Math 184 Exam 2

Spring 2021
**Question 1** (Basic Generating Function, 25 points). *Give a closed formula for the power series*

\[ 1 + (1/3)x + (1/3)(4/3)x^2/2 + \ldots + (1/3)(4/3)\cdots(k - 2/3)x^k/k! + \ldots \]

Note that

\[ (1 + x)^{-1/3} = 1 + (-1/3)x + (-1/3)(-4/3)x^2/2 + \ldots + (-1/3)(-4/3)\cdots(-k + 2/3)x^k/k! + \ldots \]

Replacing \( x \) with \(-x\), we get that the generating function in question is \((1 - x)^{-1/3}\).
Question 2 (Even and Odd Permutations, 25 points). Given the bijection discussed in class between permutations with only even length cycles and permutations with only odd length cycles, what permutation with odd length cycles corresponds to \((6415)(72)(9308)\)? [For purposes of canonical cycle notation, you should treat 0 as being smaller than 1.]

Firstly, we note that the permutation is already given in canonical cycle notation. We start with the last cycle in the permutation (9308). We note that the last number of this cycle, 8, is larger than the first number of the previous cycle, 7. Thus, this 8 must come from a singleton cycle in the corresponding permutation.

Going back to the previous cycle, (72), we note that it’s last number, 2, is smaller than the first number of the previous cycle, 6. Therefore, this number should be moved to the end of the previous cycle.

Therefore, the corresponding permutation is \((64152)(7)(8)(930)\).
**Question 3** (Sums to 100, 25 points). How many triples of integers $a, b, c$ are there with $0 \leq a, b, c \leq 100$ so that some pair of these integers add to exactly 100?

Let $S_{ab}$ be the set of such triples so that $a + b = 100$, let $S_{bc}$ be the set so that $b + c = 100$ and let $S_{ca}$ be the set so that $c + a = 100$. We wish to compute $|S_{ab} \cup S_{bc} \cup S_{ca}|$. We will do this using Inclusion-Exclusion.

$|S_{ab}| = 101^2 = 10201$. This is because there are 101 possible choices for $a$ and 101 possible choices for $c$ and then a unique choice of $b$ that causes $a + b = 100$. Similarly $|S_{bc}| = |S_{ca}| = 10201$.

$|S_{ab} \cap S_{bc}| = 101$. This is because there are 101 possible choices for $b$, and for each such choice, unique values of $a$ and $c$ causing $a + b = b + c = 100$. Similarly, $|S_{bc} \cap S_{ca}| = |S_{ca} \cap S_{ab}| = 101$.

Finally, we have that $|S_{ab} \cap S_{bc} \cap S_{ca}| = 1$. This is because the only way that $a + b = b + c = c + a = 100$ is if $a = b = c = 50$.

Thus, by Inclusion-Exclusion, we have that

$$|S_{ab} \cup S_{bc} \cup S_{ca}| = 3 \cdot 10201 - 3 \cdot 101 + 1 = 30301.$$
**Question 4** (Balls in Complicated Bins, 25 points). Let $a_n$ be the number of ways to place $n$ unlabelled balls into three labeled bins so that:

1. The balls in the first bin are painted red and blue.
2. There are an even number of balls in the second bin.
3. There are at least three balls in the third bin.

Give a formula for the generating function $\sum_{n=0}^{\infty} a_n x^n$ and justify your answer.

The number of ways to paint the $n$ balls in bin 1 red and blue is $n+1$ as the balls are indistinguishable and thus we can only count the number of red and the number of blue balls. This corresponds to the generating function

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$  

The number of ways to have $n$ balls in bin 2 is 1 if $n$ is even and 0 if $n$ is odd. This corresponds to the generating function

$$\sum_{n \text{ even}} x^n = \frac{1}{1-x^2}.$$  

The number of ways to have $n$ balls in bin 3 is 1 if $n$ is at least three and 0 otherwise. This corresponds to the generating function

$$\sum_{n=3}^{\infty} x^n = \frac{x^3}{1-x}.$$  

The generating function in question is given by the product of the above three generating functions, namely,

$$\left( \frac{1}{(1-x)^2} \right) \left( \frac{1}{1-x^2} \right) \left( \frac{x^3}{1-x} \right) = \frac{x^3}{(1-x)^3(1-x^2)}.$$