Question 1 (Anagrams, 25 points). Give a formula for the number of anagrams (rearrangements of the letters) of “yellow wood door”.

We note that this phrase has a total of 14 letters. Of these letters, ‘d’ appears twice, ‘e’ appears once, ‘l’ appears twice, ‘o’ appears five times, ‘r’ appears once, ‘w’ appears twice, and ‘y’ appears once. Using the formula in class for the number of orderings of a sequence with repeated objects, we get

\[
\frac{14!}{(2!1!2!5!1!2!1!)} = 90,810,720.
\]
Question 2 (Sterling Number Relations, 25 points). Give a formula for $S(1000, 100)$ in terms of $S(998, 98)$, $S(998, 99)$ and $S(998, 100)$.

Using the recurrence $S(n, k) = kS(n-1, k) + S(n-1, k-1)$ we have that

\[ S(1000, 100) = 100S(999, 100) + S(999, 99) \]
\[ = 100(100S(998, 100) + S(998, 99)) + (99S(998, 99) + S(998, 98)) \]
\[ = 10000S(998, 100) + 199S(998, 99) + S(998, 98). \]
Question 3 (Empty Pigeon Holes, 25 points). Suppose that \( n \) pigeons are placed into \( m \) holes with \( m \geq k(n+1) \) for some positive integer \( k \). Suppose that the holes are arranged in a line. Show that there are some \( k \) consecutive holes with no pigeons in them.

Note that there must be at least \( m - n \) empty holes. To each of these holes start at that hole and count along the line of holes until you either find the next hole with a pigeon in it or reach the end. Associate the empty hole with either that pigeon or the end. Note that \( m - n \) holes are being associated with \( n + 1 \) different things. Furthermore, \( m - n \geq k(n+1) - n > (k - 1)(n + 1) \). Thus, by the generalized pigeonhole principle, some \( k \) empty holes are all associated with the same thing (a pigeon or the end). It is easy to see that these \( k \) holes must be consecutive, empty holes.
Question 4 (Subsets with Small Differences, 25 points). How many ways can one select a subset of 10 elements from \{1, 2, 3, \ldots, 1000\} so that the biggest and smallest elements selected differ by at most 100? Justify your answer.

If the smallest element of your subset is at least 900, you have a set of 10 elements out of \{900, 901, \ldots, 1000\} and there are \binom{101}{10} such subsets. Otherwise, there are 899 choices for the smallest element, \(n\), in your subset. Once \(n\) has been selected, the other elements of your subset can be any 9 elements from \(\{n+1, n+2, \ldots, n+100\}\) and there are \binom{100}{9} ways to make that selection. Thus, by the Generalized Multiplication Rule, there are 899\(\binom{100}{9}\) such subsets with minimum element less than 900. Applying the addition rule, the total count is now:

\[
\binom{101}{10} + 899 \binom{100}{9}.
\]