CSE 101 Homework 2

Winter 2015

This homework is due Friday January 30th at the start of class. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \\LaTeX is recommend though not required.

**Question 1** (Differing Priority Queue Implementations for Dijkstra, 20 points). Consider running Dijkstra’s algorithm on the following graphs, implementing the priority queue as either an array or as a binary heap. Which one is more efficient, and what is the final runtime?

(a) A graph where \( V \) is a \( \sqrt{n} \times \sqrt{n} \) grid and so that there are edges between any vertex and its four neighbors. [5 points]

(b) A complete graph on \( n \) vertices (there is an edge between every pair of vertices). [5 points]

(c) A graph where \( V \) is a \( \sqrt{n} \times \sqrt{n} \) grid and there are edges between any pair of vertices in the same row or column. For this part, consider implementation of a priority queue using a \( d \)-ary heap for various values of \( d \). Which value of \( d \) gives the best runtime, and what is that runtime? [10 points]

**Question 2** (Shortest Paths to Nearby Vertices, 30 points). Find an algorithm to do the following: Given a graph \( G \) with positive integer edge weights, a vertex \( s \) and a positive integer \( L \) find the set of vertices within distance \( L \) of \( s \). Your algorithm should run in time \( O(|V| + |E| + L) \). Hint: Modify Dijkstra’s algorithm to use an array whose \( i^{th} \) entry holds a list of all vertices at distance \( i \) for each \( 0 \leq i \leq L \). [Algorithm 10 points, Analysis 10 points, Runtime analysis 10 points]

**Question 3** (Commodity Trading, 50 points). Charlene is a commodities trader. She trades in \( n \) different types of goods. She knows \( m \) other merchants, each of which are willing to trade one specific good for one other good at a specified exchange rate. Say that the \( i^{th} \) merchant is willing to trade one unit of good \( g_i \) for \( r_i \) units of good \( g'_i \).

(a) Given two specific goods, find an efficient algorithm by which Charlene can find a sequence of trades to exchange the first type for the second at the most favorable possible rate. Hint: Adapt Bellman-Ford. [Algorithm 5 points, Analysis 5 points]

(b) In some circumstances, it might be possible for Charlene to make some sequence of trades and eventually end up with strictly more of a good than she started with. Give an algorithm to determine whether or not this is possible. [Algorithm 5 points, Analysis 5 points]

(c) One way in which you might be able to show that the above is impossible is if there is a way to assign prices to every good in such a way that no merchant allows you to trade some collection of goods for a more valuable collection. Find a mathematical formulation of this condition [5 points] and show that it implies that it rules out sequences of trades that would allow Charlene to end up with more than she started with. [10 points]

(d) In fact, if the situation discussed in part (b) is impossible, it is always possible to assign prices as described in part (c). Show that this is the case. You may assume that given any two goods, there is some sequence of trades that allows you to exchange one for the other. Hint: pick some particular good \( g \) and find the best ways of exchanging it for each other type of good. Set prices so that all of these trades are cost-neutral. [15 points]

**Question 4** (Extra credit, 1 point). Approximately how much time did you spend working on this homework?