Connected Regions on a Randomly Colored board
Problem

Putnam 2004 A5: An nxm chessboard is colored randomly. Each square is independently assigned black or white with probability 1/2. We say that p and q are in the same monochromatic component if there is a sequence of squares, all the same color, starting at p and ending at q where consecutive square share a common side. Show that the expected number of monochromatic regions is more than mn/8.
How do we count?

• Question asks about global phenomenon (connected regions)
• Probability gives us control over local statements
Single Square Regions

• Consider single square regions

• Each of mn squares has 1/32 probability of being each type.
• Expect to see mn/16.
Two Square Regions

• See each of these with probability 1/256 at each location

• Also these

• Together see about $mn/64$ of these regions.
Other Small Regions

• Up to changing colors and rotations have:

\[ \frac{mn}{128} \text{ of these} \quad \frac{mn}{512} \text{ of these} \]

\[ \frac{mn}{512} \text{ of these} \]
The rest of the Tetris Pieces

\[ \frac{mn}{512} \text{ of these} \]

\[ \frac{mn}{512} \text{ of these} \]

\[ \frac{mn}{2048} \text{ of these} \]

\[ \frac{mn}{2048} \text{ of these} \]
How Many do we Have?

• So far we have accounted for
  $mn\left(\frac{1}{16}+\frac{1}{64}+\frac{1}{128}+\frac{1}{512}+\frac{1}{512}+\frac{1}{512}+\frac{1}{512}+\frac{1}{2048}+\frac{1}{2048}\right)$
  $= mn\left(\frac{97}{1024}\right)$
  regions with at most 4 squares.
• Still need another $mn\left(\frac{31}{1024}\right)$ regions
• Reaching diminishing returns
New Idea

• We want a new way to count regions without having to figure out what the regions actually look like

• Consider the mn squares are starting all separate

• Glue together same colored squares sharing an edge

• Number of regions = number of squares – number of glueings
Disconnected Squares
Glue along edges
Count glueings

16 - 13 = 3... Not quite right.
What’s going on?

These squares are already connected without the last edge.

- Not all glueings connect new regions.
- Number of regions at least \( \text{Number of squares} - \text{Number of Edge Glueings} + \text{Number of gluing cycles} \).
New count

• Number of squares: $mn$

• Number of edge glueings:
  $(\approx 2mn \text{ Edges}) \times (1/2 \text{ probability})$
  $\approx mn$

• Number of gluing cycles:
  $(\approx mn \text{ 2x2 squares}) \times (1/8 \text{ probability all the same color})$
  $\approx mn/8$

• Total $\approx mn/8$
Euler characteristic

• To analyze carefully, define Euler characteristic of a region:

\[ \chi := \# \text{ Squares} - \# \text{ Edges} + \# \text{ Vertices} \]

Lemma: The Euler characteristic is equal to the number of connected components minus the number of holes.
Proof idea

• Start with empty region (everything is 0 so OK)
• Add squares one at a time, right to left, bottom to top

• See how it changes Euler characteristic and number of regions, holes
One technical point

We are not going to count “kitty corner” connections. They will count as two separate corners.
Cases

• New square
  – 1 new face, 4 new edges, 4 new vertices
  – 1 new region

• Connects on one side
  – 1 new face, 3 new edges, 2 new vertices
  – No new regions/holes

• Connects on both sides
  – 1 new face, 2 new edges, 1 new vertex
  – No new regions/holes
Final case

• Connects on right and bottom to squares without fourth corner square
  – 1 new face, 2 new edges, 0 new vertices (merged kitty corners)
• If other squares connected, create hole
• If not connected, remove region
Final count

• Compute the expected sum of Euler characteristics of all monochromatic regions
• Total number of faces is mn
• Edges:
  
  1  2  1

  – 2mn locations
  – 50% prob. 1 edge, 50% prob. 2 edges
  – Expect 3mn total edges
Number of vertices

- \( mn \) vertices
- \( \frac{1}{8} \) have 1 copy
- \( \frac{3}{4} \) have 2 copies
- \( \frac{1}{8} \) have 4 copies
- Total: \( (2 + \frac{1}{8})mn \) vertices
Sum of Euler characteristics

- Expected sum of Euler characteristics
  Total faces – Total edges + Total vertices
  \[ \approx mn - 3mn + (2+1/8) mn \]
  \[ = mn/8 \]
  \[ = \text{Number of Regions} - \text{Number of holes} \]
  \[ \leq \text{Number of Regions} \]
Tightness

- This is actually an underestimate for the number of regions
- Expected number of regions
  \[ \approx \frac{mn}{8} + \text{Expected number of holes} \]
- Expect about \( \frac{mn}{256} \) holes of this kind
Summary

• Euler characteristic is a good counting tool
• Allows us to prove slightly stronger bound than required