\pi$, let $R\pi$ be its 90 degree rotation.

Note: $\pi$ contains a copy of $\rho$ iff $R\pi$ contains $R\rho$. 
Final Pattern

What about $|S_n(123)| = |S_n(321)|$?

**Theorem**: For all $n$, $|S_n(123)| = |S_n(132)| = C_n$. 
Left to Right Minima

We will use the following definition:

**Definition:** The *Left-to-Right minima* of a permutation $\pi$ are all of the indices $i$ so that $\pi(i) < \pi(j)$ for all $j < i$. 
Example

```
6 X
5 X
4 X X
3 X
2 X
1 X
```

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```
Left-to-Right Minima

We will classify permutations based on the locations (in the permutation graph) of their left-to-right minima.

One question is which sets of locations are possible.
Validity

**Definition:** A set of pairs \((i,j)\) with \(1 \leq i,j \leq n\) is valid if:

1) They can be ordered \((i_1,j_1),(i_2,j_2),..., (i_k,j_k)\) so that \(i_1 < i_2 < ... < i_k\) and \(j_1 > j_2 > ... > j_k\).

2) For each \(1 \leq a \leq n\), if \(i_{\text{last}}\) is the largest of the \(i\)’s less than or equal to \(a\), \(j_{\text{last}} \leq n+1-a\).

**Lemma:** A set of pairs can be the set of left-to-right minima of a permutation of \([n]\) only if it is valid.
Minima to Permutations

**Proposition:** For every valid set $S$ of pairs, there is exactly one 123-avoiding permutation and exactly one 132-avoiding permutation with $S$ as its set of left-to-right minima.
123-Avoiding

**Lemma:** The non-left-to-right minima in a 123-avoiding permutation are in decreasing order.

**Proof:**

Suppose this isn’t the case.

- You have \((i_1,j_1)\) and \((i_2,j_2)\) with \(i_1 < i_2\) and \(j_1 < j_2\).
- Since \((i_1,j_1)\) not a minimum, there is an \((i_0,j_0)\) with \(i_0 < i_1\) and \(j_0 < j_1\).
- Then \((i_0,j_0), (i_1,j_1), (i_2,j_2)\) is a copy of 123.
123-Avoiding

Therefore, for every valid set, there is exactly one 123-avoiding permutation with those left-to-right minimums.

• Only one since the other terms must come in decreasing order.
• Condition (2) implies none of these will be a new minimum.
• LtR-mins decreasing, others decreasing, therefore cannot have three term increase.
132-Avoiding

We will construct our 132-avoiding permutation from the left-to-right minima set one step at a time.
Lemma: When producing a 132-avoiding permutation each new value must be the smallest not-yet-used value that is larger than the previous left-to-right minimum.
Non-Minimum Value

Filled in values

Current Value

New Value

Previous LtR min

Minimum Unused

1

2

3
Theorem (14.8): \( |S_n(1234)| \leq |S_n(1324)| \).

Proof: We are going to count the number of permutations with a fixed set of Left-to-Right minima and a fixed set of Right-to-Left maxima.
Lemma: For a 1234-avoiding permutation, the “everything else” must be sorted in decreasing order.

Proof: Suppose not.
• Unsorted pair
• Smaller LtR min
• Larger RtL max
• 1234 pattern
Lemma: For every set of left-to-right minima and right-to-left maxima that admits some permutation, there is at least one consistent 1324-avoiding permutation.
Idea

Start with any permutation. If not 1324-avoiding, make it closer.
If have 1324, make replacement.

Not LtR min nor RtL max.
Proof of Result

- Each sequence of left-to-right minima and right-to-left maxima that admits a permutation has:
  - At most 1 corresponding 1234-avoiding permutation
  - At least 1 corresponding 1324-avoiding permutation.

Therefore,

\[ |S_n(1234)| \leq \#\{\text{min/max patterns}\} \leq |S_n(1324)|. \]
Strictness

**Note:** For $n \geq 7$ inequality is strict.

Need to show two 1324-avoiding perms with same min/max sequences.
Asymptotics

**Conjecture (Proved by Marcus&Tardos in 2003):**

For every permutation $\rho$, there exists a constant $C_\rho$ so that

$$|S_n(\rho)| \leq (C_\rho)^n$$

for all $n$. 
Asymptotic Result

What if we take $\rho = 123...k$?

**Theorem (14.12):**

$$|S_n(123...k)| \leq (k-1)^{2n}.$$  

**Proof:**

Split points into levels:

- $1^{st}$ level is the left-to-right minima.
- $k^{th}$ level is the left-to-right minima after remove first $k-1$ levels.
Summary

• A permutation that avoids 123...k has at most k-1 levels.
• The points in each level are sorted.

**Claim:** If for each x-coordinate and each y-coordinate you specify which level the corresponding point is on, that uniquely determines the permutation.
Putting it Together

To specify a 123...k avoiding permutation, you just need to specify a number in \{1,2,3,...,k-1\} for each row/column.

There are only \((k-1)^{2n}\) ways to do this.

**Note:** Not all such assignments work.
The only way to avoid 123...k is to have few levels. These levels must be decreasing.

**Theorem:** Any permutation of \([(n-1)(m-1)+1]\] contains a copy of 123...n or a copy of m...321.

**Proof:**
- If n or more levels have a 123...n.
- Otherwise, pigeonhole implies a level with m.
- This gives a copy of m...321.
**Definition:** Given two permutations, \( p \) and \( q \), define their sum \( p \oplus q \) to be the permutation whose graph is given as follows:
Sum Exponential Growth

**Theorem (14.15):** Let \( p \) and \( q \) be permutations with
\[
|S_n(p \oplus 1)| \leq C_p^n, \quad \text{and}
|S_n(1 \oplus q)| \leq C_q^n.
\]
Then
\[
|S_n(p \oplus 1 \oplus q)| \leq (\sqrt{C_p} + \sqrt{C_q})^{2n}.
\]
Partial Orders (Ch 16)

- Basic Definitions
- Dillworth’s Theorem
- Incidence Algebras and Mobius Inversion
Partial Orders

**Definition:** A partial order is a relation $\geq$ on a set satisfying three properties:

1. $x \geq x$ for all $x$ (reflexivity)
2. If $x \geq y$ and $y \geq z$ then $x \geq z$ (transitivity)
3. If $x \geq y$ and $y \geq x$, then $x = y$.

**Note:** There may be $x$ and $y$ that are incomparable, i.e. where neither $x \geq y$ nor $y \geq x$. 
Example I

$B_n$ is the collection of subsets of $[n]$ ordered by inclusion.

$A \geq B$ if and only if $A \supseteq B$.

$2^n$ total elements.
Example II

\( \Pi_n \) is the set of partitions of \([n]\) ordered by refinement.

A partition \( P \) is \( \geq \) a partition \( Q \) if and only if \( Q \) can be obtained from \( P \) by breaking up the parts of \( P \) into smaller parts.

For example,

\[ \{1,3,4\}\{2,5,6\} \geq \{1\}\{3,4\}\{2,5,6\} \]
Example III

\( \mathbb{N} \) ordered by divisibility.

\( a \geq b \) if and only if \( a \) is a multiple of \( b \).
Example IV

$\mathbb{R}^n$ ordered by domination.

Namely, $(x_1, x_2, ..., x_n) \geq (y_1, y_2, ..., y_n)$ if and only if $x_1 \geq y_1$, $x_2 \geq y_2$, ..., $x_n \geq y_n$. 
Example V

All permutations ordered by containment of patterns. Namely, $\pi \geq \rho$ if and only if $\pi$ contains a copy of $\rho$.

Homework exercise 2c this week essentially requires showing that the transitive property holds here.
Min/Max

**Definition:**
An element $x$ in a partial order is:

- A **maximal** element if there is no $y > x$.
- A **maximum** element if $x \geq y$ for all $y$.
- A **minimal** element if there is no $y < x$.
- A **minimum** element if $x \leq y$ for all $y$. 
Covering Elements

**Definition:** We say that $x$ covers $y$ if $x > y$, but there is no $z$ with $x > z > y$.

**Lemma:** In any finite partial order $x \geq y$ if and only if there is some chain of elements $x = x_0, x_1, x_2, ..., x_n = y$ with $x_i$ covers $x_{i+1}$.
**Hasse Diagram**

**Definition:** The Hasse Diagram of a partial order draws the elements of the partial order with lines between $x$ and $y$ if $x$ covers $y$, and with $x$ positioned vertically above $y$. 
Chains and Anti-Chains

**Definition:** A chain in a partial order is a sequence of elements

\[ x_1 < x_2 < x_3 < \ldots < x_k. \]

**Definition:** An anti-chain in a partial order is a collection of elements \( x_i \) so that no two of the \( x_i \) are comparable.
Covers By Chains

**Definition:** Given a partial order, a cover by chains is a collection of chains whose union is all of the elements of the order.

**Question:** Given a finite partial order, what is the smallest number of chains needed to cover it?
Dilworth’s Theorem

**Theorem:** If $P$ is a finite partial order the size of the largest anti-chain equals the size of the smallest chain cover.
Incidence Algebra

**Definition:** For a partial order \( P \), let
\[
\text{Int}(P) = \{(x,y): x,y \in P, x \leq y\}.
\]
The **incidence algebra** of \( P \) is the set of functions
\[
f: \text{Int}(P) \to \mathbb{R}
\]
It has a multiplication operation:
\[
(f \ast g)(x, y) := \sum_{x \leq w \leq y} f(x, w)g(w, y).
\]
Identity

The incidence algebra has an identity:
\[ \delta(x, y) = 1 \text{ if } x = y, \ 0 \text{ otherwise.} \]

\[
(f \ast \delta)(x, y) = \sum_{x \leq w \leq y} f(x, w) \delta(w, y) = f(x, y).
\]
Another useful member of the incidence algebra is the following:

\[ \zeta(x, y) = 1 \text{ for all } x \leq y. \]

**Claim:** The number of length \( k \) chains from \( x \) to \( y \) is:

\[ (\zeta - \delta)^{k-2} \zeta(x, y). \]
Proof

Let $D = \zeta - \delta$.

$D(x, y) = 1$ if $x > y$, 0 else.

$$(D \ast D \ast \ldots \ast D)(x, y)$$

$$= \sum_{x \leq w_1 \leq w_2 \leq \ldots \leq w_{k-2} \leq y} D(x, w_1) D(w_1, w_2) \cdots D(w_{k-3}, w_{k-2}) D(w_{k-2}, y)$$

$$= \sum_{x < w_1 < w_2 < \ldots < w_{k-2} < y} 1$$

$$= \text{Number of length } k \text{ chains from } x \text{ to } y.$$
Inverses

We have that \( \delta \) acts as an identity element for the incidence algebra. What about inverses?

**Lemma:** For any function \( f \) with \( f(x,x) \) non-zero for all \( x \) in \( P \), there is a \( g \) so that \( (f \ast g) = \delta \).
Mobius Function

Of particular interest, is the inverse to $\zeta$, commonly denoted by $\mu$.

We have $\mu(x,x) = 1$ and for $x < y$:

$$0 = \sum_{x \leq w \leq y} \zeta(x, w) \mu(w, y) = \sum_{x \leq w \leq y} \mu(w, y).$$

In particular:

$$\mu(x, y) = -\sum_{x < w \leq y} \mu(w, y).$$
Mobius Inversion

**Theorem:** Given two functions $f$ and $g$ on $P$ with

$$g(x) = \sum_{x \leq y} f(y),$$

we have that

$$f(y) = \sum_{y \leq x} g(x)\mu(y, x).$$
**Definition:** If $P$ and $Q$ are partial orders the product partial order is defined in the set $P \times Q$ where $(x_p, x_Q)$ is $\leq (y_p, y_Q)$ if and only if $x_p \leq y_p$ and $x_Q \leq y_Q$. 
Mobius on Products

**Theorem:** We have for partial orders $P$ and $Q$

$$
\mu_{P\times Q}((x_P, x_Q), (y_P, y_Q)) = \mu_P(x_P, y_P) \mu_Q(x_Q, y_Q).
$$
Lattices

Another special case:

**Definition:** A partial order is a lattice if for every two elements \( x, y \) we have:

- A **minimum upper bound** \( a = x \lor y \)
- A **maximum lower bound** \( b = x \land y \)

In particular,

- \( z \geq x \) and \( z \geq y \) if and only if \( z \geq a \).
- \( z \leq x \) and \( z \leq y \) if and only if \( z \leq b \).
Weisner’s Theorem

**Theorem:** Let $P$ be a lattice with maximum element $1$ and minimum element $0$. Then for any $a \neq 1$,

$$\mu(0, 1) = - \sum_{\substack{x \land a = 0 \\ x \neq 0}} \mu(x, 1).$$