This homework is due Friday April 22nd, 11pm on gradescope. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommended though not required.

**Question 1** (A*-algorithm, 40 points). Consider the graph $G$ given my an $N \times N$ grid. Namely the vertices are given by pairs $(x, y)$ with $x, y \in \{1, 2, \ldots, N\}$ and edges between pairs with $|x - x'| + |y - y'| = 1$ (all edges are length 1). Suppose that you are given two vertices $s$ and $t$ in $G$ corresponding to pairs $(x_s, y_s), (x_t, y_t)$ with $x_s - x_t = n, y_s - y_t = m$.

(a) Suppose that you run Dijkstra’s algorithm using a binary heap to compute the distance between $s$ and $t$. How long will the algorithm take asymptotically? [10 points]

(b) Modify Dijkstra’s algorithm so that it stops once it has found the distance to $t$ and does not deal with vertices too far from $s$. The runtime should now be $O((|n| + |m|)^2 \log(|n| + |m|))$ (prove this). Notice that if $|n|, |m|$ are much smaller than $N$, that this is much faster than before. This is because we do not need to analyze roads in China in order to find directions to the corner grocery store. [10 points]

(c) Suppose that you know a function $h$ so that for all edges $(u, v)$ you have $\ell(u, v) - h(u) + h(v) \geq 0$. Show that by running our modified Dijkstra’s algorithm using $\ell'$, we can compute the length of the shortest path from $s$ to $t$. Hint: relate the length of a path under $\ell'$ to the length of the same path under $\ell$. [5 points]

(d) Show that for our graph $G$, the function $h(u) = |x_u - x_t| + |y_u - y_t|$ satisfies the above property (this is the same $t$ from the problem statement). [5 points]

(e) Show that using this new algorithm, we can find a shortest $s-t$ path in time $O((|n| + 1)(|m| + 1) \log(|n| |m| + 2))$. Notice that this is much smaller if either $n$ or $m$ is close to 0. [10 points]

**Question 2** (Tramp Steamer Problem, 30 points). Problem 4.22 in the textbook.

(a) Part (a) [10 points]

(b) Part (b) [10 points]

(c) Part (c) [10 points]

**Question 3** (Combining Dijkstra and Shortest Paths in DAGs, 30 points). Let $G$ be a directed graph whose edges are assigned (not necessarily positive) edge weights. Suppose that there none of the edges of $G$ within a single strongly connected component have negative weight. In particular, all edges with negative weight must be between distinct components. Let $s$ be some vertex in $G$.

(a) Show that $G$ has no negative weight cycles. [10 points]

(b) Suppose that for some strongly connected component $C$, you already know the lengths of the shortest paths from $s$ to all vertices of other components which have edges into $C$. Devise a nearly linear time algorithm to compute the lengths of the shortest paths from $s$ to all vertices in $C$. [10 points]

(c) Devise a nearly linear time algorithm to compute the lengths of the shortest paths from $s$ to all other vertices in $G$. [10 points]

**Question 4** (Extra credit, 1 point). Approximately how much time did you spend working on this homework?