AN ELEMENTARY DERIVATION OF THE ASYMPTOTICS OF PARTITION FUNCTIONS

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Abstract. Let $S_{a,b} = \{an + b : n \geq 0\}$ where n is an integer. Let $P_{a,b}(n)$ denote the number of partitions of n into elements of $S_{a,b}$. In particular, we have the generating function,

$$\sum_{n=0}^{\infty} P_{a,b}(n) q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^{an+b})}.$$

We obtain asymptotic results for $P_{a,b}(n)$ when $\gcd(a, b) = 1$. Our methods depend on the combinatorial properties of generating functions, asymptotic approximations such as Stirling's formula, and an in depth analysis of the number of lattice points inside certain simplicies.

1. Introduction and Statement of Results.

For positive integers a and b, let $S_{a,b} = \{an+b : n \geq 0\}$ where n is an integer be the set of natural numbers, at least b, that are congruent to $a \mod b$. Let $P_{a,b}(n)$ denote the number of partitions of n into elements of $S_{a,b}$. In particular, we have the generating function,

$$\sum_{n=0}^{\infty} P_{a,b}(n) q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^{an+b})}. \tag{1.1}$$

A famous theorem of Hardy and Ramanujan is that when $a = b = 1$

$$P_{1,1}(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}}$$

as $n \to \infty$. Their proof (which marks the birth of the circle method) depends on properties of modular forms. An asymptotic formula for $P_{a,b}(n)$ for all pairs of a, b relatively prime was first attained by Ingham (see [1]) and his proof was later refined by Meinardus in [3] and [4]. The correct expression is:

$$P_{a,b}(n) \sim \Gamma\left(\frac{b}{a}\right) \pi^{b/a-1} 2^{-(3/2)-(b/2a)} 3^{-(b/2a)} a^{-(1/2)+(b/2a)} n^{-\frac{a+b}{2a}} \exp\left(\pi \sqrt{\frac{2n}{3a}}\right).$$

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Later Erdös was able to study the asymptotics using elementary methods involving recursive formulas for them. In particular, he showed in [5] that for some constant c,

$$P_{1,1}(n) \sim \frac{c}{n} e^{\pi\sqrt{2n/3}}.$$

This proof was later refined by Don Newman in [2] to obtain to full asymptotics of $P_{1,1}(n)$.

Using different methods we obtain similar results for $P_{a,b}(n)$ to within a constant multiple when $\gcd(a, b) = 1$. Our methods depend on the combinatorial properties of generating functions, asymptotic approximations such as Stirling's formula, and an in depth analysis of the number of lattice points inside certain simplicies.

In particular, we shall prove the following theorem;

Theorem 1. *If a and b are natural numbers and $\gcd(a, b) = 1$, then there are constants $0 < C^-_{a,b} < C^+_{a,b}$ such that for all sufficiently large n*

$$\frac{C^-_{a,b}}{n^{\frac{a+b}{2a}}} e^{\pi\sqrt{\frac{2n}{3a}}} < P_{a,b}(n) < \frac{C^+_{a,b}}{n^{\frac{a+b}{2a}}} e^{\pi\sqrt{\frac{2n}{3a}}}.$$

In fact, we shall compute such constants.

2. Definitions and Preliminary Lemma

We will use the following definitions. For convenience, we let

$$\sum_{\{a_n\}\{b_n\}\ldots\{z_n\}}^{C} F(\{a_n\}\{b_n\}\ldots\{z_n\}) \tag{2.1}$$

denote the sum of F over sequences of natural numbers $\{a_i\}_{1\le i\le n}, \{b_i\}_{1\le i\le n}, \ldots, \{z_i\}_{1\le i\le n}$ so that condition C is satisfied.

For series $f(q)$ and $g(q)$ let

$$f(q) \le_q g(q) \tag{2.2}$$

mean that the coefficients of the Maclaurin series in q of $g(q)$ are greater than or equal to the corresponding coefficients of the series for $f(q)$. Define $f(q) \sim_q g(q)$,$f(q) <_q g(q)$, etc. similarly. Furthermore, let

$$\sigma_{a,b}(k) = \sum_{m(an+b)=k} \frac{1}{m}, \tag{2.3}$$

where the sum is over non-negative integers n and natural numbers m so that $m(an + b) = k$. Let $\mathbb{N}_0$ denote the set of non-negative integers. Let $\gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log(n)\right) = .577\ldots$ be Euler's constant. Let $n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$. Recall Stirling's formula, which states that

$$n! \sim n^n e^{-n} \sqrt{2\pi n}. \tag{2.4}$$

Lemma 2.1. *Suppose that $a(n,k) : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}^+$ and $b(n,k) : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}^+$ are maps for which the following are true:*

(1) $\sum_{n=0}^{\infty} a(n,k)$ *and* $\sum_{n=0}^{\infty} b(n,k)$ *converge for all* $k \in \mathbb{R}$,
(2) $\left|\frac{a(n,k)}{b(n,k)} - 1\right| < C(n)$ *where* $\lim_{n\to\infty} C(n) = 0$,
(3) $\lim_{k\to\infty} \frac{a(n,k)}{a(n+1,k)} = 0$ *and* $\lim_{k\to\infty} \frac{b(n,k)}{b(n+1,k)} = 0$.

Then we have

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} a(n,k)}{\sum_{n=0}^{\infty} b(n,k)} = 1.$$

Proof of Lemma 2.1. For any $1 \geq \epsilon > 0$ find N so that $n > N$ implies $C(n) < \frac{\epsilon}{4}$. Find N_a so that for $n < N$ and $k > N_a$, $\frac{a(n,k)}{a(n+1,k)} < \frac{\epsilon}{4+\epsilon}$. For $k > N_a$, $\sum_{n=0}^{N-1} a(n,k) \leq a(N,k)\frac{\epsilon}{4} \leq \sum_{n=0}^{\infty} a(n,k)\frac{\epsilon}{4}$. Find N_b similarly. Now, for $k > \max(N_a, N_b)$

$$\frac{\sum_{n=0}^{\infty} a(n,k)}{\sum_{n=0}^{\infty} b(n,k)} = \frac{\sum_{n=0}^{\infty} a(n,k)}{\sum_{n=N}^{\infty} a(n,k)} \cdot \frac{\sum_{n=N}^{\infty} a(n,k)}{\sum_{n=N}^{\infty} b(n,k)} \cdot \frac{\sum_{n=N}^{\infty} b(n,k)}{\sum_{n=0}^{\infty} b(n,k)}.$$

By looking at the absolute values of the logs of these fractions, we find that the difference between this and 1 is at most

$$\left|\left(1 \pm \frac{\epsilon}{4}\right)^3 - 1\right| < \epsilon.$$

This proves the lemma.

□

This result will be used later in sections 6 through 9.

3. The Strategy

In order to prove Theorem 1, we shall work to prove the following theorem,

Theorem 2. *If a and b are natural numbers and if $a \leq b$, then there exist constants $0 < C_{a,b}^- < C_{a,b}^+$ so that for all sufficiently large n we have*

$$\frac{C_{a,b}^-}{n^{\frac{b}{2a}}} e^{\pi\sqrt{\frac{2n}{3a}}} < \sum_{k=1}^{n} P_{a,b}(k) < \frac{C_{a,b}^+}{n^{\frac{b}{2a}}} e^{\pi\sqrt{\frac{2n}{3a}}}.$$

To prove Theorem 2, we begin with some important observations. Recall that,

(by (1.1))
$$\sum_{n=0}^{\infty} P_{a,b}(n) q^n = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{an+b})}.$$

By taking the log of both sides we find that

$$\begin{aligned}\log\left(\sum_{n=0}^{\infty} P_{a,b}(n)q^n\right) &= \log\left(\prod_{n=0}^{\infty}\frac{1}{(1-q^{an+b})}\right)\\ &= \sum_{n=0}^{\infty}\log\left(\frac{1}{(1-q^{an+b})}\right)\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q^{m(an+b)}}{m}\\ \text{(by (2.3))} \qquad &= \sum_{n=0}^{\infty}\sigma_{a,b}(n)q^n.\end{aligned}$$

Exponentiating both sides of this and using the series expansion for e^x, we find that

$$\begin{aligned}\sum_{k=0}^{\infty} P_{a,b}(k)q^k &= e^{(\sum_{k=0}^{\infty}\sigma_{a,b}(k)q^k)}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sigma_{a,b}(k)q^k\right)^n\frac{1}{n!}.\end{aligned}$$

Equating coefficients, we find that,

$$P_{a,b}(k) = \sum_{n=0}^{\infty}\left(\sum_{\{x_n\}}^{\sum_{i=1}^n x_i=k}\prod_{i=1}^{n}(\sigma_{a,b}(x_i))\frac{1}{n!}\right).$$

Summing over k, the previous expression implies

$$\begin{aligned}\sum_{m=0}^{k} P_{a,b}(m) &= \sum_{n=0}^{\infty}\left(\sum_{\{x_n\}}^{\sum_{i=1}^n x_i\le k}\prod_{i=1}^{n}(\sigma_{a,b}(x_i))\frac{1}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}\{\beta_n\}}^{\sum_{i=1}^n \alpha_i(a\beta_i+b)\le k}\left(\prod_{i=1}^{n}\left(\frac{1}{\alpha_i}\right)\cdot\frac{1}{n!}\right)\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{\sum_{i=1}^n \alpha_i b\le k}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right)\frac{1}{n!}\cdot\left|\{(\beta_1,\beta_2,...,\beta_n)\in(\mathbb{N}_0)^n:\sum_{i=1}^{n}\alpha_i(a\beta_i+b)\le k\}\right|\right). \qquad (3.1)\end{aligned}$$

In the next few sections we shall study the last term of this expression,

$$\begin{aligned}&\left|\{(\beta_1,\beta_2,...,\beta_n)\in\mathbb{N}_0^n:\sum_{i=1}^{n}\alpha_i(a\beta_i+b)\le k\}\right| =\\ &\left|\{(\beta_1,\beta_2,...,\beta_n)\in\left(\mathbb{N}_0+\frac{1}{2}\right)^n:\sum_{i=1}^{n}(\alpha_i\beta_i)\le\frac{k-(b-\frac{a}{2})\sum_{i=1}^n\alpha_i}{a}\}\right|. \qquad (3.2)\end{aligned}$$

This is the number of half-integer lattice points within the n-dimensional simplex defined by the equations $x_i \ge 0$ $(1 \le i \le n)$ and $\sum_{i=1}^n \alpha_i x_i \le \frac{k-(b-\frac{a}{2})\sum_{i=1}^n\alpha_i}{a}$. In sections 4 and 5 we will compute upper and lower bounds respectively for (3.2). In section 6 we will incorporate this into (3.1) to remove the β dependence. In sections 7 and 8 we will calculate upper and lower bounds respectively on this new expression, removing the α dependence. In section 9 we shall come up with asymptotic bounds for the expressions that came out of sections 7 and 8, proving Theorem 2 in the process. Finally, in section 10 we will use Theorem 2 to prove Theorem 1.

4. Upper Bounds For (3.2)

In this section we derive upper bounds for the number of half-integer lattice points within certain right simplicies and hence obtain upper bounds for (3.2).

Lemma 4.1. *For any sequence of positive real numbers* $\{S_i\}_{1\leq i\leq n}$ *if* $1 \leq m \leq n$, *then*

$$\left|\{(x_1,...,x_n) \in \left(\mathbb{N}_0 + \frac{1}{2}\right)^n : \sum_{i=1}^{n} \frac{x_i}{S_i} \leq 1\}\right| \leq \left(\frac{\prod_{i=1}^{n} S_i}{n!}\right)\left(1 + \frac{n}{S_m}\right).$$

We note that this is equivalent to the statement that the number of half-integer lattice points inside the right simplex with legs parallel to the axis, meeting at the origin, and of lengths S_i is at most the volume of the simplex plus the area of any one of the faces.

Proof of Lemma 4.1. Let S be such a simplex. Let the plane $x_m = 0$ be N. Let the face of S defined by $S \cap N$ be F. Define a lattice prism to be the set of points defined by $y_i \leq x_i \leq y_i + 1$ for $1 \leq i \leq n$ and $i \neq m$ for some integers y_i. Let the center of such a lattice prism be the point $(y_1 + \frac{1}{2}, ..., y_{m-1} + \frac{1}{2}, 0, y_{m+1} + \frac{1}{2}, ..., y_n + \frac{1}{2})$. For a lattice prism L, with center C define $P(L)$ to be the number of half-integer lattice points in $S \cap L$. Define $V(L)$ to be the volume of $S \cap L$. Let the area of $F \cap L$ be $A(L)$. We wish to show that

$$P(L) \leq A(L) + V(L). \tag{4.1}$$

If $P(L) = 0$, then the result follows trivially because $P(L) = 0 \leq A(L) + V(L)$. Otherwise, $P(L) \geq 1$ so $C + \frac{1}{2}e_m \in S$, where e_m denotes the unit vector in the m direction. This implies that $C \in S$. When a hypercube is cut by a plane, the half containing the center has the larger volume (since it and its reflection about the center cover the cube). Therefore, $A(L) \geq \frac{1}{2}$. The height of S above the N is some linear function, M, of the position in the N plane. Since all lattice points in $S \cap L$ are in the same column, $P(L)$ is the greatest integer k so that $C + e_m(k - \frac{1}{2}) \in S$. Therefore, $P(L)$ is at most the height of S above N at C plus one half, or $P(L) \leq M(C) + \frac{1}{2}$. Now, $V(L) = \int_{F\cap L} M(x)dx$. Since M is negative outside of F, $V(L) \geq \int_{L\cap N} M(x)dx$. Since M is linear, this equals $M(C)$. So, $V(L) \geq M(C)$. This means that

$$P(L) \leq M(C) + \frac{1}{2} \leq V(L) + A(L),$$

proving (4.1).

By summing (4.1) over all lattice prisms, we get that

$$\sum_L P(L) \leq \sum_L V(L) + \sum_L A(L).$$

Each term in the first sum is the number of half-integer lattice points in $S \cap L$ for any lattice prism L. Since the lattice prisms tessellate space and do not overlap over any half-integer lattice point, this is merely $\left|\{(x_1, ..., x_n) \in \left(\mathbb{N}_0 + \frac{1}{2}\right)^n : (x_1, ..., x_n) \in S\}\right|$, which equals

$$\left|\{(x_1, ..., x_n) \in \left(\mathbb{N}_0 + \frac{1}{2}\right)^n : \sum_{i=1}^{n} \frac{x_i}{S_i} \leq 1\}\right|.$$

The second term is the sum of the volumes of the intersections of S with all lattice prisms. Since lattice prisms tessellate space and overlap only on surfaces of no volume, this is the volume of S, which is $\left(\frac{\prod_{i=1}^n S_i}{n!}\right)$. The last term is the sums of the areas of the intersections of F with lattice prisms. Again since lattice prisms tessellate space and since the intersection of their overlaps with F has smaller dimension, this is just the area of F, which is $\left(\frac{\prod_{1\leq i\leq n}^{i\neq m} S_i}{(n-1)!}\right)$. Substituting in these values, we get that

$$\begin{aligned}\left|\{(x_1, ..., x_n) \in \left(\mathbb{N}_0 + \frac{1}{2}\right)^n : \sum_{i=1}^{n} \frac{x_i}{S_i} \leq 1\}\right| &\leq \left(\frac{\prod_{i=1}^n S_i}{n!}\right) + \left(\frac{\prod_{1\leq i\leq n}^{i\neq m} S_i}{(n-1)!}\right) \\ &= \left(\frac{\prod_{i=1}^n S_i}{n!}\right)\left(1 + \frac{n}{S_m}\right).\end{aligned}$$

□

5. Lower Bounds For (3.2)

In this section, we derive lower bounds on the number of half-integer lattice points within a simplex, and hence lower bounds on (3.2). We shall prove

Lemma 5.1. *For positive real numbers S_i, where $1 \leq i \leq n$, we have*

$$\left|\{(x_1, ..., x_n) \in \mathbb{N}_0^2 \times \left(\mathbb{N}_0 + \frac{1}{2}\right)^{n-2} : \sum_{i=1}^{n} \frac{x_i}{S_i} \leq 1\}\right| \geq \left(\frac{\prod_{i=1}^n S_i}{n!}\right)\left(1 - \frac{n^2}{12}\sum_{i=1}^{n} \frac{1}{S_i^2}\right).$$

This is to say that the number of half-integer (except for the first two coordinates) lattice points within a simplex with sides along the axis that meet at the origin of length S_i , is at least the volume of the simplex times some correction term. Call such points l-points.

Proof of Lemma 5.1. First note that in two dimensions, the number of lattice points inside the triangle with vertices $(0,0)$, $(x,0)$, and $(0,y)$ is at least $\frac{x\cdot y}{2}$. This is true because all the unit squares to the upper left of such lattice points, cover the triangle.

Let $K = \frac{S_1 S_2}{2}$. Let S be such a simplex. Let N be the hyperplane defined by $x_1 = x_2 = 0$. Let F be the $(n-2)$-dimensional face of S defined by $S \cap N$. Let a lattice prism be the set of points defined by $y_i \leq x_i \leq y_i + 1$ $(3 \leq i \leq n)$ for some integers y_i. Let the center of such a lattice prism be $(0, 0, y_3 + \frac{1}{2}, y_4 + \frac{1}{2}, ..., y_n + \frac{1}{2})$. For a lattice prism L with center C, let $P(L)$ be the number of l-points in $L \cap S$. Let $V(L)$ be the volume of $L \cap S$. Let $A(L)$ be the area of $L \cap F$. Let $E(L) = \frac{V(L) - P(L)}{A(L)}$. Define the linear function M as $M(x_3, ..., x_n) = 1 - \sum_{i=3}^{n} \frac{x_i}{S_i}$. Note that the area of the section of S of the form $(a, b, x_3, ..., x_n)$ is KM^2. We now wish to put upper bounds on $E(L)$.

First we will consider the case when $C \in F$. In this case, $A(L) \geq \frac{1}{2}$. By our previous discussion, $P(L) \geq KM^2(C)$ because that is the area in the triangle above C. So,

$$\begin{aligned}
E(L) &\leq 2(V(L) - P(L)) \\
&\leq 2\int_{F\cap L} KM^2(x)dx - 2KM^2(C) \\
&\leq 2\int_{N\cap L} KM^2(x)dx - 2KM^2(C) \\
&= 2\int_{N\cap L} \frac{1}{2}KM^2(x) + \frac{1}{2}KM^2(2C-x) - KM^2(C)dx \\
&= 2K\int_{N\cap L} \frac{1}{2}M^2(x) + \frac{1}{2}(2M(C)-M(x))^2 - M^2(C)dx \\
&= 2K\int_{N\cap L} M^2(x) - 2M(x)M(C) + M^2(C)dx \\
&= 2K\int_{N\cap L} (M(x)-M(C))^2dx. \qquad (5.1)
\end{aligned}$$

Notice that this value is independent of the choice of L.

Now, when $C \notin F$, $E(L) = \frac{V(L)}{A(L)}$, which is the average area above points in $L \cap F$. Notice that the closer C is to the boundary of F, the larger this is. Therefore, this is at most twice the volume above a unit square centered on the boundary of F. This is less than the quantity in (5.1), so therefore, (5.1) is an upper bound on $E(L)$.

Let Q be the set of points where $x_1 = x_2 = 0$ and $|x_i| \leq \frac{1}{2}$ for $3 \leq i \leq n$. We have that

$$\begin{aligned}
E(L) &\leq 2K\int_Q \left(\sum_{i=3}^{n} \frac{x_i}{S_i}\right)^2 dx \\
&= \left(\prod_{i=3}^{n} S_i\right) \cdot \left(\frac{4K}{n!}\right) \cdot \left(\sum_{i=3}^{n} \frac{x_i}{S_i}\right)^n |_{-1/2}^{+1/2}|_{-1/2}^{+1/2}\cdots|_{-1/2}^{+1/2},
\end{aligned}$$

where $|_{-1/2}^{+1/2}|_{-1/2}^{+1/2}\cdots|_{-1/2}^{+1/2}$ is the difference of the evaluations of the expression between plus and minus one half over all of the x_i. The last factor here is a polynomial of degree n in the $\frac{1}{S_i}$. Furthermore, it is odd in every $\frac{1}{S_i}$. Therefore, all terms but those of the form $\frac{1}{S_h^2}\prod_{i=3}^{n}\frac{1}{S_i}$ cancel out. These terms have a contribution of $\frac{1}{2^n} \cdot \binom{n}{1,1,...,1,3}$ from each of the 2^{n-2} terms. Therefore,

$$\begin{aligned}
E(L) &\leq \left(\prod_{i=3}^{n} S_i\right) \cdot \frac{4K}{n!} \cdot \frac{n!}{6} \cdot \frac{2^{n-2}}{2^n} \sum_{i=3}^{n} \left(\frac{1}{S_i^2}\prod_{j=3}^{n}\frac{1}{S_i}\right) \\
&= \frac{K}{6}\sum_{i=3}^{n}\frac{1}{S_i^2} \\
&\leq \frac{1}{12}S_1S_2\sum_{i=1}^{n}\frac{1}{S_i^2}. \qquad (5.2)
\end{aligned}$$

Let $K' = \frac{1}{12}S_1S_2\sum_{i=1}^{n}\frac{1}{S_i^2}$

By (5.2) we have $E(L) \leq K'$. Therefore, $P(L) \geq V(L) - K'A(L)$. Summing over lattice squares L, we get

$$\sum_L P(L) \geq \sum_L V(L) - K' \sum_L A(L).$$

By our discussion in the previous section, these terms are the number of l-points in S, the volume of S, and K' times the area of F respectively. Therefore,

$$\begin{aligned}\left|\{(x_1, ..., x_n) \in \mathbb{N}_0^2 \times \left(\mathbb{N}_0 + \frac{1}{2}\right)^{n-2} : \sum_{i=1}^n \frac{x_i}{S_i} \leq 1\}\right| &\geq \prod_{i=1}^n S_i \frac{1}{n!} - K' \prod_{i=3}^n S_i \frac{1}{(n-2)!} \\ &= \left(\prod_{i=1}^n S_i \frac{1}{n!}\right)\left(1 - \frac{n(n-1)}{12} \sum_{i=1}^n \frac{1}{S_i^2}\right) \\ &\geq \left(\frac{\prod_{i=1}^n S_i}{n!}\right)\left(1 - \frac{n^2}{12} \sum_{i=1}^n \frac{1}{S_i^2}\right).\end{aligned}$$

□

6. Simplification of (3.1)

We shall now use the results from the last two sections to simplify (3.1). Let C be the condition that $b\sum_{i=1}^n \alpha_i \leq k$. Let C' be C with the further condition that at most one α_i equals one. Let C'' be C and that $\alpha_i \geq 2$ for all $i \geq 2$. Let C''' be the condition C and not C' or, in other words, C and at least two of the α_i are equal to 1.

Let S_α denote $\sum_{i=1}^n \alpha_i$.

Recall, that expression (3.1) is

$$\sum_{n=0}^{\infty} \left(\sum_{\{\alpha_n\}}^{C} \left(\prod_{i=1}^n \frac{1}{\alpha_i}\right) \frac{1}{n!} \cdot \left|\{(\beta_1, \beta_2, ..., \beta_n) \in (\mathbb{N}_0)^n : \sum_{i=1}^n \alpha_i(a\beta_i + b) \leq k\}\right| \right).$$

Notice that

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C'}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right)\frac{1}{n!}\cdot\left|\{(\beta_1,\beta_2,...,\beta_n)\in(\mathbb{N}_0)^n:\sum_{i=1}^{n}\alpha_i(a\beta_i+b)\leq k\}\right|\right)\leq$$

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C''}n\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right)\frac{1}{n!}\cdot\left|\{(\beta_1,\beta_2,...,\beta_n)\in(\mathbb{N}_0)^n:\sum_{i=1}^{n}\alpha_i(a\beta_i+b)\leq k\}\right|\right)=$$

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C''}n\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right)\frac{1}{n!}\cdot\left|\{(\beta_1,\beta_2,...,\beta_n)\in\left(\mathbb{N}_0+\frac{1}{2}\right)^n:\sum_{i=1}^{n}\alpha_i(\beta_i)\leq\frac{k-(b-\frac{a}{2})S}{a}\}\right|\right)\leq$$

(By Lemma 4.1)

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C''}n\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k}{a}\right)^n\frac{1}{(n!)^2}\left(1+\frac{aS}{k-(b-a/2)S}\right)\right)\leq$$

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C''}n\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k}{a}\right)^n\frac{1}{(n!)^2}\left(1+\frac{ak/b}{k-(b-a/2)k/b}\right)\right)\leq$$

(Because the sums over α_i separate)

$$3\sum_{n=0}^{\infty}\left(n\frac{\pi^2}{6}\left(\frac{\pi^2}{6}-1\right)^{n-1}\left(\frac{k}{a}\right)^n\frac{1}{(n!)^2}\right)=$$

$$O\left(\sum_{n=0}^{\infty}\left(n\left(\frac{k}{a}\left(\frac{\pi^2}{6}-1\right)\right)^n\frac{1}{(n!)^2}\right)\right)=$$

(Because $\binom{2n}{n}\leq 4^n$)

$$O\left(\sum_{n=0}^{\infty}\left(n\left(\frac{4k}{a}\left(\frac{\pi^2}{6}-1\right)\right)^n\frac{1}{(2n)!}\right)\right)=$$

$$O\left(ke^{\sqrt{\frac{2k(\pi^2-6)}{3a}}}\right). \tag{6.1}$$

Let

$$\epsilon_1(k)=ke^{\sqrt{\frac{2k(\pi^2-6)}{3a}}}. \tag{6.2}$$

Now, let

$$G(k)=\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k-(b-a/2)S}{a}\right)^n\frac{1}{(n!)^2}\right). \tag{6.3}$$

By our prior discussion,

$$(3.1)=\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_n\}}^{C'''}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right)\frac{1}{n!}\cdot\left|\{(\beta_1,\beta_2,...,\beta_n)\in(\mathbb{N}_0)^n:\sum_{i=1}^{n}\alpha_i(a\beta_i+b)\leq k\}\right|\right)+O\left(\epsilon_1(k)\right)\leq$$

(By Lemma 4.1)

$$\sum_{n=0}^{\infty}\left(\sum_{\{\alpha_i\}}^{C'''}\left(\prod\frac{1}{\alpha_i^2}\right)\left(\frac{k-(b-a/2)S}{a}\right)^n\frac{1}{(n!)^2}\left(1+\frac{na}{k-(b-a/2)S}\right)\right)+O\left(\epsilon_1(k)\right)\leq$$

$$3G(k)+O\left(\epsilon_1(k)\right). \tag{6.4}$$

Now let us consider (3.2) under the condition that at least two of the α_i (WLOG α_1 and α_2) are equal to 1.

$$(3.2) = \left| \{(\beta_1, \beta_2, ..., \beta_n) \in (\mathbb{N}_0)^n : \sum_{i=1}^{n} \alpha_i(a\beta_i + b) \leq k\} \right|$$

$$= \left| \{(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}_0^2 \times \left(\mathbb{N}_0 + \frac{1}{2}\right)^{n-2} : \sum_{i=1}^{n} \alpha_i\beta_i \leq \frac{k-1-(b-a/2)S}{a}\} \right|$$

(By Lemma 5.1)

$$\geq \left(\frac{k-1-(b-a/2)S}{a}\right)^n \prod_{i=1}^{n} \frac{1}{\alpha_i} \frac{1}{n!} \left(1 - \frac{n^2}{12} \sum_{i=1}^{n} \left(\frac{a\alpha_i}{k-1-(b-a/2)S}\right)^2\right).$$

Using this we find that

$$(3.1) \geq -\frac{1}{12} \sum_{n=0}^{\infty} \left(\sum_{\{\alpha_n\}}^{C'''} \left(n^2 \left(\prod_{i=1}^{n} \frac{1}{\alpha_i^2} \right) \left(\sum_{i=1}^{n} \alpha_i^2 \right) \frac{1}{(n!)^2} \left(\frac{k-1-(b-a/2)S}{a} \right)^{n-2} \right) \right)$$
$$+ G(k-1) + O\left(\epsilon_1(k)\right)$$
$$\geq -\frac{1}{12} \sum_{n=0}^{\infty} \left(\sum_{\{\alpha_n\}}^{C'''} \left(n \left(\prod_{i=1}^{n-1} \frac{1}{\alpha_i^2} \right) \frac{1}{((n-1)!)^2} \left(\frac{k-1-(b-a/2)S}{a} \right)^{n-2} \right) \right)$$
$$+ G(k-1) + O\left(\epsilon_1(k)\right)$$
$$\geq -\frac{1}{12} \cdot \frac{b-a/2}{a} \sum_{n=0}^{\infty} \left(\sum_{\{\alpha_{n-1}\}}^{C'''} \left(\frac{n}{n-1} \left(\prod_{i=1}^{n-1} \frac{1}{\alpha_i^2} \right) \frac{1}{((n-1)!)^2} \left(\frac{k-1-(b-a/2)S}{a} \right)^{n-1} \right) \right)$$
$$+ G(k-1) + O\left(\epsilon_1(k)\right)$$

(6.5)

$$\geq \frac{5}{6} G(k-1) + O\left(\epsilon_1(k)\right).$$

Combining (6.4) and (6.5) we get

$$3G(k) + O\left(\epsilon_1(k)\right) \geq \sum_{i=0}^{k} P_{a,b}(i) \geq \frac{5}{6} G(k-1) + O\left(\epsilon_1(k)\right). \tag{6.6}$$

Let

$$y = \frac{\pi^2 k}{6a}.$$

The summand in $G(k)$ is

$$\sum_{\{\alpha_n\}}^{C} \left(\left(\prod_{i=1}^{n} \frac{1}{\alpha_i^2} \right) \left(\frac{k-(b-a/2)S}{a} \right)^n \frac{1}{(n!)^2} \right) \leq$$
$$\sum_{\{\alpha_n\}}^{C} \left(\left(\prod_{i=1}^{n} \frac{1}{\alpha_i^2} \right) \left(\frac{k}{a} \right)^n \frac{1}{(n!)^2} \right) \leq$$
$$\frac{y^n}{(n!)^2}.$$

If we sum this over $n \in \mathbb{N}$ where $n < \sqrt{y} - 2y^{3/8}$ or $n > \sqrt{y} + 2y^{3/8}$ we get at most

$$\frac{y^{\sqrt{y}}}{(\sqrt{y}!)^2}\left(\sum_{i=0}^{\infty}\left(\frac{y}{y+2y^{7/8}+y^{3/4}}\right)^{i+y^{3/8}} + \left(\frac{y-2y^{7/8}+y^{3/4}}{y}\right)^{i+y^{3/8}}\right) \lesssim$$

(By (2.4))
$$e^{2\sqrt{y}}\left(1-y^{-1/8}\right)^{3/8} \lesssim$$
$$O\left(e^{2\sqrt{y}-y^{1/4}}\right).$$

Let

$$\epsilon_2(k) = e^{2\sqrt{y}-y^{1/4}}.$$

Then we have that

$$G(k) = \sum_{|n-\sqrt{y}|<y^{3/8}}\left(\sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k-(b-a/2)S}{a}\right)^n\frac{1}{(n!)^2}\right) + O\left(\epsilon_2(k)\right). \tag{6.7}$$

7. Upper bounds on $G(k)$

We have from (6.7) that

$$G(k) = \sum_{|n-\sqrt{y}|<y^{3/8}}\left(\sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k-(b-a/2)S}{a}\right)^n\frac{1}{(n!)^2}\right) + O\left(\epsilon_2(k)\right).$$

We will examine the summand for n such that $|n-\sqrt{y}| < 2y^{3/8}$. This is

$$\begin{aligned}
\sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k-(b-a/2)S}{a}\right)^n\frac{1}{(n!)^2} &= \sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k}{a}\right)^n\left(1-\frac{(b-a/2)S}{k}\right)^n\frac{1}{(n!)^2}\\
&\le \sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}\right)\left(\frac{k}{a}\right)^n e^{-\frac{n(b-a/2)S}{k}}\frac{1}{(n!)^2}\\
&= \sum_{\{\alpha_i\}}^{C}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i^2}e^{-\frac{n(b-a/2)\alpha_i}{k}}\right)\left(\frac{k}{a}\right)^n\frac{1}{(n!)^2}\\
&\le \left(\sum_{i=1}^{\infty}\frac{e^{-\frac{n(b-a/2)i}{k}}}{i^2}\right)^n\left(\frac{k}{a}\right)^n\frac{1}{(n!)^2}. \qquad (7.1)
\end{aligned}$$

Let

$$h(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

This converges absolutely for $|x| \leq 1$. Furthermore we have that

$$h(x) = \int_0^x \frac{-\log(1-t)}{t} dt$$

(Because $h(1) = \zeta(2) = \frac{\pi^2}{6}$)

$$\begin{aligned} &= \frac{\pi^2}{6} - \int_x^1 \frac{-\log(1-t)}{t} dt \\ &\leq \frac{\pi^2}{6} - \int_x^1 -\log(1-t) dt \\ &= \frac{\pi^2}{6} + (1-x)\log(1-x) + (1-x). \end{aligned}$$

Let $m = (\frac{b}{a} - \frac{1}{2})$. Using this we find that

$$\begin{aligned} (7.1) &\leq \left(\frac{\pi^2}{6} + (1 - e^{-\frac{n(b-a/2)}{k}}) \log(1 - e^{-\frac{n(b-a/2)}{k}}) + 1 - e^{-\frac{n(b-a/2)}{k}} \right)^n \left(\frac{k}{a} \right)^n \frac{1}{(n!)^2} \\ &\leq \left(y + \frac{n(b-a/2)}{a} \log \left(\frac{n(b-a/2)}{k} \right) + \frac{n(b-a/2)}{a} \right)^n \frac{1}{(n!)^2} \\ &= \left(1 + m\frac{n}{y} \log \left(\frac{n(b-a/2)}{k} \right) + m\frac{n}{y} \right)^n \frac{y^n}{(n!)^2} \\ &\leq e^{m\frac{n^2}{y} \log\left(\frac{n(b-a/2)}{k} \right) + m\frac{n^2}{y}} \frac{y^n}{(n!)^2}. \end{aligned}$$

Now since $\frac{n^2}{y} = 1 + O\left(y^{-1/8}\right)$, we have that

$$\begin{aligned} (7.1) &\leq \left(\frac{n(b-a/2)}{k} \right)^{m(1+O\left(y^{-1/8}\right))} e^{m(1+O\left(y^{-1/8}\right))} \frac{y^n}{(n!)^2} \\ &\lesssim \left(\frac{6ae(b-a/2)}{\pi^2} \right)^m \left(\frac{n}{y} \right)^m \frac{y^n}{(n!)^2} \\ &\lesssim \left(\frac{6ae(b-a/2)}{\pi^2} \right)^m n^{-m} \frac{y^n}{(n!)^2}. \end{aligned}$$

So it turns out that

$$G(k) \lesssim N_1 \sum_{n=1}^{\infty} n^{-m} \frac{y^n}{(n!)^2} \tag{7.2}$$

where

$$N_1 = \left(\frac{6ae(b-a/2)}{\pi^2} \right)^{(\frac{b}{a} - \frac{1}{2})}.$$

8. Lower bounds on $G(k)$

We have from (6.7) that

$$G(k) = \sum_{|n-\sqrt{y}|<y^{3/8}} \left(\sum_{\{\alpha_i\}}^{C} \left(\prod_{i=1}^{n} \frac{1}{\alpha_i^2} \right) \left(\frac{k-(b-a/2)S}{a} \right)^n \frac{1}{(n!)^2} \right) + O\left(\epsilon_2(k)\right).$$

We will examine the summand for n such that $|n-\sqrt{y}| < 2y^{3/8}$. This is

$$\sum_{\{\alpha_n\}}^{C} \left(\prod_{i=1}^{n} \frac{1}{\alpha_i^2} \right) \left(\frac{k-(b-a/2)S}{a} \right)^n \frac{1}{(n!)^2} \geq \sum_{\{\alpha_n\}}^{\alpha_i \leq \frac{k}{nb}} \left(\prod_{i=1}^{n} \frac{k-(b-a/2)S}{a\alpha_i^2} \right) \frac{1}{(n!)^2}$$

(By the Arithmetic-Geometric Mean Inequality)

$$\geq \sum_{\{\alpha_n\}}^{\alpha_i \leq \frac{k}{nb}} \left(\prod_{i=1}^{n} \frac{k-n(b-a/2)\alpha_i}{a\alpha_i^2} \right) \frac{1}{(n!)^2}$$

$$(8.1) \qquad = \left(\sum_{i=1}^{\frac{k}{nb}} \frac{k-n(b-a/2)i}{ai^2} \right)^n \frac{1}{(n!)^2}.$$

Notice that

$$\sum_{i=1}^{\frac{k}{nb}} \frac{k-n(b-a/2)i}{ai^2} = \frac{k}{a} \sum_{i=1}^{\frac{k}{nb}} \frac{1}{i^2} - \frac{n(b-a/2)}{a} \sum_{i=1}^{\frac{k}{nb}} \frac{1}{i}$$

(Remembering that $m = \frac{b}{a} - \frac{1}{2}$)

$$\geq \frac{k}{a} \left(\frac{\pi^2}{6} - \frac{nb}{k} \right) - nm \left(\log\left(\frac{k}{nb} \right) + \gamma \right)$$

(Remembering that $y = \frac{\pi^2 k}{6a}$)

$$= y - \frac{nb}{a} - nm \log\left(\frac{k}{nb} \right) - nm\gamma$$

$$= y \left(1 - \frac{nb}{ay} - \frac{nm}{y} \log\left(\frac{k}{nb} \right) - \frac{nm}{y}\gamma \right).$$

Substituting this into (8.1) we get

$$\left(1 - \frac{nb}{ay} - \frac{nm}{y} \log\left(\frac{k}{nb} \right) - \frac{nm}{y}\gamma \right)^n \frac{y^n}{(n!)^2}.$$

Now, since $\frac{nb}{ay} + \frac{nm}{y} \log\left(\frac{k}{nb}\right) + \frac{nm}{y}\gamma = O\left(\frac{\log(n)}{n}\right)$ and since $\frac{n^2}{y} = 1 + O\left(y^{-1/8}\right)$,

$$(8.1) \geq e^{-(1+O(y^{-1/8}))(\frac{b}{a}+m\log(k)-m\log(nb)+m\gamma)+O\left(\frac{\log^2(n)}{n}\right)} \frac{y^n}{(n!)^2}$$

$$\gtrsim e^{-\frac{b}{a}} e^{-m\gamma} \left(\frac{nb}{k} \right)^m \frac{y^n}{(n!)^2}$$

$$= e^{-\frac{b}{a}} e^{-m\gamma} \left(\frac{6a}{b\pi^2} \right)^m \left(\frac{n}{y} \right)^m \frac{y^n}{(n!)^2}$$

$$\sim N_2 n^{-m} \frac{y^n}{(n!)^2},$$

where

$$N_2 = e^{-\frac{b}{a}} e^{-m\gamma} \left(\frac{6a}{b\pi^2} \right)^m .$$

Therefore, by (6.7)

$$G(k) \gtrsim N_2 \sum_{|n-\sqrt{y}|<2y^{3/8}} n^{-m} \frac{y^n}{(n!)^2} + O\left(\epsilon_2(k)\right).$$

But since $n^{-m}\frac{y^n}{(n!)^2} < \frac{y^n}{(n!)^2}$, by previous arguments we have that

$$G(k) \gtrsim N_2 \sum_{n=1}^{\infty} n^{-m} \frac{y^n}{(n!)^2} + O\left(\epsilon_2(k)\right). \tag{8.2}$$

9. Final analysis of $G(k)$ and proof of Theorem 2

Let

$$F(y) = \sum_{n=1}^{\infty} n^{-m} \frac{y^n}{(n!)^2}.$$

By (7.2) and (8.2) we have

$$N_2 F(y) + O\left(\epsilon_2(k)\right) \lesssim G(k) \lesssim N_1 F(y). \tag{9.1}$$

We now use Lemma 2.1 and (2.4) to find that

$$\begin{aligned}
F(y) &= \sum_{n=1}^{\infty} n^{-m} \frac{y^n}{(n!)^2} \\
&\sim \sum_{n=1}^{\infty} n^{-2n-m-1} e^{2n} \frac{y^n}{2\pi} \\
&= \sum_{n=1}^{\infty} (2n+m+1/2)^{-2n-m-1} \left(\frac{2n+m+1/2}{n} \right)^{2n+m+1} e^{2n} \frac{y^n}{2\pi} \\
&= \sum_{n=1}^{\infty} (2n+m+1/2)^{-2n-m-1} 2^{2n+m+1} \left(1 + \frac{2m+1}{4n} \right)^{2n+m+1} e^{2n} \frac{y^n}{2\pi} \\
&\sim \sum_{n=1}^{\infty} (2n+m+1/2)^{-2n-m-1} e^{2n+m+1/2} (2\pi)^{-1/2} 2^{m+1} (2\pi)^{-1/2} (4y)^n \\
&\sim \frac{1}{\sqrt{\pi}} y^{-m/2-1/4} \sum_{n=1}^{\infty} \frac{(2\sqrt{y})^{2n+m+1/2}}{(2n+m+1/2)!}. \qquad (9.2)
\end{aligned}$$

If

$$H(z) = \sum_{n=1}^{\infty} \frac{z^{2n+m+1/2}}{(2n+m+1/2)!},$$

then (9.2) implies that

$$F(y) \sim \pi^{-1/2} y^{-b/2a} H(2\sqrt{y}).$$

This leaves us to find an asymptotic formula for $H(z)$.

Notice that $H(z)$ is uniquely defined by the differential equation with initial conditions

$$H''(z) - H(z) = \frac{z^{m+1/2}}{(m+1/2)!}$$

And $H(0) = H'(0) = 0$. The solution to this equation is

$$\begin{aligned} H(z) &= \frac{1}{2}e^{z}\int_0^z \frac{t^{m+1/2}e^{-t}}{(m+1/2)!}dt - \frac{1}{2}e^{-z}\int_0^z \frac{t^{m+1/2}e^{t}}{(m+1/2)!}dt \\ &= \frac{1}{2}e^{z}\int_0^z \frac{t^{m+1/2}e^{-t}}{(m+1/2)!}dt + O\left(z^{m+3/2}\right) \\ &\sim \frac{e^z}{2}. \end{aligned} \tag{9.3}$$

Proof of Theorem 2. Combining (9.2) and (9.3) we get that

$$F(y) \sim \left(\frac{1}{2\sqrt{\pi}y^{\frac{b}{2a}}}\right)e^{2\sqrt{y}}.$$

Substituting $y = k\frac{\pi^2}{6a}$, it turns out that

$$F(y) \sim N_3 k^{-\frac{b}{2a}} e^{\pi\sqrt{\frac{2k}{3a}}},$$

where

$$N_3 = \frac{1}{2\sqrt{\pi}}\left(\frac{\pi^2}{6a}\right)^{\frac{b}{2a}}.$$

Note that for large k that this is much bigger than either ϵ_1 or ϵ_2. Combining this with (9.1) we get that

$$N_2N_3k^{-\frac{b}{2a}}e^{\pi\sqrt{\frac{2k}{3a}}} \lesssim G(k) \lesssim N_1N_3k^{-\frac{b}{2a}}e^{\pi\sqrt{\frac{2k}{3a}}}.$$

Now since $F(k\frac{\pi^2}{6a}) \sim F((k-1)\frac{\pi^2}{6a})$, by (6.6) we have that

$$\frac{5}{6}N_2N_3k^{-\frac{b}{2a}}e^{\pi\sqrt{\frac{2k}{3a}}} \lesssim \sum_{i=1}^{k} P_{a,b}(i) \lesssim 3N_1N_3k^{-\frac{b}{2a}}e^{\pi\sqrt{\frac{2k}{3a}}}.$$

This proves the theorem for any constants

$$C^-_{a,b} < \frac{5}{6}\frac{1}{2\sqrt{\pi}}\left(\frac{\pi^2}{6a}\right)^{\frac{b}{2a}} e^{-\frac{b}{a}}e^{-(b/a-1/2)\gamma}\left(\frac{6a}{b\pi^2}\right)^{(b/a-1/2)}$$

and

$$C^+_{a,b} > 3\frac{1}{2\sqrt{\pi}}\left(\frac{\pi^2}{6a}\right)^{\frac{b}{2a}}\left(\frac{6ae(b-a/2)}{\pi^2}\right)^{(\frac{b}{a}-\frac{1}{2})}.$$

□

Remark: Note that when $a = 1$ and $b = 2$, $\sum_{i=1}^{n} P_{a,b}(i) = P_{1,1}(n)$. So we have just proved that

$$\frac{.0036...}{n}e^{\pi\sqrt{2n/3}} \lesssim P_{1,1}(n) \lesssim \frac{5.4...}{n}e^{\pi\sqrt{2n/3}}.$$

While Hardy and Ramanujan's asymptotic formula gives

$$\frac{.14...}{n}e^{\pi\sqrt{2n/3}} \sim P_{1,1}(n).$$

10. Proof of Theorem 1

Notice that since for a and b relatively prime, any natural number larger than $2b(a+b)$ can be written uniquely as the sum of a multiple of b less than $a(a+b)$ plus a multiple of $a+b$ less than $b(a+b)$ plus a multiple of $b(a+b)$,

$$\frac{q^{2a(a+b)}}{1-q} \leq_q \frac{(1+q^b+q^{2b}+...+q^{b(a+b-1)})(1+q^{a+b}+q^{2(a+b)}+...+q^{(a+b)(a-1)})}{1-q^{a(a+b)}} \leq_q \frac{1}{1-q}.$$

Recall that $\leq_q$ says that the coefficients of one series are bigger than the corresponding coefficients of the other. This last expression implies that

$$\frac{q^{2b(a+b)}}{(1-q)(1-q^{b(a+b)})} \leq_q \frac{1}{(1-q^b)(1-q^{a+b})} \leq_q \frac{1}{(1-q)(1-q^{b(a+b)})}.$$

We also have that

$$\frac{q^{b(a+b)}}{b(a+b)(1-q)^2} \leq_q \frac{1}{(1-q)(1-q^{b(a+b)})} \leq_q \frac{1}{b(a+b)(1-q)^2}.$$

And since

$$\frac{1}{1-q} = \frac{Pl(q)}{1-q^{a+b}},$$

where $Pl(q)$ is a polynomial with $Pl(1) = a+b$, we have that

$$Q_1(q)\frac{1}{(1-q)(1-q^{a+b})} \leq_q \frac{1}{(1-q^b)(1-q^{a+b})} \leq_q Q_2(q)\frac{1}{(1-q)(1-q^{a+b})}, \tag{10.1}$$

where $Q_1(q)$ and $Q_2(q)$ are polynomials in q the sum of whose coefficients are b with highest degree at most $3b(a+b)$.

Proof of Theorem 1. By (10.1) we have

$$\frac{bq^{3b(a+b)}}{1-q}\prod_{n=1}^{\infty}\frac{1}{1-q^{a+nb}} \leq_q \prod_{n=0}^{\infty}\frac{1}{1-q^{a+nb}} \leq_q \frac{b}{1-q}\prod_{n=1}^{\infty}\frac{1}{1-q^{a+nb}},$$

or that

$$b\sum_{i=1}^{n-3b(a+b)} P_{a,a+b}(i) \leq P_{a,b}(n) \leq b\sum_{i=1}^{n} P_{a,a+b}(i).$$

By Theorem 2, we have for any positive constants satisfying

$$C^-_{a,b} < b\frac{5}{6}\frac{1}{2\sqrt{\pi}}\left(\frac{\pi^2}{6a}\right)^{\frac{a+b}{2a}} e^{-\frac{a+b}{a}}e^{-(b/a+1/2)\gamma}\left(\frac{6a}{(a+b)\pi^2}\right)^{(b/a+1/2)}$$

and

$$C^+_{a,b} > 3b\frac{1}{2\sqrt{\pi}}\left(\frac{\pi^2}{6a}\right)^{\frac{a+b}{2a}}\left(\frac{6ae(b+a/2)}{\pi^2}\right)^{(\frac{b}{a}+\frac{1}{2})}.$$

that

$$\frac{C^-_{a,b}}{n^{\frac{a+b}{2a}}}e^{\pi\sqrt{\frac{2n}{3a}}} < P_{a,b}(n) < \frac{C^+_{a,b}}{n^{\frac{a+b}{2a}}}e^{\pi\sqrt{\frac{2n}{3a}}}$$

for all sufficiently large n.

□

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