CSE 203A Homework 3

Fall 2017

This homework is due in class Monday December 4th at the start of class. If you will miss class for some reason, either get another student to turn your paper in for you or arrange some other method to get it to me on time.

Please make sure that your homework is:

• Labelled with your name.
• Either clearly handwritten or typed.
• Stapled together and with page numbers if more than one page long.

You are encouraged to discuss these problems with fellow students, although I expect that your writeups should be done individually. Intentionally searching for problem solutions online will be considered cheating.

Make sure to justify all of your answers with a mathematical proof. For algorithms you should both prove correctness (with appropriately small probability of error) as well as appropriate bound on runtime.

Question 1 (Cover Times of Regular Graphs, 25 points).
(a) Let G be a connected d-regular undirected graph on n vertices. Consider the random walk on G. Show that the commute time $C_{v,w}$ between any pair of vertices $v$ and $w$ is $O(n^2)$. Hint: First bound the diameter of G. [15 points]
(b) Show that the cover time of G starting from any vertex is $O(n^2 \log(n))$. [10 points]

Question 2 (Counting and Sampling, 25 points).
Let $S$ be a subset of \{0,1\}^n (think of the set of things that we would want to count in some # P problem). Let a subcube $C$ be a subset of \{0,1\}^n obtained by fixing some of the $n$ coordinates and letting the remaining coordinates by any combination of 0s and 1s. We consider the problems of counting elements from $S$ subject to restrictions and of sampling.

(a) Show that if you have a polynomial time algorithm that can count the number of points in $S \cap C$ for any subcube $C$, you can produce a polynomial time algorithm for sampling uniformly from $S$. [10 points]
(b) Show that if you have a polynomial time algorithm for sampling uniformly from $S \cap C$ for any subcube $C$ for which the intersection is non-trivial, that you have a randomized FTPAS for approximating the number of points in $S$. [15 points]

Question 3 (Triangle Coloring, 25 points).
Let $G$ be a 3-colorable graph. Give a polynomial time randomized algorithm for coloring the vertices with two colors so that there is no monochromatic triangle. Hint: Consider a fixed 3-coloring $C$ of $G$. Perform a random walk on 2-colorings of $G$ and keep track of the number of vertices whose colors agree with their coloring under $C$ (noting that one of the three colors in $C$ will be impossible to match).

Question 4 (High Degree Regular Graphs and Expanders, 25 points).
Consider $G$ a connected, d-regular graph on $n$ vertices.

(a) Show that if $d > 2n/3$ that $h(G) = \Omega(n)$. [10 points]
(b) Give an example of a graph with $d = n/2 - 1$ where $h(G) = O(1/n)$. [10 points]
(c) What can you say about the cover times of these graphs? [5 points]