Hardness of Learning Halfspaces with Massart Noise

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Abstract

We study the complexity of PAC learning halfspaces in the presence of Massart (bounded) noise. Specifically, given labeled examples \((x, y)\) from a distribution \(D\) on \(\mathbb{R}^n \times \{-1, 1\}\) such that the marginal distribution on \(x\) is arbitrary and the labels are generated by an unknown halfspace corrupted with Massart noise at rate \(\eta < 1/2\), we want to compute a hypothesis with small misclassification error. Characterizing the efficient learnability of halfspaces in the Massart model has remained a longstanding open problem in learning theory.

Recent work gave a polynomial-time learning algorithm for this problem with error \(\eta + \epsilon\). This error upper bound can be far from the information-theoretically optimal bound of \(\text{OPT} + \epsilon\). More recent work showed that exact learning, i.e., achieving error \(\text{OPT} + \epsilon\), is hard in the Statistical Query (SQ) model. In this work, we show that there is an exponential gap between the information-theoretically optimal error and the best error that can be achieved by a polynomial-time SQ algorithm. In particular, our lower bound implies that no efficient SQ algorithm can approximate the optimal error within any polynomial factor.

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1 Introduction

1.1 Background and Motivation

A halfspace, or Linear Threshold Function (LTF), is any function $f : \mathbb{R}^m \to \{\pm 1\}$ of the form $f(x) = \text{sign}(w \cdot x - \theta)$, for some weight vector $w \in \mathbb{R}^m$ and threshold $\theta \in \mathbb{R}$. (The function $\text{sign} : \mathbb{R} \to \{\pm 1\}$ is defined as $\text{sign}(t) = 1$ if $t \geq 0$ and $\text{sign}(t) = -1$ otherwise.) Halfspaces are a fundamental class of Boolean functions that have been extensively studied in computational complexity and learning theory over several decades. [MP68, Yao90, GHR92, STC00, O’D14]. The problem of learning an unknown halfspace is as old as the field of machine learning, starting with the Perceptron algorithm [Ros58, Nov62], and has been one of the most influential problems in this field with techniques such as SVMs [Vap98] and AdaBoost [FS97] coming out of its study.

In the realizable PAC model [Val84], i.e., when the labels are consistent with the target function, halfspaces are efficiently learnable via Linear Programming (see, e.g., [MT94]). In the presence of noisy data, the computational complexity of learning halfspaces depends on the underlying noise model. Here we study the complexity of learning halfspaces with Massart noise. In the Massart (or bounded noise) model, the label of each example $x$ is flipped independently with probability $\eta(x) \leq \eta$, for some parameter $\eta < 1/2$. The flipping probability is strictly bounded above by $1/2$, but can depend on the example $x$ in a potentially adversarial manner. The following definition encapsulates the PAC learning problem in this noise model.

**Definition 1.1 (PAC Learning with Massart Noise).** Let $\mathcal{C}$ be a concept class of Boolean-valued functions over $X = \mathbb{R}^m$, $\mathcal{D}_x$ be a fixed but unknown distribution over $X$, and $0 \leq \eta < 1/2$ be the noise parameter. Let $f : X \to \{\pm 1\}$ be the unknown target concept with $f \in \mathcal{C}$. A Massart example oracle, $\text{EX}^{\text{Mas}}(f, \mathcal{D}_x, \eta)$, works as follows: Each time $\text{EX}^{\text{Mas}}(f, \mathcal{D}_x, \eta)$ is invoked, it returns a labeled example $(x, y)$, where $x \sim \mathcal{D}_x$, $y = f(x)$ with probability $1 - \eta(x)$ and $y = -f(x)$ with probability $\eta(x)$, for some unknown function $\eta(x) : X \to [0, 1/2]$ with $\eta(x) \leq \eta$ for all $x \in X$. Let $\mathcal{D}$ denote the joint distribution on $(x, y)$ generated by the Massart example oracle. A PAC learning algorithm is given i.i.d. samples from $\mathcal{D}$ and its goal is to output a hypothesis $h : X \to \{\pm 1\}$ such that with high probability the error $\text{Pr}_{(x,y) \sim \mathcal{D}}[h(x) \neq y]$ is as small as possible. We will use $\text{OPT} = \inf_{g \in \mathcal{C}} \text{Pr}_{(x,y) \sim \mathcal{D}}[g(x) \neq y]$ to denote the optimal misclassification error.

We emphasize here that, throughout this paper, we focus on improper learning, where the learning algorithm is allowed to output any polynomially evaluable hypothesis.

The Massart noise model is a natural semi-random input model that was formulated in [MN06]. An equivalent noise model had already been defined in the 80s by Sloan and Rivest [Slo88, Slo92, RS94] (under the name “malicious misclassification noise”) and a very similar definition had been proposed and studied even earlier by Vapnik [Vap82]. The sample complexity of PAC learning halfspaces with Massart noise is well-understood. For example, it is known (see, e.g., [MN06]) that for any concept class $\mathcal{C}$ of VC dimension $d$, $O(d/\epsilon^2)$ samples information-theoretically suffice to compute a hypothesis with misclassification error $\text{OPT} + \epsilon$. We note here that, by definition, we have that $\text{OPT} = \mathbb{E}_{x \sim \mathcal{D}_x}[\eta(x)] \leq \eta$. In particular, this implies that halfspaces on $\mathbb{R}^m$ are learnable in the Massart model with $O(m/\epsilon^2)$ samples.

In sharp contrast, our understanding of the algorithmic aspects of PAC learning various natural concept classes with Massart noise is startlingly poor and has remained a tantalizing open problem in computational learning theory since the 1980s. In [Slo88] (see also [Slo92]), Sloan defined the malicious misclassification noise model (an equivalent formulation of Massart noise) and asked whether there exists an efficient learning algorithm for Boolean disjunctions, i.e., ORs of Boolean literals — a very special case of halfspaces — in this model. About a decade later, Edith Cohen [Coh97]...
asked the same question for the general class of halfspaces. The problem remained open, even for weak learning, and was highlighted in Avrim Blum’s FOCS 2003 tutorial [Blu03]. Surprisingly, until fairly recently, it was not even known whether there exists an efficient algorithm that achieves misclassification error 49% for Massart halfspaces with noise rate upper bound of $\eta = 1%$.

Recent work [DGT19] made the first algorithmic progress on this learning problem. Specifically, [DGT19] gave a poly($m, 1/\epsilon$) time learning algorithm for Massart halfspaces with error guarantee of $\eta + \epsilon$, where $\eta$ is the upper bound on the Massart noise rate. This is an absolute error guarantee which cannot be improved in general (since it may well be the case that OPT = $\eta$). However, the error bound of $\eta + \epsilon$ can be very far from the information-theoretically optimal bound of OPT + $\epsilon$. (Recall that OPT $\leq \eta$, but it could be the case that OPT $\ll \eta$.) Follow-up work [CKMY20] showed that exact learning, i.e., obtaining the optimal error of OPT + $\epsilon$, when OPT is very close to 1/2, requires super-polynomial time in the Statistical Query (SQ) model [Kea98]. Unfortunately, the [CKMY20] SQ lower bound is very fragile, in the sense that it does not rule out any constant factor (relative) approximation.

This recent progress notwithstanding, a very large gap remains in our understanding of the approximate efficient learnability of halfspaces in the presence of Massart noise.

Is there a polynomial-time learning algorithm for Massart halfspaces achieving a relative (distribution-independent) error guarantee?

Specifically, is there a polynomial-time algorithm with misclassification error $C \cdot$ OPT + $\epsilon$, for some constant $C > 1$? If not, can we efficiently achieve a polynomial approximation ratio, i.e., misclassification error of $O$(OPT$^c$) + $\epsilon$, for some universal constant $0 < c < 1$? More generally, what is the best error (as a function of OPT and $\eta$) that can be achieved in polynomial time?

As the main contribution of this paper, we prove strong negative results ruling out any constant factor or even any polynomial factor (relative) approximation in the Statistical Query (SQ) model. Before we formally state our contributions, we require some background on SQ algorithms.

**Statistical Query Model.** Statistical Query (SQ) algorithms are a class of algorithms that are only allowed to query expectation of bounded functions of the underlying distribution rather than directly access samples. The SQ model was introduced by Kearns [Kea98] in the context of supervised learning as a natural restriction of the PAC model [Val84] and has been extensively studied in learning theory. A recent line of work [FGR+13, FPV15, FGV17, Fel17] generalized the SQ framework for search problems over distributions. The reader is referred to [Fel16] for a survey.

One can prove lower bounds on the complexity of SQ algorithms via an appropriate notion of Statistical Query dimension. Such a complexity measure was introduced in [BFJ+94] for PAC learning of Boolean functions and has been generalized to the unsupervised setting in [FGR+13, Fel17]. A lower bound on the SQ dimension of a learning problem provides an unconditional lower bound on the computational complexity of any SQ algorithm for the problem.

The class of SQ algorithms is fairly broad: a wide range of known algorithmic techniques in machine learning are known to be implementable using SQs. These include spectral techniques, moment and tensor methods, local search (e.g., Expectation Maximization), and many others (see, e.g., [CKL+06, FGR+13, FGV17]). In the context of PAC learning classes of Boolean functions (the topic of this paper), with the exception of learning algorithms using Gaussian elimination (in particular for the concept class of parities, see, e.g., [BKW03]), all known algorithms with non-trivial performance guarantees are either SQ or are implementable using SQs. Finally, we acknowledge very recent work [BBH+20] which established a near-equivalence between the SQ model and low-degree algorithms under mild assumptions.
1.2 Our Contributions

Our main result shows that any SQ algorithm that achieves a constant or even a polynomial relative approximation for the problem of PAC learning Massart halfspaces (in the distribution-independent setting) requires at least super-polynomial number of statistical queries of inverse super-polynomial accuracy. Formally, we establish the following theorem:

**Theorem 1.2** (Main Result). Let \( \text{OPT} > 0 \) and \( M \in \mathbb{Z}_+ \) be such that \( \log(M)/(\log \log(M))^3 \) is at least a sufficiently large constant multiple of \( \log(1/\text{OPT}) \). There exists a parameter \( \tau = M^{-\Omega(\log(M)/(\log \log(M))^3 \log(1/\text{OPT}))} \) such that no SQ algorithm can learn a halfspace on \( \mathbb{R}^M \) in the presence of Massart noise with \( \eta = 1/3 \) within error better than \( 1/\text{polylog}(M) \) with \( 1/\tau \) queries of accuracy \( \tau \). This holds even if the optimal halfspace classifier has error at most OPT.

As an immediate corollary of Theorem 1.2, by taking \( \log(1/\text{OPT}) = \sqrt{\log(M)} \), we obtain a quasi-polynomial SQ lower bound against learning a hypothesis with error better than \( 1/\text{polylog}(M) \), even when error OPT (which is exponentially smaller) is possible. This a fortiori implies that it is SQ-hard to learn a hypothesis with error \( O(\text{OPT}) \) or even \( \text{poly}(\text{OPT}) \).

It is worth comparing Theorem 1.2 to the hardness result of Daniely [Dan16] for PAC learning halfspaces in the agnostic model. Daniely’s result is qualitatively similar to our Theorem 1.2 with two differences: (1) The lower bound in [Dan16] only applies against the (much more challenging) agnostic model. (2) In the agnostic setting, it is hard to learn halfspaces within error significantly better than 1/2, rather than error 1/\text{polylog}(M) in the Massart setting. (We remind the reader that there exists an efficient Massart halfspace learning algorithm that achieves error arbitrarily close to \( \eta < 1/2 \) [DGT19].) Theorem 1.2 proves an SQ lower bound for a much more benign noise model at the cost of allowing somewhat better error in polynomial time.

1.3 Related and Prior Work

We have already provided some background on the Massart noise model. Here we summarize the most relevant literature on learning halfspaces in related noise models.

Random Classification Noise (RCN) [AL88] is the special case of Massart noise where each label is flipped with probability exactly \( \eta < 1/2 \). Halfspaces are known to be efficiently learnable to optimal accuracy in the (distribution-independent) PAC model with RCN [BFKV96, BFKV97]. In fact, it is well-known that any SQ learning algorithm [Kea98] can be transformed to an RCN noise tolerant learning algorithm — a fact that no longer holds in the presence of Massart noise. Roughly speaking, the ability of the Massart adversary to choose whether to flip a given label and, if so, with what probability, makes the algorithmic problem in this model challenging.

The agnostic model [Han92, KSS94] is the more challenging model where an adversary is allowed to arbitrarily corrupt an arbitrary OPT < 1/2 fraction of the labels. In the distribution-independent setting, even weak agnostic learning of halfspaces (i.e., obtaining a hypothesis with any non-trivial accuracy) is known to be intractable. A long line of work (see, e.g., [GR06, FGKP06]) has established NP-hardness of weak agnostic proper learning. (See [Fel15] for a survey on hardness of proper learning results.) More recently, [Dan16] gave super-polynomial lower bounds for improper learning, under certain average-case assumptions, and simultaneously established SQ lower bounds for the problem. Concretely, [Dan16] showed that no polynomial-time SQ algorithm foragnostically learning halfspaces on \( \mathbb{R}^m \) can compute a hypothesis with error \( 1/2 - 1/m^c \), for some constant \( c > 0 \), even for instances with optimal error \( \text{OPT} = 2^{-\log^{1-c}(m)} \), for some constant \( \nu \in (0, 1/2) \).

We also note that exact learning in the agnostic model is hard even under the uniform distribution on the hypercube [KKMS08, DFT+15] and the Gaussian distribution [KK14, GGK20, DKZ20].
In the Massart model, the recent work [CKMY20] showed an SQ lower bound of $m^{\Omega(\log(1/\epsilon))}$ for learning halfspaces to error $OPT + \epsilon$, when $OPT$ is close to $1/2$. Specifically, [CKMY20] pointed out a connection between SQ learning with Massart noise and the Correlational Statistical Query (CSQ) model, a restriction of the SQ model defined in [BF02] (see also [Fel08, Fel11]). Given this connection, the SQ lower bound of [CKMY20] follows directly from a CSQ lower bound established by Feldman [Fel11].

Finally, we note that $\text{poly}(m, 1/\epsilon)$ time learning algorithms for homogeneous halfspaces with optimal error guarantees in the presence of Massart noise (and other benign noise models) have been established when the marginal distribution on examples is log-concave or, more generally, it satisfies appropriate (anti-)anti-concentration properties and tail bounds [ABHU15, ABHZ16, ZLC17, YZ17, ZSA20, DKTZ20a, DKTZ20b, DKK+20]. The hardness result obtained in this paper provides additional motivation for such distributional assumptions. Without some assumptions on the distribution on examples, even obtaining weak relative approximations is computationally hard.

**Broader Context.** This work is part of the broader direction of understanding the computational complexity of robust high-dimensional learning in the distribution-independent setting. A long line of work, see, e.g., [KLS09, ABL17, DKK+16, LRV16, DKK+17, DKK+18, DKS18, KKM18, DKS19, DKK+19] and the recent survey [DK19], has given efficient robust learners for a range of high-dimensional estimation tasks (both supervised and unsupervised) in the presence of a small constant fraction of adversarial corruptions. These algorithmic results inherently rely on the assumption that the clean data is drawn from a "well-behaved" distribution. On the other hand, the recent work [DGT19] established that efficient robust learners with non-trivial error guarantees are achievable even in the distribution-independent setting, under the more “benign” Massart model.

This result provided compelling evidence that there are realistic noise models in which efficient algorithms are possible without imposing assumptions on the good data distribution. Conceptually, the result of this paper shows that, even in such benign noise models, there can be strong computational limitations in learnability — in the sense that it is computationally hard to achieve even weak relative approximations to the optimal error.

### 1.4 Overview of Techniques

Here we provide a detailed overview of our approach. At a high level, our proof makes use of the SQ lower bound technique developed in [DKS17]. Roughly speaking, that work [DKS17] established the following result: Let $A$ be one-dimensional distribution that matches the first $k$ moments with the standard Gaussian $G$. Suppose we want to distinguish between the standard high-dimensional Gaussian $N(0, I)$ on $\mathbb{R}^d$ and a distribution that is a copy of $A$ in a random direction and is a standard Gaussian in the orthogonal complement. Then any SQ algorithm for this hypothesis testing task requires super-polynomial time (roughly, $d^{O(k)}$ time).

In the context of the current paper, we will in fact require a generalization of the latter generic result that holds even if the one-dimensional distribution $A$ nearly matches the first $k$ moments with $G$. Such a statement (Proposition 2.5) can easily be derived from the techniques of [DKS17]. As we will explain in the proceeding discussion, to employ this methodology in our context, we require a number of conceptual and technical ideas.

To leverage the aforementioned result in our circumstances, we would like to establish the existence of a distribution $(X, Y)$ on $\mathbb{R} \times \{\pm 1\}$ that corresponds to a halfspace with Massart noise such that both the distributions of $X$ conditioned on $Y = 1$ ($X \mid Y = 1$) and $X$ conditioned on $Y = -1$ ($X \mid Y = -1$) approximately match their first $k$ moments with the standard Gaussian.

Unfortunately, achieving this goal directly is impossible for the following reason. It is not hard to see that any distribution $X$ that even approximately matches a constant number of low-order
moments with the standard Gaussian will have $\mathbf{E}[f(X)] \approx \mathbf{E}[f(G)]$ for any halfspace $f$. To see this, we can use the known fact from $[\text{DGJ}^+\text{10}]$ that $f$ can be sandwiched between low-degree polynomials $f_+ \geq f \geq f_-$ with $\mathbf{E}[f_+(G) - f_-(G)]$ small. This means that if both conditional distributions $X \mid Y = 1$ and $X \mid Y = -1$ approximately match their low-degree moments with $G$, then $\mathbf{E}[f(X)\mid Y = 1]$ will necessarily be close to $\mathbf{E}[f(X)\mid Y = -1]$, which cannot hold with Massart noise. In order to circumvent this difficulty, we will instead prove a super-polynomial SQ lower bound against learning degree-$d$ polynomial threshold functions (PTFs) under the Gaussian distribution with Massart noise, for an appropriate (super-constant) value of the degree $d$. Since a degree-$d$ PTF on $X$ is equivalent to an LTF on $X^{\otimes d}$ — a random variable in $m^d$ dimensions — we will thus obtain an SQ lower bound for the original halfspace Massart learning problem. A similar idea was used in $[\text{Dan\text{16}}]$ to prove SQ lower bounds in the agnostic model.

The challenge is, of course, to construct the required moment-matching distributions in one dimension. Even for our reformulated PTF learning problem, it remains unclear whether this is even possible. For example, let $f(x) = \text{sign}(p(x))$ be a degree-$d$ PTF. Then it will be the case that $\mathbf{E}[p(X)\mid Y = 1] = \mathbf{E}[p(X)f(X)(1 - 2\eta(X))] = \mathbf{E}[(p(X))(1 - 2\eta(X))] > 0$. This holds despite the fact that $\mathbf{E}[p(X)\mid Y = 1] \approx \mathbf{E}[p(X)\mid Y = -1] \approx \mathbf{E}[p(G)]$. If $\mathbf{E}[p(G)] > 0$, it will be the case that $\mathbf{E}[p(X)\mid Y = -1]$ will be positive, despite the fact that the conditional distribution of $X \mid Y = -1$ is almost entirely supported on the region where $p(X) < 0$. Our construction will thus need to take advantage of finding points where $|p(X)|$ is very large.

Fortunately for us, something of a miracle occurs here. Consider a discrete univariate Gaussian $G_\delta$ with spacing $\delta$ between its values. It is not hard to show that $G_\delta$ approximately matches moments with the standard Gaussian $G$ to error $\exp(-\Omega(1/\delta^2))$. On the other hand, all but a tiny fraction of the probability mass of $G_\delta$ is supported on $d = \tilde{O}(1/\delta)$ points. We use this observation to construct one of our conditional distributions. In more detail, we will use a $\delta$-spaced discrete Gaussian, widened slightly by convolving it with a narrow Gaussian. We note that the univariate degree-$d$ PTF that we are approximating will be a union of $d/2$ narrow intervals around the most significant peaks of this distribution. This construction gives us a distribution almost entirely supported on the negative values of a degree-$d$ PTF, while still matching moments with a standard Gaussian to error approximately $\exp(-\tilde{\Omega}(d^2))$. Since this error is still small relative to the underlying dimension of the induced LTF problem, i.e., $m^d$, this will provide us with the desired super-polynomial SQ lower bounds.

So far, we have described our construction of one of the conditional distributions (namely, the distribution of $X \mid Y = -1$). Additionally, we require a distribution for the other conditional $X \mid Y = 1$ that nearly matches many of its low-degree moments with the Gaussian $G$ and has much larger density when our PTF, $f(x)$ is 1, and much smaller density when $f(x)$ is $-1$. We achieve this as follows: We start with a standard Gaussian distribution. By making our final joint distribution on $(X, Y)$ output $Y = 1$ with substantially higher probability than $Y = -1$, it is not hard to arrange that the conditional probability that $Y = 1$ is large when $f(X) = 1$. However, we still need to deal with the case where $f(X) = -1$ (which happens on a union of short intervals). To deal with this, we show how to move the mass from these short intervals to nearby regions, in such a way as to not affect any of the low-order moments. To achieve this, we make use of an LP duality argument.

The one-dimensional moment matching construction described above and its proof of correctness are given in Proposition 2.6. Given this construction, we can use (an adaptation of) the SQ machinery in $[\text{DKSI\text{17}}]$ to complete the proof of Theorem 1.2.
1.5 Preliminaries

We use $E[X]$ for the expectation of random variable $X$ and $Pr[E]$ for the probability of event $E$.

**Basics on Statistical Query Algorithms.** We will use the framework of Statistical Query (SQ) algorithms for problems over distributions introduced in [FGR13]. We start by defining a decision problem over distributions.

**Definition 1.3** (Decision Problem over Distributions). We denote by $B(D, D)$ the decision (or hypothesis testing) problem in which the input distribution $D'$ is promised to satisfy either (a) $D' = D$ or (b) $D' \in D$, and the goal of the algorithm is to distinguish between these two cases.

We define SQ algorithms as algorithms that do not have direct access to samples from the distribution, but instead have access to an SQ oracle. We consider the following standard oracle.

**Definition 1.4** (STAT Oracle). For a tolerance parameter $\tau > 0$ and any bounded function $f : \mathbb{R}^n \to [-1, 1]$, \text{STAT}(\tau) returns a value $v \in [E_{x \sim D}[f(x)] - \tau, E_{x \sim D}[f(x)] + \tau]$.

We note that [FGR13] introduced another related oracle, which is polynomially equivalent to STAT. Since we prove super-polynomial lower bounds here, there is no essential distinction between these oracles. To define the SQ dimension, we need the following definitions.

**Definition 1.5** (Pairwise Correlation). The pairwise correlation of two distributions with probability density functions $D_1, D_2 : \mathbb{R}^m \to \mathbb{R}_+$ with respect to a distribution with density $D : \mathbb{R}^m \to \mathbb{R}_+$, where the support of $D$ contains the supports of $D_1$ and $D_2$, is defined as $\chi_D(D_1, D_2) \overset{\text{def}}{=} \int_{\mathbb{R}^m} D_1(x)D_2(x)/D(x)dx - 1$.

We remark that when $D_1 = D_2$ in the above definition, the pairwise correlation is identified with the $\chi^2$-divergence between $D_1$ and $D$, i.e., $\chi^2(D_1, D) \overset{\text{def}}{=} \int_{\mathbb{R}^m} D_1(x)^2/D(x)dx - 1$.

**Definition 1.6**. We say that a set of $s$ distributions $D = \{D_1, \ldots, D_s\}$ over $\mathbb{R}^m$ is $(\gamma, \beta)$-correlated relative to a distribution $D$ if $|\chi_D(D_i, D_j)| \leq \gamma$ for all $i \neq j$, and $|\chi_D(D_i, D_j)| \leq \beta$ for $i = j$.

We are now ready to define our notion of dimension.

**Definition 1.7** (Statistical Query Dimension). For $\beta, \gamma > 0$, a decision problem $B(D, D)$, where $D$ is a fixed distribution and $D$ is a family of distributions over $\mathbb{R}^m$, let $s$ be the maximum integer such that there exists a finite set of distributions $D_D \subseteq D$ such that $D_D$ is $(\gamma, \beta)$-correlated relative to $D$ and $|D_D| \geq s$. We define the *Statistical Query dimension* with pairwise correlations $(\gamma, \beta)$ of $B$ to be $s$ and denote it by $SD(B, \gamma, \beta)$.

Our proof bounds below the Statistical Query dimension of the considered learning problem. This implies lower bounds on the complexity of any SQ algorithm for the problem using the following standard result.

**Lemma 1.8** (Corollary 3.12 in [FGR13]). Let $B(D, D)$ be a decision problem, where $D$ is the reference distribution and $D$ is a class of distributions. For $\gamma, \beta > 0$, let $s = SD(B, \gamma, \beta)$. For any $\gamma' > 0$, any SQ algorithm for $B$ requires at least $s \cdot \gamma'/(\beta - \gamma)$ queries to the $\text{STAT}(\sqrt{\gamma} + \gamma')$ oracle.
2 Proof of Main Result

In this section, we prove Theorem 1.2. The structure of this section is as follows: In Section 2.1 we review the SQ framework from [DKS17] with the necessary modifications required for our setting. In Section 2.2 we establish the existence of the one-dimensional distributions with the desired moment-matching properties. Finally, in Section 2.3 we put everything together to complete the proof of Theorem 1.2. Section 2.2 is the main technical contribution of this paper.

2.1 Background on SQ Bound Machinery

We start with the following definition:

**Definition 2.1** (High-Dimensional Hidden Direction Distribution). For a distribution $A$ on the real line with probability density function $A(x)$ and a unit vector $v \in \mathbb{R}^m$, consider the distribution over $\mathbb{R}^m$ with probability density function $P_v^A(x) = A(v \cdot x) \exp\{-\|x - (v \cdot x)v\|^2/2\}/(2\pi)^{(m-1)/2}$. That is, $P_v$ is the product distribution whose orthogonal projection onto the direction of $v$ is $A$, and onto the subspace perpendicular to $v$ is the standard $(m-1)$-dimensional normal distribution.

We consider the following condition:

**Condition 2.2.** Let $k \in \mathbb{Z}_+$ and $\nu > 0$. The distribution $A$ is such that (i) the first $k$ moments of $A$ agree with the first $k$ moments of $N(0, 1)$ up to error at most $\nu$, and (ii) $\chi^2(A, N(0, 1))$ is finite.

Note that Condition 2.2 (ii) above implies that the distribution $A$ has a pdf, which we will denote by $A(x)$. We will henceforth blur the distinction between a distribution and its pdf.

Our main result in this subsection makes essential use of the following key lemma:

**Lemma 2.3** (Correlation Lemma). Let $k \in \mathbb{Z}_+$. If the univariate distribution $A$ satisfies Condition 2.2 (i), then for all $v, v' \in \mathbb{R}^m$, with $|v \cdot v'|$ less than a sufficiently small constant, we have that

$$|\chi_{N(0, I)}(P_v^A, P_{v'}^A)| \leq |v \cdot v'|^{k+1} \chi^2(A, N(0, 1)) + \nu^2. \quad (1)$$

This lemma is a technical generalization of Lemma 3.4 from [DKS17], which applied under exact moment matching assumptions. The proof is deferred to Appendix A.

We will also use the following standard fact:

**Lemma 2.4.** For any constant $c > 0$ there exists a set $S$ of $2^{\Omega(c)}$ unit vectors in $\mathbb{R}^m$ such that any pair $u, v \in S$, with $u \neq v$, satisfies $|u \cdot v| < c$.

It is not difficult to show that a set of random unit vectors satisfies the above property. We note that [DKS17] made use of a very similar lemma (Lemma 3.7 in that work) with somewhat different parameters.

The following result is the basis for our SQ lower bounds:

**Proposition 2.5.** Let $A$ and $B$ be distributions on $\mathbb{R}$ satisfying Condition 2.2 with parameters $k$ and $\nu$ and let $p \in (0, 1)$. For $m \in \mathbb{Z}_+$ and a unit vector $v \in \mathbb{R}^m$, define the distribution $P_v^{A, B, p}$ on $\mathbb{R}^m \times \{\pm 1\}$ that returns a sample from $(P_v^A, 1)$ with probability $p$ and a sample from $(P_v^B, -1)$ with probability $1 - p$. Let $\tau = \nu^2 + 2^{-k}(\chi^2(A, N(0, 1)) + \chi^2(B, N(0, 1)))$. Then any SQ algorithm that learns a hypothesis $h$ such that $Pr_{(X,Y) \sim P_v^{A, B, p}}[h(X) \neq Y] < \min(p, 1 - p) - 2\sqrt{\tau}$ must either make queries of accuracy better than $2\sqrt{\tau}$ or must make at least $2^{\Omega(m)}\tau/(\chi^2(A, N(0, 1)) + \chi^2(B, N(0, 1)))$ statistical queries.
Proof of Proposition 2.6. Let $S$ be a set of $2^\Omega(m)$ unit vectors in $\mathbb{R}^m$ whose pairwise inner products are all less than a sufficiently small universal constant. Such a set exists by Lemma 2.3. We note that any SQ algorithm that can compute such an $h$ can (with one additional query to estimate the $\Pr[h(X) \neq Y]$) distinguish between (i) the distribution $P_{\delta,B,p}^{A,B,p}$, for $v$ randomly chosen from $S$, and (ii) the distribution $G'$ on $\mathbb{R}^m \times \{\pm 1\}$, where for $(X, Y) \sim G'$ we have that $X$ is a standard Gaussian $G \sim N(0, I)$, and $Y$ is independently 1 with probability $p$ and $-1$ with probability $1-p$. This is because for any $h$ we have that $\Pr[(X,Y) \sim G'[h(X) \neq Y]] \geq \min(p,1-p)$. We will prove that this is impossible for an SQ algorithm with the desired parameters using Lemma 1.8.

For this, we need to show that for $u, v \in S$ we have that $|\chi_{\mathcal{G}'}(P_{\delta,B,p}^{A,B,p}, P_{u}^{A,B,p})|$ is small. Since $G', P_{v}^{A,B,p}$, and $P_{u}^{A,B,p}$ all assign $Y = 1$ with probability $p$, it is not hard to see that

$$
\chi_{\mathcal{G}'}(P_{v}^{A,B,p}, P_{u}^{A,B,p}) = p \chi_{\mathcal{G}}|_{Y=1} ((P_{v}^{A,B,p} | Y = 1), (P_{u}^{A,B,p} | Y = 1)) + (1-p) \chi_{\mathcal{G}}|_{Y=-1} ((P_{v}^{A,B,p} | Y = -1), (P_{u}^{A,B,p} | Y = -1)) = p \chi_{\mathcal{G}}(P_{v}^{A}, P_{u}^{A}) + (1-p) \chi_{\mathcal{G}}(P_{v}^{B}, P_{u}^{B}).
$$

By Lemma 2.3 it follows that

$$
\chi_{\mathcal{G}'}(P_{v}^{A,B,p}, P_{u}^{A,B,p}) \leq \nu^2 + 2^{-k}(\chi^2(A, N(0, 1)) + \chi^2(B, N(0, 1))) = \tau.
$$

A similar computation shows that

$$
\chi_{\mathcal{G}'}(P_{v}^{A,B,p}, P_{u}^{A,B,p}) = \chi^2(P_{v}^{A,B,p}, G') \leq \chi^2(A, N(0, 1)) + \chi^2(B, N(0, 1)).
$$

An application of Lemma 1.8 for $\gamma = \gamma' = \tau$ and $\beta = \chi^2(A, N(0, 1)) + \chi^2(B, N(0, 1))$ completes the proof. \hfill \Box

2.2 Construction of Univariate Moment-Matching Distributions

Here we give our univariate approximate moment-matching construction (Proposition 2.6), which is the main technical contribution of this paper.

Notation. We will use the term pseudodistribution to mean a non-negative measure. We will use $G$ for the 1-dimensional standard Gaussian distribution and $g(x)$ for its probability density.

The main result of this section is captured in the following proposition.

Proposition 2.6. Let $d, k \in \mathbb{Z}_+$ and $\delta, \epsilon, \zeta \in \mathbb{R}_+$ with $\delta > \epsilon$ and such that (i) $\epsilon \sqrt{\log(1/\zeta)/\delta}$ is less than a sufficiently small constant multiple of $1/k^2$, (ii) $\delta$ is less than a small constant multiple of $d^{-1/2}$, and (iii) $d$ is at least a sufficiently large constant multiple of $\sqrt{\log(1/\zeta)/\delta}$. There exist pseudodistributions $\mathcal{D}_+$ and $\mathcal{D}_-$ over $\mathbb{R}$ and a union $J$ of $d$ intervals such that:

1. $\mathcal{D}_+ = 0$ on $J$ and $\mathcal{D}_+ > 2\mathcal{D}_-$ on $\bar{J}$.
2. All but $\zeta$ fraction of the measure of $\mathcal{D}_-$ lies in $J$.
3. The distributions $\mathcal{D}_+/\|\mathcal{D}_+\|_1$ and $\mathcal{D}_-/\|\mathcal{D}_-\|_1$ have their first $k$ moments matching those of $G$ within error at most $k! \exp(-\Omega(1/\delta^2))$.
4. $\mathcal{D}_+$ is pointwise at most $O(\delta/\epsilon)$ $G$ and $\|\mathcal{D}_+\|_1 = \Theta(\delta/\epsilon)$.
5. $\mathcal{D}_- = O(\delta/\epsilon)G$ pointwise.

The rest of this subsection is devoted to the proof of Proposition 2.6.
Proof of Proposition 2.7. We begin by defining \( G_\delta \) to be the \( \delta \)-spaced discrete Gaussian distribution. In particular \( G_\delta \) assigns probability mass \( \delta g(n\delta) \) to the point \( n\delta \) for each \( n \in \mathbb{Z} \). The following simple lemma shows that the moments of \( G_\delta \) approximately match the moments of the standard Gaussian \( G \).

Lemma 2.7. For \( t \geq 0 \), \( G_\delta \) and \( G \) have \( t \)th moments differing by at most \( t!O(\delta^t) \exp(-\Omega(1/\delta^2)) \).

The proof of this lemma is deferred to Appendix [13]. The proof proceeds by analyzing the Fourier transform of \( G_\delta \) and using the fact that the \( t \)th moment of a pseudodistribution is proportional to the \( t \)th derivative of its Fourier transform at 0.

We now define the measure \( D_\varepsilon \) to be the sum of \( \sqrt{1-\varepsilon^2} \) times a copy of \( G_\delta \) and \( \varepsilon \) times an independent copy of \( G \). For \( t \leq k \), we can calculate the \( t \)th moment of \( D_\varepsilon \) as follows:

\[
E[D^t] = \sum_{i=0}^{t} \binom{t}{i} (1-\varepsilon^2)^i \varepsilon^{t-i} E[G^i_\delta]E[G^{t-i}]
\]

\[
= \sum_{i=0}^{t} \binom{t}{i} (1-\varepsilon^2)^i \varepsilon^{t-i} (E[G^i] + i!O(\delta^i) \exp(-\Omega(1/\delta^2)))E[G^{t-i}]
\]

\[
\leq \sum_{i=0}^{t} \binom{t}{i} (1-\varepsilon^2)^i \varepsilon^{t-i} E[G^i]E[G^{t-i}] + t! \exp(-\Omega(1/\delta^2)) \sum_{i=0}^{t} \binom{t}{i} O(\delta^i) \varepsilon^{t-i} O(\sqrt{\varepsilon})^{t-i}
\]

\[
= E[G^t] + t! \exp(-\Omega(1/\delta^2))O(\delta + \varepsilon\sqrt{t}).
\]

By assumption (i), we have that \( e^{1/2} \beta \leq \delta \), and by assumption (ii) \( \delta \) is sufficiently small so that the term \( O(\delta + \varepsilon\sqrt{t}) \) is less than 1. This implies the desired moment matching condition for \( D_\varepsilon \).

We define \( J \) to be the union of intervals of length \( L = C\varepsilon \sqrt{\log(1/\varepsilon^2)} \), for \( C \) a sufficiently large universal constant, centered at the points \( n\delta \sqrt{1-\varepsilon^2} \), for integers \( -[d/2] \leq n \leq [d/2] \). By Gaussian concentration, it follows that for each of the components of \( D_\varepsilon \), centered at \( n\delta \sqrt{1-\varepsilon^2} \) for such an \( n \), all but a \( \varepsilon/2 \)-fraction of their measure lies in \( J \). Furthermore, since \( d \) is a sufficiently large constant multiple of \( \sqrt{\log(1/\varepsilon^2)} \), all but a \( \varepsilon/2 \) fraction of the measure comes from these intervals.

We will define \( D_\varepsilon \) to be roughly equal to some suitable multiple \( M \) of \( G \). This definition will satisfy the moment matching conditions and will cause \( D_\varepsilon \geq 2D_\varepsilon \) outside of \( J \), as desired. However, we will need to tweak this to guarantee that \( D_\varepsilon < D_\varepsilon /2 \) on \( J \). In fact, we will make \( D_\varepsilon \) equal to 0 on \( J \). To achieve this, we require the following lemma:

Lemma 2.8. Let \( x, c \in \mathbb{R}_+ \). Let \( D \) be a pseudodistribution defined on \([-x,x]\) such that the probability density function of \( D \) varies by at most a factor of 2 in this range. If \( c \) is less than a sufficiently small constant multiple of \( 1/k^2 \), there exists a pseudodistribution \( D' \) on \([-x,x]\) such that:

(i) \( D' \) is 0 on \([-cx, cx]\).

(ii) \( D' \) and \( D \) agree on their first \( k \) moments.

(iii) \( D/2 \leq D' \leq 3D/2 \) on \([-x,x]\)\([-cx, cx]\).

Proof. It suffices to prove this lemma for \( x = 1 \). Note that the desired conditions on \( D' \) define an LP. Thus, there is a feasible solution unless the dual LP has a solution. The latter statement would amount to there being a polynomial \( p \) of degree at most \( k \) such that \( \int_{-c}^c p(x)D(x) \geq \int_{[-1,1]} \int_{[-c,c]} p(x)|D(x)/2 \). This in particular would imply that

\[
c \sup_{[-1,1]} |p(x)| \gg \int_{-1}^1 |p(x)| dx,
\]
or equivalently that \( c\|p\|_\infty \gg \|p\|_1 \). By Hölder’s inequality, this implies that \( \sqrt{c}\|p\|_\infty \gg \|p\|_2 \). Note that the measure on \([-1, 1]\) is identical to the projection of the measure on the 2-sphere. Therefore, lifting \( p \) to the 2-sphere, we get a degree at most \( k \) polynomial in the 2-sphere such that the max of \( |p| \) is \( \Omega(c^{-1/2}) \) times the \( L_2 \)-norm of \( p \) over the 2-sphere. We write \( p \) as a sum of the orthogonal polynomials \( \phi_i \) on the sphere, as \( p(x) = \sum_i a_i \phi_i(x) \). Then we have that \( \|p\|_2^2 = \sum_i a_i^2 \). From this it is easy to see that \( |p(x)| \leq \sqrt{\sum_i \phi_i(x)^2} \|p\|_2 \). However, \( \sum_i \phi_i(x)^2 \) can easily be seen to be a spherically symmetric function with average value equal to the dimension of the space of degree at most \( k \) polynomials on the sphere, which is \( O(k^d) \). Therefore, this function is a constant function, which is bounded by \( O(k^2) \). Hence, for any such \( p \), we have that \( \|p\|_\infty = O(k\|p\|_2) \), which gives a contradiction when \( c \) is a sufficiently small constant multiple of \( 1/k^2 \). This completes the proof of Lemma 2.8.

We are now ready to explain how to construct \( \mathcal{D}_+ \). We start with \( \mathcal{D}_+ = M \mathcal{G} \), for \( M > 0 \) a sufficiently large constant multiple of \( \delta/\epsilon \). Then, for each integer \( -|d/2| \leq n \leq |d/2| \), we apply Lemma 2.8 to \( \mathcal{D}_+ \) restricted to the interval \([\sqrt{1-\epsilon^2}(n-1/2)\delta, \sqrt{1-\epsilon^2}(n+1/2)\delta]\) with \( c = C\epsilon\log(1/\zeta)/(2\delta) \), to replace \( \mathcal{D}_+ \) by another pseudodistribution with matching first \( k \) moments but with no support on the appropriate interval of \( J \). Note that this causes the intervals of length \( 2c\epsilon \) to correspond exactly to the intervals of \( J \). Note that the above lemma can be applied because the pdf of \( G \) on this interval varies by at most a factor of

\[
\exp(O(\delta^2((n + 1/2)^2 - (n - 1/2)^2))) = \exp(O(n\delta^2)) = \exp(O(d\delta^2)).
\]

This is at most 2 since \( \delta \) is less than a small constant multiple of \( d^{-1/2} \). Furthermore, since \( \epsilon\log(1/\zeta)/\delta \) is less than a small constant multiple of \( 1/k^2 \), we have that \( c \) is less than a suitable multiple of \( 1/k^2 \). Thus, we satisfy the hypotheses of Lemma 2.8. The final \( \mathcal{D}_+ \) matches \( k \) moments with \( M \mathcal{G} \), has no support on \( J \), and elsewhere has size \( \Theta(M \mathcal{G}) \). It only remains to verify the ratio of \( \mathcal{D}_+ \) to \( \mathcal{D}_- \) on \( J \). In fact, we claim that \( \mathcal{D}_- \) is substantially less than \( M \mathcal{G} \) everywhere.

It is not hard to see that the probability density function of \( \mathcal{D}_- \) at \( x \) is

\[
\sum_{n \in \mathbb{Z}} \frac{\delta}{\epsilon \sqrt{2\pi}} \exp \left( -(x - n\delta \sqrt{1-\epsilon^2}/2^2 - (n\delta/2)^2) \right).
\]

Observe that the term in the exponent is a quadratic in \( n\delta \). It is not hard to see that this quadratic has maximum value \(-x^2/2\) and the \( n^2 \) term is multiplied by a constant less than \(-1\). From this one can see that the entire sum is at most \( O(\delta/\epsilon \exp(-x^2/2)) = O(\delta/\epsilon g(x)) \). Therefore, since \( M \) is a sufficient constant multiple of \( \delta/\epsilon \), the relative sizes of the distributions hold as desired.

This completes the proof of Proposition 2.10.

### 2.3 Putting Everything Together: Proof of Theorem 1.2

We now have the necessary ingredients to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Our proof will make use of the hidden-direction SQ framework of Section 2.1 and will crucially rely on the one-dimensional construction of Proposition 2.10.

Let \( C > 0 \) be a sufficiently large universal constant. Let \( m \) be a positive integer on the order of \( C\log(1/\tau) \) and \( d \) a positive integer on the order of \( C\sqrt{\log(1/\text{OPT})} \log(1/\tau) \log \log(1/\tau) \). Observe that

\[
\binom{2d + m}{m} \leq m^{2d} = \exp(O(C\sqrt{\log(1/\text{OPT})} \log(1/\tau)(\log \log(1/\tau))^2)).
\]
We note that if \( \log(1/\tau) \) is a sufficiently small multiple of \( \log^2(M)/(\log \log(M))^3 \log(1/OPT) \), it is less than \( M \). Thus, by decreasing \( M \) if necessary, we can assume that \( M = (2d^2 + m) \). We define the Veronese mapping \( f_{2d} : \mathbb{R}^m \to \mathbb{R}^M \), such that the coordinate functions of \( f_{2d} \) are exactly the monomials in \( m \) variables of degree at most \( 2d \).

We can now formally construct the hard distribution. Define measures \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) on \( \mathbb{R}^m \) as given by Proposition 2.6 with \( \zeta = \text{OPT}, \delta \) a sufficiently large constant multiple of \( \sqrt{\log(1/\zeta)}/d \), \( k \) a small multiple of \( \log^2(1/\delta)/\log \log(1/\delta) \), and \( \epsilon \) a small constant multiple of \( \delta/(\sqrt{\log(1/\epsilon)}/d^2) \). It is easily verified that these parameters satisfy the assumptions of Proposition 2.6. For a unit vector \( v \in \mathbb{R}^m \), define \( D^+_{v,D_+} \) and \( D^-_{v,D_-} \) as in Proposition 2.6 with \( p = \|D^+\|_1/(\|D_+\|_1 + \|D_-\|_1) \). Define a distribution \( (X',Y') \) on \( \mathbb{R}^m \times \{\pm 1\} \) by drawing \((X,Y)\) from \( D^+_{v,D_+} \) and letting \( X' = f_{2d}(X) \) and \( Y' = Y \).

We begin by verifying that this distribution is a Massart LTF distribution with optimal misclassification error OPT and Massart noise rate upper bound of \( \eta = 1/3 \). To begin with, let \( g : \mathbb{R}^m \to \{\pm 1\} \) be defined as \( g(x) = -1 \) if and only if \( v \cdot x \in J \). Note that \( g \) is a degree-2d PTF. Therefore, there exists some LTF \( L : \mathbb{R}^M \to \{\pm 1\} \) such that \( g(x) = L(f_{2d}(x)) \) for all \( x \). We note that our distribution returns \((X',Y')\) with \( Y' = L(X') \), unless it picked a sample corresponding to a sample of \( \mathcal{D}_- \) coming from \( \bar{J} \), which happens with probability at most \( \zeta = \text{OPT} \). Additionally, suppose that our distribution returned a sample with \( X' = f_{2d}(X) \) for some \( X \in \mathbb{R}^m \). By construction, conditioned on this, we have that \( Y' = 1 \) with probability proportional to \( \mathcal{D}_+(v \cdot X) \), and \( Y' = -1 \) with probability proportional to \( \mathcal{D}_-(v \cdot X) \). We note that if \( L(f_{2d}(X)) = 1 \) then \( v \cdot X \in J \), so this ratio is at least \( 2:1 \), and if \( L(f_{2d}(X)) = -1 \), then \( v \cdot X \in J \), so \( \mathcal{D}_+(v \cdot X) = 0 \). This implies that the pointwise probability of error \( \eta(X') \) is at most \( 1/3 \).

On the other hand, it is easy to see that finding a hypothesis that guesses \( Y' \) given \( X' \) is equivalent to finding a hypothesis for \( Y \) given \( X \) (since \( Y = Y' \) and there is a known 1-1 mapping between \( X \) and \( X' \)). The pointwise bounds on \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) imply that

\[
\chi^2(\mathcal{D}_+,G) + \chi^2(\mathcal{D}_-,G) = O(\delta/\epsilon)^2 = \text{polylog}(M).
\]

The parameter \( \nu \) in Proposition 2.5 is \( k! \exp(-\Omega(1/\delta^2)) = \exp(-\Omega(1/\delta^2)) \). We note that \( k \) is a small multiple of \( \log(1/\delta)^2/\log \log(1/\delta) \), which is a large multiple of

\[
d^2/\log(1/OPT)/\log \log(1/\delta) = C \log(1/\tau) \log \log(1/\delta)/\log \log(1/\delta) \gg C \log(1/\tau).
\]

Since \( \log^2(1/\delta) \) is larger than this, our \( \nu \) is at most a large power of \( \tau \). Thus, the \( \tau \) parameter relevant to our application of Proposition 2.5 is \( \nu^2 + 2^{-k}(\chi^2(\mathcal{D}_+,G) + \chi^2(\mathcal{D}_-,G)) \), which is at most a large power of our real \( \tau \). It is furthermore easy to see that \( \min(p,1-p) = 1-p = \Theta(\epsilon/\delta) = 1/\text{polylog}(M) \).

Therefore, Proposition 2.5 implies that it is impossible for an SQ algorithm to learn a hypothesis with error better than \( 1/\text{polylog}(M) \) without either using queries of accuracy better than \( \tau \) or making at least \( 2^{\Omega(m)} \tau/\text{polylog}(M) > 1/\tau \) many queries. This completes the proof of Theorem 1.2. \( \Box \)
References


Appendix

A Proof of Lemma 2.3

Let $\theta$ be the angle between $v$ and $v'$. By making an orthogonal change of variables, we can reduce to the case where $v = (1, 0, \ldots, 0)$ and $v' = (\cos(\theta), \sin(\theta), 0, 0, \ldots, 0)$. Then by definition we have that $\chi^2_{N(0,1)}(P_v, P_{v'}) + 1$ is

$$\int_{\mathbb{R}^m} \left( \frac{A(x_1)A(x_1 + \sin(\theta)x_2)g(x_2)g(\sin(\theta)x_1 - \cos(\theta)x_2)}{g(x_1)g(x_2)} \right) g(x_3) \cdots g(x_m) dx_1 \cdots dx_m.$$  

Noting that the integral over $x_3, \ldots, x_m$ separates out, we are left with

$$\int_{\mathbb{R}^2} \left( \frac{A(x)A(x + \sin(\theta)y)g(y)g(\sin(\theta)x - \cos(\theta)y)}{g(x)g(y)} \right) dx dy.$$  

Integrating over $y$ gives

$$\int_{\mathbb{R}} \frac{A(x)}{g(x)} \left( \int A(x + \sin(\theta)y)g(y)g(\sin(\theta)x - \cos(\theta)y) dy \right) dx = \int \frac{A(x)U_{\cos(\theta)}A(x)}{g(x)} dx ,$$

where $U_t$ is the Ornstein-Uhlenbeck operator. We will simplify our computations by expressing the various quantities in terms of the eigenbasis for this operator.

In particular, let $h_n(x) = He_n(x)/\sqrt{n!}$ where $He_n(x)$ is the probabilist’s Hermite polynomial. We note the following basic facts about them:

1. $\int_{\mathbb{R}} h_i(x)h_j(x)g(x)dx = \delta_{i,j}.$

2. $U_t(h_n(x)g(x)) = t^n h_n(x)g(x).$

We can now write $A(x)$ in this basis as

$$A(x) = \sum_{n=0}^{\infty} a_n h_n(x)g(x).$$

From this, we obtain that

$$\chi^2(A, N(0, 1)) = \int_{\mathbb{R}} \left( \sum_{n=0}^{\infty} a_n h_n(x)g(x) \right)^2 /g(x)dx$$

$$= \int_{\mathbb{R}} \sum_{n,m=0}^{\infty} a_n a_m h_n(x)h_m(x)g(x)dx$$

$$= \sum_{n=0}^{\infty} a_n^2.$$  

Furthermore, we have that

$$\int_{\mathbb{R}} h_s(x)A(x)dx = \int_{\mathbb{R}} \sum_{n=0}^{\infty} a_n h_s(x)h_n(x)g(x)dx = a_s.$$  

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For \( 1 \leq s \leq k \), we have that
\[
h_s(x) = \sqrt{s!} \sum_{t=0}^{\lfloor s/2 \rfloor} \frac{(-1)^t x^{s-2t}}{2^t t!(n-2t)!}.
\]

We therefore have that
\[
a_s = \sum_{t=0}^{\lfloor s/2 \rfloor} \left( \frac{\sqrt{s!}(-1)^t x^{s-2t}}{2^t t!(s-2t)!} \right) \mathbb{E}[A^{s-2t}].
\]

Note that the above is close to
\[
\sum_{t=0}^{\lfloor s/2 \rfloor} \left( \frac{\sqrt{s!}(-1)^t x^{s-2t}}{2^t t!(s-2t)!} \right) \mathbb{E}[G^{s-2t}] = \mathbb{E}[h_s(G)] = 0.
\]

In particular, the difference between the two quantities is at most
\[
\nu \sum_{t=0}^{\lfloor s/2 \rfloor} \left( \frac{\sqrt{s!}}{2^t t!(s-2t)!} \right).
\]

It is easy to see that the denominator is minimized when \( t = s/2 - O(\sqrt{s}) \). From this it follows that this sum is \( 2^O(s) \nu \). Therefore, we have that \( a_s = 2^O(s) \nu \), for \( 1 \leq s \leq k \). Furthermore, \( a_0 = \int A(x) dx = 1 \). Thus, we have that
\[
\chi_{N(0,1)}(P_v, P_{v'}) + 1 = \int_{\mathbb{R}} \frac{A(x)U_{v,v'}A(x)}{g(x)} dx
\]
\[
= \int_{\mathbb{R}} \left( \sum_{n=0}^{\infty} a_n h_n(x)g(x) \right) \left( \sum_{n'=0}^{\infty} a_n' (v \cdot v')^{n'} h_n'(x)g(x) \right) /g(x)dx
\]
\[
= \int_{\mathbb{R}} \sum_{n,n'=0}^{\infty} a_n a_n' (v \cdot v')^{n'} h_n(x)h_n'(x)g(x)dx
\]
\[
= \sum_{n=0}^{\infty} a_n^2 (v \cdot v')^n
\]
\[
= 1 + \sum_{n=1}^{k} a_n^2 (v \cdot v')^n + \sum_{n=k+1}^{\infty} a_n^2 (v \cdot v')^n.
\]

Therefore,
\[
|\chi_{N(0,1)}(P_v, P_{v'})| \leq O(\nu^2) \sum_{n=1}^{k} 2^{O(n)} |v \cdot v'|^n + |v \cdot v'|^{k+1} \sum_{n=0}^{\infty} a_n^2
\]
\[
\leq \nu^2 + |v \cdot v'|^{k+1} \chi^2(A, N(0,1)).
\]

This completes our proof.

**B Proof of Lemma 2.7**

We consider the Fourier transform of \( G_\delta \). Note that \( G_\delta \) is the pointwise product of \( G \) with a mesh of delta-functions. Therefore, its Fourier transform is the convolution of their Fourier transforms.
The Fourier transform of $G$ is $\sqrt{2\pi}G$. The Fourier transform of the net of delta-functions is a new net with spacing $1/\delta$. Thus, we have that the Fourier transform of $G_\delta$ at $\xi$ is

$$\sum_{n \in \mathbb{Z}} \sqrt{2\pi}g(\xi + n/\delta).$$

The $t^{th}$ moment of a pseudodistribution is proportional to the value of the $t^{th}$ derivative of its Fourier transform at $\xi = 0$. For $G$, this is $\sqrt{2\pi}g^{(t)}(0)$. For $G_\delta$, it is equal to this term plus

$$\sum_{n \in \mathbb{Z}, n \neq 0} \sqrt{2\pi}g^{(t)}(n/\delta).$$

Computing the derivative of $g$ using Cauchy’s integral formula (integrating around a circle of radius $1/(2\delta)$ centered at $n/\delta$), we find that

$$|g^{(t)}(n/\delta)| = t!O(\delta)^t \exp(-\Omega(n/\delta)^2).$$

Taking a sum over $n$ yields our result.