The Number of Ways of Expressing \( t \) as a Binomial Coefficient

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An Interesting Note:

10 is both a triangular number and a tetrahedral number.
Why does this hold?

\[ 10 = \binom{5}{2} = \binom{5}{3} \]

Is it equal to any other binomial coefficient?

\[ = \binom{10}{1} = \binom{10}{9} \]
Are there any non-trivial relations?

\[ 3003 = \binom{78}{2} = \binom{15}{5} = \binom{14}{6} \]

Trying to produce solutions of the form

\[ \binom{n-1}{m} = \binom{n}{m-1} \]

Yields a Pell Equation with Solutions

\[ n = F_{2k} F_{2k+1} \]

\[ m = F_{2k-1} F_{2k} \]
Stating the Problem More Precisely

Define $N(t) = \left| \left\{ (n, m) \in \mathbb{Z}^2 : \binom{n}{m} = t \right\} \right|$.

• Studied by Singmaster in [2] (1971)
• $N(3003) = 8$
• $N(t) \geq 6$ for infinitely many $t$
• \underline{Conj}: $N(t) = O(1)$
What are Reasonable Bounds?

• Singmaster showed in [2] that

\[ N(t) = O(\log(t)) \]

• Consider \( n > 2m \) (we do this from now on)

\[
 t = \binom{n}{m} \geq \binom{2m}{m} \geq 2^m 
\]

\[ \implies m = O(\log t) \]
An improvement

• In [3] Erdös et al. proved that

\[ N(t) = O \left( \frac{\log(t)}{\log \log(t)} \right) \]

• Prime in \((n - n^{5/8}, n)\) for \(n \gg 1\).
• Split into cases based on \(n > (\log t)^{6/5}\)
For $n > (\log t)^{6/5}$

- Use Approximation $\binom{n}{m} \geq \left( \frac{n}{m} \right)^m$

$$t = \binom{n}{m} \geq \left( \frac{(\log t)^{6/5}}{m} \right)^m \geq (\log t)^{m/5}$$

$$\implies m = O \left( \frac{\log t}{\log \log t} \right)$$
For $n < (\log t)^{6/5}$

- Approximation yields $m > n^{5/8}$
- $\exists$ prime, $P$, s.t. $n-m < P < n$
- $P$ divides $t$
- Pick largest $N$, all others satisfy $P < n < N$
- At most $M$ solutions
- $M = O(N^{5/8}) = O((\log t)^{3/4})$
- Using the strongest conjectures on gaps between primes could give

$$N(t) = O \left((\log t)^{2/3+\epsilon}\right)$$
Another way to handle this case

- Consider all solutions \( t = \binom{n_1}{m_1} = \ldots = \binom{n_k}{m_k} \)
- Order so \( n_1 > n_2 > \ldots > n_k \)
  \( m_1 < m_2 < \ldots < m_k \),
- As \( n \) decreases and \( m \) increases, the effect of changing \( n \) decreases and the effect of changing \( m \) increases.
- \( (n_i-n_{i+1})/(m_{i+1}-m_i) \) increasing.
- These fractions are distinct
Another way (continued)

• Differences of \((n_i, m_i)\) are distinct
• Only \(O(s^2)\) have \(\Delta n, \Delta m < s\)
• Total Change in \(n-m > ck^{3/2}\)
• \(k^{3/2} = O((\log t)^{6/5})\)
• \(k = O((\log t)^{4/5})\).
What did we do?

- n convex function of m
- Lattice points on graph of ANY convex function
- Idea: Consider higher order derivatives.
Problems to Overcome

• How do we use the derivative data?
• How do we obtain the derivative data?
Lemma If $f$ and $g$ are $C^n$-functions so that $f(x) = g(x)$ at $x_1 < x_2 < \ldots < x_{n+1}$, then $f^{(n)}(y) = g^{(n)}(y)$ for some $y \in (x_1, x_{n+1})$

• Generalization of Rolle’s Theorem
• Proof by induction & Rolle’s Theorem
Using Data, Proof of Lemma

\[ f^{(n)}(x) = g^{(n)}(x) \]

\[ f^{(2)}(x) = g^{(2)}(x) \]

\[ f^{(1)}(x) = g^{(1)}(x) \]

\[ f^{(0)}(x) = g^{(0)}(x) \]
Using Derivative Data (Continued)

• Consider function \( f \), \( f(m_i) = n_i \)
• \( g \) polynomial interpolation of \( f \) at \( m_1, m_2, \ldots, m_{k+1} \)
• \( g^{(k)} \) constant
• \( f^{(k)} \) is this constant at some point
Using Derivative Data (cont.)

\[ g(x) = \sum_i n_i \frac{\prod_{j \neq i} (x - m_j)}{\prod_{j \neq i} (m_i - m_j)} \]

\[ \frac{1}{k!} \frac{\partial^k}{\partial x^k} g(x) = \frac{A}{B} \]

\[ B = \text{LCM}_i \prod_{i \neq i} (m_i - m_j) \]

\[ \left| \frac{1}{k!} \frac{\partial^k}{\partial x^k} g(x) \right| \text{ is more than } \frac{1}{B}, \text{ Or equals 0.} \]
Using Derivative Data (cont.)

- $k^{th}$ derivate of $f$ small but non-zero
- Fit polynomial to $k+1$ points separated by $S$
- $\Rightarrow k^{th}$ derivative over $k!$ either 0 or more than $S^{-k(k+1)/2}$. 
Derivative Data

- Define \( \left( \frac{f(x)}{x} \right) = t \)
- Make smooth using \( \Gamma \)
- Estimate with Sterling’s approximation

\[
f(z) = \exp\left( \frac{\log t + \log \Gamma(z + 1)}{z} \right) + \frac{z - 1}{2} + O\left( \frac{z^2}{f(z)} \right)
\]

uniformly when \( f(z) > |2z| \),

which holds when \( \exp\left( \frac{\log t + \log \Gamma(z + 1)}{z} \right) > |6z| \).
Estimating Derivatives

• Main Term: Derivatives of
  \[ \left[ \log t + \Gamma(z+1) \right] / z \] and take exp of power series
• \((z - 1) / 2\) is easy
• Error term: Cauchy Integral Formula

If \( f(x)^{7/4} > x^2 3^k + 1 k! \),

\[
0 < \left| \frac{1}{k!} \frac{d^k}{dx^k} f(x) \right| < 2 f(x) e^{2 \log f(x) / \log x} x^{-k} (\log x)^k
\]
A Useful Parameter

Define \( f(x) = x^\alpha(x) \).

If \( \alpha > 1.15 \), \( \alpha \sim \frac{\log t}{x \log x} + 1 \).

If \( x^{(7\alpha/4) - 2} > 3^{k+1} k! \),

\[
0 < \left| \frac{1}{k!} \frac{d^k}{dx^k} f(x) \right| < 2x^{\alpha-k} e^{2\alpha} (\log x)^k.
\]
Split into Cases

1. \( \alpha < 1.15 \)
2. \( 1.15 < \alpha < \log \log t/(24 \log \log \log t) \)
3. \( \log \log t/(24 \log \log \log t) < \alpha < (\log \log t)^4 \)
4. \( (\log \log t)^4 < \alpha \)
Case 1: \( \alpha < 1.15 \)

- Already covered
- \( O((\log t)^{3/4}) \) solutions
Case 2: $1.15 < \alpha < \log \log t/(24 \log \log \log t)$

- Set $k = (\log \log t)/(12 \log \log \log t)$
- Technical conditions satisfied
- $k+1$ adjacent solutions $m_i$ of separation $S$

\[
S - k(k+1)/2 < 2x^{k-\alpha}e^{2\alpha(\log x)^k}
\]

\[
S = \exp \left( \Omega \left( \frac{\log x}{k} \right) \right) = \Omega \left( (\log \log t)^4 \right)
\]

\[
O(k \log t/S + k) = O \left( \frac{(\log t)(\log \log \log t)}{(\log \log t)^3} \right)
\]
Case 3: \( \log \log t/(24 \log \log \log t) < \alpha < (\log \log t)^4 \)

We need a slightly better analysis to bound

\[
B = \text{LCM}_i \prod_{j \neq i} (m_i - m_j)
\]

We use \( \text{LCM} \prod_{i=1}^{k} r_i \)

LCM over all sequences of \( k \) distinct \( r_i \neq 0 \),

\( |r_i| < S \)
Bounding B

- Count multiples of each prime
- \( p^n \) divides at most \( \min(k, 2S/p^n) \) \( r_i \)'s
- Use Prime Number Theorem

\[
\log B \leq \sum_{p^n} \log p \min \left( k, \left\lfloor \frac{2S}{p^n} \right\rfloor \right) \\
= k \sum_{p^n < 2S/k} \log p + \sum_{2S/k < p^n < 2S} 2S \left( \frac{\log p}{p} \right) \\
< 3S \log k
\]
Using Bound

• $k = 2\alpha$
• Technical Conditions Satisfied

$$e^{-3S \log k} < 2x^{\alpha-k} e^{2\alpha} (\log x)^k$$

$$\frac{S}{k} = \Omega \left( \frac{\log x}{\log \alpha} \right) = \Omega \left( \frac{\log \log t}{\log \log \log \log t} \right).$$

$$\left( \frac{k}{S} \right) O \left( \frac{(\log t)(\log \log \log t)}{(\log \log t)^2} \right) = O \left( \frac{(\log t)(\log \log \log t)^2}{(\log \log t)^3} \right)$$
Case 4: \((\log \log t)^4 < \alpha\)

\[ m = O \left( \frac{\log t}{\alpha} \right) \]

\[ O \left( \frac{(\log t)}{(\log \log \log t)^4} \right) \] solutions.
Conclusions

\[ N(t) = O \left( \frac{(\log t)(\log \log \log \log t)^2}{(\log \log t)^3} \right) \]

- Know where to look to tighten this bound
- Can use technique for other problems

\[ \limsup_{t \to \infty} \left| \{ (n, m) \in \mathbb{Z}^2 : n!m! = t \} \right| = 6 \]
References

