Announcements

• Exam 1 Solutions online
• HW 4 Due on Sunday
Last Time

• Planar graphs
  – Can draw in the plane without crossing edges
• Faces
  – Regions bounded by edges
  – One infinite face
• Euler’s Formula
  – For connected graphs, $v - e + f = 2$. 
Sides to a Face

If $G$ is a connected planar graph, any face (including the infinite one) will be bounded by a loop of edges.

The number of *sides* of the face is the number of edges in this loop.
Example

You can have weird examples like this:

Note that sides 1/17, 4/8, and 10/15 are really the same edge listed twice.
Today

• Dual Handshake Lemma
• Applications of Euler’s Formula
  – Edge Bounds
  – Non-planarity of $K_5$ and $K_{3,3}$
• Fary’s Theorem
Face Bounds

To really make use of Euler’s Formula, it is important to get an idea of how many faces there are.

There is a way of counting these that is somewhat dual to the Handshake Lemma.
Dual Handshake Lemma

Lemma: For a connected, planar graph,

\[ \sum_{\text{Faces } f} \text{Sides}(f) = 2|E|. \]

• Note similarity to Handshake Lemma. Sides of faces instead of degrees of vertices.
• Proof similar.
Proof

• Count the number of pairs of an edge as a side of a face (be careful to count edges that are double sides twice).
• Each edge on two faces (or one face twice).
• Each face $f$ has $\text{Sides}(f)$ edges.
A Key Observation

Every face has \textit{at least three} sides.

\[ 2e = \sum_{\text{Faces } f} \text{Sides}(f) \geq 3f. \]

(unless \(|V| = 2\))
More Generally,

If $G$ only has faces with at least $k$ sides then

$$e \geq k \cdot f / 2.$$
Theorem (1.33): If $G$ is a connected planar graph with $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$
Proof

We know:
• $v - e + f = 2$.
• $e \geq 3f/2$.

So:

$$2 = v - e + f \leq v - e + 2e/3 = v - e/3$$

Rearranging, we find:

$$e \leq 3v - 6.$$
Question: Side Bound

If each face has at least k sides, what is the maximum number of edges?

A) \( \frac{v}{1-\frac{2}{k}} - 6 \)

B) \( k(v-2) \)

C) \( \frac{(v-2)}{1-\frac{2}{k}} \)

D) \( 3v-6 \)

\[ 2 = v - e + f \leq v - e + \frac{2e}{k} \]
\[ = v - e\left(1-\frac{2}{k}\right) \]
K_5 Non-Planar

**Theorem (1.34):** The K_5 is non-planar.

**Proof:** If it were, we would have

\[ e \leq 3v - 6 = 9 \]

But \( e = 10 \). Contradiction!
K₃,₃ Non-Planar

**Theorem (1.32):** K₃,₃ is non-planar.

**Proof:** K₃,₃ is bipartite, so it has no odd cycles. Therefore, if planar any face has at least 4 sides.

If planar,

\[ e \leq 2v-4 = 8. \]

But e=9. Contradiction!
Minimum Degree

**Theorem (1.35):** If \( G \) is a finite, connected planar graph, its vertices have minimum degree at most 5.

**Proof:** Otherwise, each vertex has degree 6 or more.

Handshake Lemma implies

\[ 2e = \Sigma d(v) \geq 6v. \]

But then

\[ 3v - 6 \geq e \geq 3v. \]

Contradiction!
Triangulations

We note that our edge bound of $3v - 6$ has an equality case if and only if all faces are triangles.

We can always ensure that this is the case if we add more edges.

**Lemma:** For any planar embedding of a graph $G$ there is a way to add more edges to $G$ to get a new planar graph $G'$ in which all faces are triangles.
Proof

- Add edges until triangulated.
- Consider a face F with at least 4 sides.
- Add edge between non-adjacent vertices.
- Cannot if already outside edge.
- Cannot have on both ends.
Fary’s Theorem

**Theorem (V. 7.4.2):** Any finite (simple) planar graph $G$ has a plane embedding where all of the edges are straight line segments.
Proof Strategy

- Induct on $v$.
  - If $v \leq 3$, easy to draw.
- Assume $G$ is connected (ow/ draw each component separately)
- Find a vertex $v$ of low degree.
- Draw $G-v$ with straight lines.
- Re-insert $v$ into drawing.