Announcements

• Homework 2 Due Sunday

• Students interested in problem solving practice should attend discussion sections.
Today

- Start Chapter 1.4
  - Introduction to Eulerian graphs.
Paths and Cycles (Ch 1.4)

• Eulerian Circuits
  – Definition
  – Classification of Eulerian graphs
  – Algorithms

• Hamiltonian cycles
  – Definition
  – Hardness
  – Some conditions
The Bridges of Konisberg

The city of Konisberg had 7 bridges as shown. People liked to go on walks touring the bridges. The mathematician Euler was asked if there was a tour that crossed each bridge exactly once.
Graph Theory

- Turn city into a (multi)graph.
  - Vertices = land areas
  - Edges = bridges

- Want: A walk that uses each edge exactly once.

Stuck!
Definitions

An *Eulerian circuit* is a circuit that uses every edge of a graph exactly once.

An *Eulerian trail* similarly uses each edge exactly once, but does not start and end at the same vertex.

A graph is *Eulerian* if it contains an Eulerian circuit and *semi-Eulerian* if it contains an Eulerian trail.
Question

The graph to the right is:
A) Eulerian
B) Semi-Eulerian
C) Neither
Questions we want to answer

• Which graphs are Eulerian / semi-Eulerian?
• How do we construct Eulerian circuits/trials?
Observation I

(semi-)Eulerian graphs must be connected!
(except for isolated vertices)

If there is no path from \( u \) to \( v \), certainly no Eulerian circuit/trail that connects both of them either.
Observation II

How does Eulerian circuit interact with vertex $v$?

Each time we take edge into $v$, then take different edge out of $v$. 
Observation II

If $G$ is Eulerian, $\deg(v)$ must be even!
Conclusion

So if G is Eulerian then it must:
• Be connected (except for isolated vertices)
• Have all vertices of even degree

Is this enough?

Yes!

**Theorem (1.20):** A finite, connected graph G is Eulerian if and only if all vertices have even degree.
Question: Eulerian Graphs

Which of these graphs are Eulerian?

A

B

C

D

E
Proof

• “Only if” already done.
  – If G has Eulerian circuit, each time the circuit passes through v it uses two of its edges.
  – Since Eulerian, eventually uses all of v’s edges
  – Therefore, deg(v) must be even.
  – This must hold for all vertices v.

• Need to prove that this is sufficient
  – Show how to construct an Eulerian circuit.
Constructing a Circuit I

As a first step, we will show you can construct a circuit.

Basic facts:

• Every degree is even
• Every non-isolated vertex has degree at least 2.
Constructing a Circuit II

- Start at any non-isolated vertex $v$.
- Construct a trail by adding new edges until you get stuck.
Claim: Can *only* get stuck at v.
Once you get stuck, you will have a circuit!

Proof:
• Suppose you got stuck at some other w.
• Each time you pass through w, use up 2 edges.
• Takes another edge to reach w. If at w used an odd number of edges.
• At least one left!
Constructing a Circuit IV
Are we done?

We have a circuit. Does it necessarily cover all the edges?

No.
How do we fix this?

Two ideas:

• Can find more circuits
• Glue circuits together
More Circuits

• Existing circuit uses an even number of edges at each vertex.
• Removing those edges, have an even number left at each vertex.
• Can create new circuit in remainder.
Combining Circuits

• Have two circuits that share a vertex.
• Turn them into one big circuit.
Final Algorithm

• Find a circuit.
• If all of $G \rightarrow$ done.
• Otherwise, find $v$ on circuit with unused edge.
• Find additional circuit through $v$.
• Merge with existing circuit.
• Repeat
Analysis

• By connectivity, if circuit isn’t all of G, contains some vertex with an extra edge. (Otherwise your circuit would be a full connected component)

• Each step increases the number of edges in your circuit. Eventually, you must get all of G.